A note on the cardinal factorial

by

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Abstract. A model of set theory without choice is exhibited which satisfies "For all infinite x, $2^x = x!$".

Dawson and Howard [1] compare the cardinals 2^x (the cardinal of the power set) and $x!$ (the cardinal of the symmetric group) in set theory without the axiom of choice. They show that $x<x!$ for $x\geqslant 3$, $2^x\leqslant x!$ for x such that $2\cdot x = x$, and $2^x = x!$ for x such that $x^2 = x$. They also illustrate by examples in models of set theory that all 3 possibilities of inequality between 2^x and $x!$ can occur. These possibilities are; $2^x<x!$, $x!<2^x$, and 2^x is incomparable with $x!$.

A question which Dawson and Howard pose without answering concerns the strength of the statement "For all infinite x, $x! = 2^x$". They ask whether this statement is an equivalent to the axiom of choice. In this note we answer the question in the negative by producing a model for ZF set theory in which the statement is true and the axiom of choice is false.

1. The model. The model is introduced in [3] § II as an example of a ZF model without choice in which there is a class of sets consisting of exactly one representative for each cardinal number. It is defined as $M = U[I]$ where U is a model of ZF, class choice, and the generalized continuum hypothesis and where $I = \bigcup_{n\in\omega} I_n$, $I_{-1} = 2$, and I_{n+1} is a countable set of independent generic functions from ω onto I_n. Notice that $\langle I_n^n\colon n\in\omega\rangle \in M$, $I_0 = I\cap 2^\omega$, $I_{n+1} = I\cap I_n^\omega$.

M can be regarded as an intermediate model between U and V where V is the extension of U by a generic filter for a countable partially ordered set. It follows that V satisfies class choice and the generalized continuum hypothesis and that alephs and cofinalities are preserved between U, M and V. We adopt the convention that a set theoretical concept is assumed to be defined relative to M unless a relativization to U or V is indicated by the appropriate superscript. A concept, such as ω_α, whose meaning is independent of its relativization to U, M, or V is not superscripted.

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a. Remark (In M). If 2 sets of ordinals have the same cardinal then there is a $1:1$ onto map between them defined from parameters in U.

Proof. Map the two sets $1:1$ and onto the ordinals of their order types. The two ordinals have the same cardinal in M, hence in U, so there is a $1:1$ onto map between them in U.

2. The support structure of M. If G is a finite subset of I we say that $x \in \nabla G$ if x is definable in M with parameters in $G \cup \{I\} \cup U$. Since $M = U[I]$ and the transitive closure of $I \subset U \cup I$ it follows that every $x \in M$ belongs to some ∇G. I has a canonical linear ordering in M (it is defined inductively using $I_{n+1} \subset I_n^{(\omega)}$) so there is a canonical function $T(G, \alpha)$ such that for fixed G, $T(G, \alpha)$ maps On (the class of ordinals) onto ∇G.

Every member of I_{n+1} is a map from ω onto I_n. It follows that if $G \subset I_{n+1}$ ∇G contains an enumeration of $\bigcup_{m \leqslant n} I_m$. Thus if $x \in M$ there is an $n \in \omega$ and $G \subset I_n$ such that $x \in \nabla G$. The support lemma ([3] II 14) shows that if n is taken to be least and $|G|$ is taken to be minimal then G is uniquely determined. This G is denoted G_x. A corollary of the density lemma ([3] II 13b) is that if $f \in I \cap \nabla G$ and $G \subset I_n$ then $f \in G \cup \bigcup_{m<n} I_n$. From this:

a. THEOREM [3]. *The axiom of choice fails in M. In fact $\langle I_n\colon n \in \omega\rangle$ is a countable sequence of countable sets which fails to have a choice function.*

b. THEOREM [3]. *The ordering theorem is true in M. In fact if J_n is the set of finite subsets of I_n and $J = \bigcup_{n \in \omega} J_n$ then there is a $1:1$ onto function $T^*\colon J \times On \to M$ definable from I in M.*

c. THEOREM [3]. *There is a function, Ψ, definable from I in M such that for every infinite x, $\Psi(x)$ is a countable (in M) subset. $\Psi(x)$ is easily produced from the fact that for some least n, $x \cap \bigcup_{G \in J_n} \nabla G$ is infinite.*

3. On 2^{ω_α} in M. Since each $G \in J$ is coded by a real. (An easy induction on n established this for $f \in I_n$ and a coding trick; together with Theorem 2b, extends this to G.) and since $(2^{\omega_\alpha})^V = \omega_{\alpha+1}$ it follows that $|\{a \in 2^{\omega_\alpha}\colon G_a = G\}| = \omega_{\alpha+1}$. ($|x|$ denotes the cardinal number of x.) Remark 1a now gives:

a. LEMMA (In M). $2^{\omega_\alpha} = |J \times \omega_{\alpha+1}|$.

For any $x \in M$ and $G \in J$ set $\omega(G, x) = |\{y \in x\colon G_y = G\}|$. Also set $\omega(x) = \operatorname{Sup}_{G \in J} \omega(G, x)$ and $\omega^*(x) = \operatorname{Max}(\omega, \omega(x))$.

b. LEMMA (In M). *If $2^{\omega_\alpha} \leqslant |x|$ and $\omega(x) \leqslant \omega_{\alpha+1}$ then $2^{\omega_\alpha} = |x|$.*

Proof. By the Cantor–Bernstein theorem one need only show $|x| \leqslant 2^{\omega_\alpha}$. By Remark 1a and the fact that $\omega(x) \leqslant \omega_{\alpha+1}$ there is a $1:1$ map from x into $J \times \omega_{\alpha+1}$. This suffices by Lemma 3a above.

4. THE MAIN THEOREM (In M). *If x is infinite $2^x = x! = 2^{\omega^*}(x)$.*

Proof. We will apply Lemma 3b. Let $\omega^*(x) = \omega_\alpha$. To see that $\omega(2^x) \leqslant \omega_{\alpha+1}$ notice that $\omega^*(x) = |x|^V$ so that $(2^x)^V = \omega_{\alpha+1}$. $2^x \subset (2^x)^V$ so $|z| \leqslant \omega_{\alpha+1}$ for any well ordered $z \subset x$. This applies in particular to $\{y \in x\colon G_Y = G\}$. A similar argument shows that $\omega(X!) \leqslant \omega_{\alpha+1}$.

It remains to prove $2^{\omega^*(x)} \leqslant 2^x$ and $2^{\omega^*(x)} \leqslant x!$. We consider first the case in which x has a subset y, with cardinal $\omega^*(x)$. This case includes $\omega^*(x) = \omega$ by Theorem 2c. In this case $2^y \subset 2^x$ so $2^{\omega^*(x)} \leqslant 2^x$. Also, since $y^2 = y$, $2^{\omega^*(x)} = 2^y = y! \leqslant x!$.

Unfortunately, owing to the lack of choice in M, one cannot dismiss the possibility that x has no subset of cardinal $\omega^*(x)$. In this case $\{\omega(G, x)\colon G \in J\}$ is a countable set of ordinals by Remark 1a and the fact that J is a countable union of countable sets. Let $\omega^{(0)} < \omega^{(1)} < \omega^{(2)} < \ldots < \omega^*(x)$ be a sequence of uncountable cardinals from among the $\omega(G, x)$ with limit $\omega^*(x)$. (Note that $\omega^*(x) = \omega(x)$ since $\omega^*(x) \neq \omega$.) Let $A_n = \{G \in J\colon \omega(G, x) = \omega^{(n)}\}$. Let $y = \bigcup_{n \in \omega} (A_n \times \omega^{(n)})$. Another use of Remark 1a permits the conclusion that $|y| \leqslant |x|$, hence that $2^y \leqslant 2^x$ and $y! \leqslant x!$.

It now suffices to show $2^{\omega^*(x)} \leqslant 2^y$ and $2^{\omega^*(x)} \leqslant y!$. Actually $2^{\omega^*(x)} \leqslant 2^y$ will do because $y = 2 \cdot y$ so $2^y \leqslant y!$. To see that $y = 2 \cdot y$ one has only to notice that the canonically defined 1-1 onto maps between $\omega^{(n)}$ and $2 \cdot \omega^{(n)}$ give canonicaly defined 1-1 onto maps between $A_n \times \omega^{(n)}$ and $2 \times A_n \times \omega^{(n)}$. These patch together to give a 1-1 onto map between y and $2 \times y$.

It is also not difficult to see that $2^{\omega^*(x)} \leqslant 2^y$. Map $a \subset \omega^*(x)$ to $\bigcup_{n \in \omega} (A_n \times (\omega^{(n)} \cap a))$. If $a \neq b$ then, as $\omega^*(x) = \operatorname{Sup} \omega^{(n)}$, some $a \cap \omega^{(n)} \neq b \cap \omega^{(n)}$. Thus a and b have different images.

5. Concluding remarks.

a. $2^x = x!$ can now be seen not to imply $x^2 = x$ or even $2 \times x = x$. As was shown in [3] II 19, the set I fails to satisfy $2 \times I = I$ in M.

b. M satisfies "For every infinite x there is a well ordered $\omega^*(x)$ such that $2^x = 2^{\omega^*(x)}$. This is an interesting property but it does not imply $2^x = x!$ because a similar argument establishes this statement in the model $U[I_0]$ which is the Halpern–Lévy model of [2]. $U[I_0]$ can be seen, by methods similar to those of [1], to satisfy $I_0! < 2^{I_0}$.

c. Our arguments have made little use of the particular definition of $x!$. Indeed let $\mathscr{F}$ be any set valued operation which satisfies:

1) The predicate $y \in \mathscr{F}(x)$ is absolute (at least from M to V).

2) ZF proves $|y| \leqslant x \Rightarrow |\mathscr{F}(y)| \leqslant |\mathscr{F}(x)|$ and $|2x| = |x| \Rightarrow 2^x \leqslant |\mathscr{F}(x)|$ for infinite x.

3) ZF with choice proves $2^x = |\mathscr{F}(x)|$ for infinite x.

The statement "For every infinite x, $2^x = |\mathscr{F}(x)|$.", holds in M (and therefore is not an equivalent to the axiom of choice). Examples of $\mathscr{F}$, apart from $x!$, are x^x and $x^x - x!$.

References

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Some results on uniform spaces with linearly ordered bases

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Abstract. The paper is concerned with uniform spaces having a base linearly ordered by inclusion of entourages (or by refinement of uniform coverings, respectively). By a well-known fact, these spaces coincide with the socalled ω_μ-metric spaces, for which several well-known metrization theorems can be extended. — Amongst several applications, two characterizations of compact metric spaces and of separable metric spaces are derived.

§ 1. Introduction. The uniform structure $\mathfrak{U}_d$ of any metric space (X, d) obviously has a linearly ordered base $\mathfrak{B}$ consisting of entourages

$$U_n = \{(x, y) |\ d(x, y) < 1/n\}, \quad n = 1, 2, \ldots$$

More generally, it is interesting to study uniform spaces $(X, \mathfrak{U})$ with linearly ordered bases $\mathfrak{B}$ ($U_i < U_j$ iff $U_i \supset U_j$ for $U_i, U_j \in \mathfrak{B}$). Such spaces have been investigated by many authors and under several aspects: R. Sikorski [37], F. Hausdorff [12, p. 285 ff], L. W. Cohen and C. Goffman [5], F. W. Stevenson and W. J. Thron [39], Shu-Tang Wang [42], P. Nyikos and H. C. Reichel [27], [28], A. Hayes [15], P. Nyikos [25], R. Paintandre [29], E. M. Alfsen and O. Njastad [2], M. Fréchet [9].

If $(X, \mathfrak{U})$ is a uniform space with a linearly ordered base $\mathfrak{B}$ and $\aleph_\mu$ is the least power of such a base, then there exists an equivalent well ordered base of power $\aleph_\mu$ ([39]). (Obviously, such a space is metrizable iff $\mu = 0$). Moreover, F. W. Stevenson and W. J. Thron [39] showed that any such space $(X, \mathfrak{U})$ is ω_μ-metrizable in the sense of R. Sikorski. That means: there is a linearly ordered abelian group G which has a decreasing ω_μ-sequence converging to 0 in the order topology, and a "distance function" $\varrho\colon X \times X \to G$ satisfying the usual axioms for a metric on X, which generates the topology of X. (Here ω_μ denotes the μth infinite cardinal). — Conversely, any ω_μ-metric ϱ on X induces a uniformity $\mathfrak{U}_\varrho$ on X, a base of which consists of all sets $U_a = \{(x, y) |\ \varrho(x, y) < a\}$, $a \in G$, $a > 0$. Many properties of metric spaces have their analogues in the theory of ω_μ-metric spaces, however the uncountable case usually regards different proofs and methods (see for example: [27], [28], [42]).