A note on the cardinal factorial

by

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Abstract. A model of set theory without choice is exhibited which satisfies "For all infinite \( x \), \( 2^x = x^x \).

Dawson and Howard [1] compare the cardinals \( 2^x \) (the cardinal of the power set) and \( x! \) (the cardinal of the symmetric group) in set theory without the axiom of choice. They show that \( x < x! \) for \( x \geq 3 \), \( 2^x < x! \) for \( x \) such that \( 2 \times x = x \), and \( 2^x = x! \) for \( x \) such that \( x^2 = x \). They also illustrate by examples in models of set theory that all 3 possibilities of inequality between \( 2^x \) and \( x! \) can occur. These possibilities are: \( 2^x < x! \), \( x! < 2^x \), and \( 2^x \) is incomparable with \( x! \).

A question which Dawson and Howard pose without answering concerns the strength of the statement "For all infinite \( x \), \( x! = 2^x \). They ask whether this statement is equivalent to the axiom of choice. In this note we answer the question in the negative by producing a model for ZF set theory in which the statement is true and the axiom of choice is false.

1. The model. The model is introduced in [3] § II as an example of a ZF model without choice in which there is a class of sets consisting of exactly one representative for each cardinal number. It is defined as \( M = U[I] \) where \( U \) is a model of ZF, class choice, and the generalized continuum hypothesis and where \( I = \bigcup_{\alpha \in \omega} I_{\alpha} \), \( I_{\omega} = 2 \), and \( I_{\alpha \cdot 2} \) is a countable set of independent generic functions from \( \alpha \) onto \( I \). Notice that \( \langle I_{\alpha}; \alpha \in \omega \rangle \in M \), \( I_0 = I \cap 2^x \), \( I_{\omega \cdot 1} = I \cap I_0^x \).

\( M \) can be regarded as an intermediate model between \( U \) and \( V \) where \( V \) is the extension of \( U \) by a generic filter for a countable partially ordered set. It follows that \( V \) satisfies class choice and the generalized continuum hypothesis and that alephs and cofinalities are preserved between \( U, M \) and \( V \). We adopt the convention that a set theoretical concept is assumed to be defined relative to \( M \) unless a relativization to \( U \) or \( V \) is indicated by the appropriate superscript. A concept, such as \( \omega_\alpha \), whose meaning is independent of its relativization to \( U, M, \) or \( V \) is not superscripted.

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a. Remark (in M). If 2 sets of ordinals have the same cardinal then there is
a 1:1 onto map between them defined from parameters in U.

Proof. Map the two sets 1:1 and onto the ordinals of their order types. The
two ordinals have the same cardinal in M, hence in U, so there is a 1:1 onto
map between them in U.

2. The support structure of M. If G is a finite subset of I we say that x ∈ VG
if x is definable in M with parameters in G ∪ {I} ∪ U. Since M = U[U] and
the transitive closure of I ⊆ U ∪ I follows that every x ∈ M belongs to some VG. I has
a canonical linear ordering in M (it is defined inductively using I_{x+1} ⊆ \mathbb{I}_x) so
there is a canonical function T(G,x) such that for fixed G, T(G,x) maps On the (class of
ordinals) onto VG.

Every member of I_{x+1} is a map from ω onto I_x. It follows that if G⊆I_{x+1}
VG contains an enumeration of \bigcup I_\omega. Thus if x ∈ M there is an n ∈ ω and G ⊆ I_n
such that x ∈ VG. The support lemma ([3] II 14) shows that if n is taken to be least
and [G] is taken to be minimal then G is uniquely determined. This G is denoted G_n.
A corollary of the density lemma ([3] II 13b) is that if n ∈ E then f|G ⊆ E and G ⊆ I_n.
From this:

a. Theorem [3]. The axiom of choice fails in M. In fact (J_x: n ∈ \omega) is a countable
sequence of countable sets which fails to have a choice function.

b. Theorem [3]. The ordering theorem is true in M. In fact if J_x is the set of
finite subsets of I_x and J = \bigcup J_x then there is a 1:1 onto function T*: J x \mathbb{N} → M
x \omega definable from I in M.

c. Theorem [3]. There is a function, \mathcal{F}, definable from I in M such that for the
infinite x, \mathcal{F}(x) is a countable (in M) subset. \mathcal{F}(x) is easily produced from the fact
that for some least n, x ∈ \bigcup VG is infinite.

3. On 2^\omega in M. Since each G ∈ E is coded by a real. (An easy induction on n
established this for f|G and a coding trick; together with Theorem 2b, extends this
to G.) Since 2^{\omega^2} = \omega_{x+1} it follows that \{[α ∈ 2^\omega: G_n = G] = \omega_{x+1}. (x)
denotes the cardinal number of x.) Remark la now gives:

a. Lemma (in M). 2^\omega = [J x \omega_{x+1}].

For any x ∈ M and G ∈ J set o(G, x) = [y ∈ x: G_y = G]. Also set o(x) = Sup(o(G, x)) and o^*(x) = Max(o, o(x)).

b. Lemma (in M). If 2^{\omega} \subseteq \omega(x) and o(x) \subseteq \omega_{x+1} then 2^{\omega} \subseteq \omega(x).

Proof. By the Cantor–Bernstein theorem one need only show [x] \subseteq 2^{\omega}. By
Remark la and the fact that o(x) \subseteq \omega_{x+1} there is a 1:1 map from x into J x \omega_{x+1},
This suffices by Lemma 3a above.

4. The Main Theorem (in M). If x is infinite 2^\omega = x = 2^{\omega}(x).

Proof. We will apply Lemma 3b. Let o^*(x) = o_x. To see that 2^{\omega}(x) \subseteq o_{x+1}
notice that o^*(x) = [x] so that (2^{\omega}, o_{x+1}, 2^{\omega}(x), \omega, \omega_{x+1}) so [x] \subseteq o_{x+1} for any well
ordered x \in x. This applies in particular to \{y ∈ x: G_y = G\}. A similar argument
shows that o(X) \subseteq o_{x+1}.

It remains to prove 2^{\omega}(x) \subseteq 2^\omega and 2^\omega \subseteq 2^{\omega}(x). We consider first the case in
which x has a subset y, with cardinal o^*(x). This case includes o^*(x) = \omega by
Theorem 2c. In this case 2^{\omega} \subseteq 2^\omega. Also, since \omega^2 = \omega,
2^{\omega}(x) = 2 = y \subseteq x.

Unfortunately, owing to the lack of choice in M, one cannot dismiss the possibility
that x has no subset of cardinal o^*(x). In this case (o(G, x): G ∈ J) is a countable
set of ordinals by Remark 1a and the fact that J is a countable union of countable
sets. Let o_{\omega} < o_{\omega+1} < o_{\omega+2} \cdots < o_\omega(x) be a sequence of uncountable cardinals
from among the o(G, x) with limit o^*(x). (Note that o^*(x) = o(x) since
o^*(x) \neq o_x.) Let A_x = \{G ∈ J: o(G, x) = o_\omega(x)\}. Let y = \bigcup A_x. Another
use of Remark la permits the conclusion that [y] \subseteq [x], hence that 2^{\omega} \subseteq 2^\omega and y \subseteq x.

It now suffices to show 2^\omega \subseteq 2^{\omega} and 2^{\omega}(x) \subseteq 2^\omega. Actually 2^{\omega}(x) \subseteq 2^\omega
will do because y = 2 = y so 2^{\omega} \subseteq y. To see that y = 2 = y one has only to notice that the
canonical-defined 1:1 onto maps between o_\omega and 2^\omega gives canonically 1:1 onto
maps between A_x x o_\omega and 2^\omega x A_x x o_\omega. These patch together to give a 1:1
onto map between y and 2 x y.

It is also not difficult to see that 2^{\omega}(x) \subseteq 2^\omega. Map a o^*(x) to (A_x x (o^*(x) \land n)).

If a \neq b then, as o^*(x) = Sup(o^*(x)), some a \wedge o^*(x) \neq b \wedge o^*(x). Thus a and b have
different images.

5. Concluding remarks.
a. 2^\omega = x can now be seen not to imply 2^\omega \subseteq x or even 2 \times x = x. As was
shown in [3] II 19, the set I fails to satisfy 2 \times x = I in M.

b. M satisfies "For every infinite x there is a well ordered o^*(x) such that
2^\omega \subseteq o^*(x)." This is an interesting property but it does not imply 2^\omega = x
because a similar argument establishes this in the model U[I_\omega] which is the
Halpern–Lévy model of [2]. U[I_\omega] can be seen, by methods similar to those of [1],
to satisfy I_\omega \subseteq 2^{\omega^2}.

c. Our arguments have made little use of the particular definition of x! Indeed
let \mathfrak{F} be any set valued operation which satisfies:

1) The predicate y ∈ \mathfrak{F}(x) is absolute (at least from M to V).

2) ZF proves |y| \subseteq x \Rightarrow \mathfrak{F}(y) \subseteq \mathfrak{F}(x) and |2x| = |x| \Rightarrow 2^\omega \subseteq \mathfrak{F}(x) for
infinite x.

3) ZF with choice proves 2^\omega = \mathfrak{F}(x) for infinite x.

The statement "For every infinite x, 2^\omega \subseteq \mathfrak{F}(x)" holds in M (and therefore
is not an equivalent to the axiom of choice). Examples of \mathfrak{F}, apart from x!, are x^x
and x^x - x!."
References

[2] J. D. Halpern and A. Lévy, The Boolean prime ideal theorem does not imply the axiom of

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Some results on uniform spaces
with linearly ordered bases

by

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Abstract. The paper is concerned with uniform spaces having a base linearly ordered by in-
cclusion of entourages (or by refinement of uniform coverings, respectively). By a well-known fact,
these spaces coincide with the so-called \( \omega \_n \)-metric spaces, for which several well-known me-
trization theorems can be extended. — Amongst several applications, two characterizations of compact
metric spaces and of separable metric spaces are derived.

\( \S \) 1. Introduction. The uniform structure \( \mathcal{U}_x \) of any metric space \( (X, d) \) obviously
has a linearly ordered base \( \mathcal{B} \) consisting of entourages

\[ U_n = \{(x, y) \mid d(x, y) < 1/n\}, \quad n = 1, 2, \ldots \]

More generally, it is interesting to study uniform spaces \( (X, \mathcal{U}) \) with linearly ordered
bases \( \mathcal{B} \) (\( U_i < U_j \) iff \( U_i \supseteq U_j \) for \( U_i, U_j \in \mathcal{B} \)). Such spaces have been investigated by
many authors and under several aspects: R. Sikorski [37], F. Hausdorff [12, p. 285 ff],
L. W. Cohen and C. Goffman [5], F. W. Stevenson and W. I. Thron [39], Shu-Tang
Wang [42], P. Nyikos and H. C. Reichel [27, [28], A. Hayes [15], P. Nyikos [25],
R. Paindree [29], E. M. Alfsen and O. Njastad [2], M. Fréchet [9].

If \( (X, \mathcal{U}) \) is a uniform space with a linearly ordered base \( \mathcal{B} \) and \( \kappa \) is the least
power of such a base, then there exists an equivalent well ordered base of power \( \kappa \) \((39)\). (Obviously,
such a space is metrizable iff \( \mu = 0 \)). Moreover, F. W. Stevenson
and W. I. Thron [39] showed that any such space \( (X, \mathcal{U}) \) is \( \omega \_\alpha \)-metrizable
in the sense of R. Sikorski. That means: there is a linearly ordered abelian group \( G \)
which has a decreasing \( \omega \_\alpha \)-sequence converging to 0 in the order topology, and
a "distance function" \( g: X \times X \to G \) satisfying the usual axioms for a metric on \( X \),
which generates the topology of \( X \). (Here \( \omega \_\alpha \) denotes the \( \mu \)th infinite cardinal).

Conversely, any \( \omega \_\alpha \)-metric \( g \) on \( X \) induces a uniformity \( \mathcal{U}_x \) on \( X \), a base of which
consists of all sets \( U_a = \{(x, y) \mid g(x, y) < a\}, \quad a \in G, \ a > 0 \). Many properties of metric
spaces have their analogues in the theory of \( \omega \_\alpha \)-metric spaces, however the un-
countable case usually regards different proofs and methods (see for example: \[27, 
[28], [42]\).