[3] J. G. Dhombres and R. Ger, *Équations de Cauchy conditionelles*, Comptes Rendus Acad. Sc. Paris 280 (1975), pp. 513–515.
[4] — — *Conditional Cauchy equations*, Glasnik Matematički (to appear).
[5] L. Dubikajtis, C. Ferens, R. Ger and M. Kuczma, *On Mikusiński's functional equation*, Ann. Polon. Math. 28 (1973), pp. 39–47.
[6] R. Ger, *On some functional equations with a restricted domain*, Fund. Math. 89 (1975), pp. 131–149.

Accepté par la Rédaction le 29. 9. 1975

C-S-maximal superassociative systems

by

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Abstract. Let $(A, \varkappa)$ be an n-dimensional superassociative system, C the set of its constants and S the set of its selectors. Further assume, that for any $i = 1, \ldots, n$ there exists at least one ith selector in $(A, \varkappa)$. The problem of determining all the pairs $(n, |C|)$, for which $(C \cup S, \varkappa)$ is already a maximal irreducibly generated subalgebra of $(A, \varkappa)$ is solved.

This paper is devoted to the study of certain superassociative systems. The notion of superassociativity was introduced by K. Menger, who was the first to point out the importance of considering such algebras (cf. [1]).

Let n be some positive integer and for each set X denote its cardinality by $|X|$. Now we define an *n-dimensional superassociative system* or an *n-system* — as we shall call it briefly — to be an algebra $(A, \varkappa)$ of type $n+1$ such that the equality

$$\varkappa\varkappa x_0 \ldots x_n y_1 \ldots y_n = \varkappa x_0 \varkappa x_1 y_1 \ldots y_n \ldots \varkappa x_n y_1 \ldots y_n$$

holds for any $x_0, \ldots, x_n, y_1, \ldots, y_n \in A$. $(A, \varkappa)$ is called *trivial*, if $|A| \leqslant 1$. A subalgebra of $(A, \varkappa)$ is an algebra (B, λ) of type $n+1$ such that B is a subset of A and $\lambda x_0 \ldots x_n = \varkappa x_0 \ldots x_n$ for any $x_0, \ldots, x_n \in B$. By a *constant* of $(A, \varkappa)$ we mean some element c of A, for which $\varkappa c x_1 \ldots x_n = c$ for any $x_1, \ldots, x_n \in A$, and denote the set of all constants by C. An element $s_i \in A$, $1 \leqslant i \leqslant n$, is called an ith *selector* of $(A, \varkappa)$ provided that $\varkappa s_i x_1 \ldots x_n = x_i$ for any $x_1, \ldots, x_n \in A$. Let S_i denote the set of all ith selectors of $(A, \varkappa)$. We put $S := S_1 \cup \ldots \cup S_n$ and call the elements of S selectors of $(A, \varkappa)$. Further we define an n-tuple $(s_1, \ldots, s_n) \in A^n$ to be a *complete system of selectors* for $(A, \varkappa)$ provided that 1) s_i is an ith selector of $(A, \varkappa)$ for any $i = 1, \ldots, n$ and 2) the equality $\varkappa x s_1 \ldots s_n = x$ holds for any $x \in A$. An element a of A is called *symmetric*, if the equation $\varkappa a x_1 \ldots x_n = \varkappa a x_{\pi(1)} \ldots x_{\pi(n)}$ holds for any $x_1, \ldots, x_n \in A$ and for any permutation π of the set $\{1, \ldots, n\}$. An *irreducibly generated* (*i.g.*) n-system is an n-system $(A, \varkappa)$ such that we have $\varkappa x_0 \ldots x_n \in \{x_0, \ldots, x_n\}$ for any $x_0, \ldots, x_n \in A$. I. g. n-systems were also considered by H. Skala (cf. [2]). $C \cup S$ obviously induces an i.g. subalgebra of $(A, \varkappa)$. Applying Zorn's Lemma to this special case we see that there exist maximal i.g. subalgebras of $(A, \varkappa)$. Now there is the question, whether $(C \cup S, \varkappa)$ is already a maximal i.g. subalgebra of $(A, \varkappa)$ or not. If the first comes true, we shall call $(A, \varkappa)$ *C-S-maximal.* Obviously each trivial n-system is C-S-maximal. Therefore let us assume $(A, \varkappa)$ not to be trivial. Moreover, we make the following assumption: For any $i = 1, \ldots, n$ there should exist some ith selector, say s_i, in $(A, \varkappa)$. We note that in this case $s_1 \neq s_2$ holds, if $n \geqslant 2$; for otherwise we should have

$$x = \varkappa s_1 xy \ldots y = \varkappa s_2 xy \ldots y = y \quad \text{for any } x, y \in A.$$

The aim of this paper is to determine all the pairs $(n, |C|)$, for which $(A, \varkappa)$ must be C-S-maximal, and those, for which it needs not be.

PROPOSITION 1. *If $n = 1$ or if $n = 2$ and $|C| \leqslant 2$ or if $n = 3$ and $|C| = 0$ then $(A, \varkappa)$ needs not be C-S-maximal.*

Proof. We distinguish the following cases: 1) $n = 1$, $|C| = 0$, 2) $n = 1$, $|C| \geqslant 1$ 3) $n = 2$, $|C| \leqslant 2$ and 4) $n = 3$, $|C| = 0$. In the first case we define a binary operation $\varkappa$ on the three-element set $A := \{a, b, s_1\}$ as follows: $\varkappa ab := b$, $\varkappa ax := a$ otherwise, $\varkappa ba := a$, $\varkappa bx := b$ otherwise and $\varkappa s_1 x := x$ for any $x \in A$. Now we turn to the second case. Let η be some cardinal, $\eta \geqslant 1$, C some set with $|C| = \eta$ and a, s_1 two distinct elements not belonging to C. Now we consider the set $A := C \cup \{a, s_1\}$ with the following binary operation $\varkappa$ defined on it: $\varkappa ac := c$ for any $c \in C$, $\varkappa ax := a$ otherwise, $\varkappa cx := c$ for any $c \in C$ and for any $x \in A$ and $\varkappa s_1 x := x$ for any $x \in A$. In the third case let m be an integer, $0 \leqslant m \leqslant 2$. Moreover, let a, c_1, c_2, s_1, s_2 be five distinct elements and denote by C the empty set or $\{c_1\}$ or $\{c_1, c_2\}$ according as $m = 0$ or 1 or 2, respectively. On the set $A := C \cup \{a, s_1, s_2\}$ we define a ternary operation $\varkappa$ as follows: $\varkappa ac_1 x = \varkappa axc_1 := c_1$ for any $x \in A$ if $m \geqslant 1$, $\varkappa ac_2 x = \varkappa axc_2 := x$ for any $x \in A$ if $m = 2$, $\varkappa as_i s_i := s_i$ for any $i = 1, 2$, $\varkappa axy := a$ otherwise, $\varkappa cxy := c$ for any $c \in C$ and for any $x, y \in A$ and $\varkappa s_i x_1 x_2 := x_i$ for any $i = 1, 2$ and for any $x_1, x_2 \in A$. Finally, we turn to the fourth case and consider the four-element set $A := \{a, s_1, s_2, s_3\}$ with the following quarternary operation $\varkappa$ defined on it: $\varkappa axyy = \varkappa ayxy = \varkappa ayyx := y$ for any $x, y \in A$, $\varkappa axyz := a$ otherwise and $\varkappa s_i x_1 x_2 x_3 := x_i$ for any $i = 1, 2, 3$ and for any $x_1, x_2, x_3 \in A$.

Remark. In any of the n-systems defined in the proof above the n-tuple $(s_1, \ldots, s_n)$ even constitutes a complete system of selectors.

LEMMA. *If $(A, \varkappa)$ is not C-S-maximal then there exists some symmetric element a of A such that $a \notin C \cup S$, $(C \cup S \cup \{a\}, \varkappa)$ is an i.g. subalgebra of $(A, \varkappa)$ and $\varkappa ax_1 \ldots x_n \neq a$ holds for any $i = 1, \ldots, n$ and for any*

$$x_1, \ldots, x_n \in C \cup \bigcup \{S_l \mid 1 \leqslant l \leqslant n,\, l \neq i\}.$$

Proof. By definition of C-S-maximality there exists an i.g. subalgebra $(B, \varkappa)$ of $(A, \varkappa)$ such that B contains $C \cup S$ as a proper subset. Now choose some fixed element a of $B \setminus (C \cup S)$. We put $D := C \cup S \cup \{a\}$. Then obviously $(D, \varkappa)$ is an i.g. subalgebra of $(A, \varkappa)$. Let π be some permutation of the set $\{1, \ldots, n\}$. Suppose there exists an integer i, $1 \leqslant i \leqslant n$, such that $\varkappa as_{\pi(1)} \ldots s_{\pi(n)} = s_i$. Then

$$\varkappa ax_1 \ldots x_n = \varkappa\varkappa as_{\pi(1)} \ldots s_{\pi(n)} x_{\pi^{-1}(1)} \ldots x_{\pi^{-1}(n)} = \varkappa s_i x_{\pi^{-1}(1)} \ldots x_{\pi^{-1}(n)} = x_{\pi^{-1}(i)}$$

for any $x_1, \ldots, x_n \in A$, whence $a \in S$, which contradicts the choice of a. Hence $\varkappa as_{\pi(1)} \ldots s_{\pi(n)} = a$ and therefore

$$\varkappa ax_1 \ldots x_n = \varkappa\varkappa as_{\pi(1)} \ldots s_{\pi(n)} x_1 \ldots x_n = \varkappa ax_{\pi(1)} \ldots x_{\pi(n)} \quad \text{for any } x_1, \ldots, x_n \in A.$$

Thus, the symmetry of a is proved. Now suppose there exists an integer i, $1 \leqslant i \leqslant n$, and elements $a_1, \ldots, a_n \in C \cup \bigcup \{S_l \mid 1 \leqslant l \leqslant n, l \neq i\}$ such that $\varkappa aa_1 \ldots a_n = a$. We show by means of induction on l that for any $l = 1, \ldots, n$ the statement

$$(1) \qquad \varkappa ax_1 \ldots x_n = \varkappa a \ldots ax_{l+1} \ldots x_n \quad \text{for any } x_1, \ldots, x_n \in A$$

proves true. For $l = 1$ the validity of (1) is shown by the following calculation:

$$\begin{aligned} \varkappa ax_1 \ldots x_n &= \varkappa\varkappa aa_1 \ldots a_n x_2 \ldots x_i x_1 x_{i+1} \ldots x_n \\ &= \varkappa a\varkappa a_1 x_2 \ldots x_i x_1 x_{i+1} \ldots x_n \ldots \varkappa a_n x_2 \ldots x_i x_1 x_{i+1} \ldots x_n \\ &= \varkappa a\varkappa a_1 x_2 \ldots x_i ax_{i+1} \ldots x_n \ldots \varkappa a_n x_2 \ldots x_i ax_{i+1} \ldots x_n \\ &= \varkappa\varkappa aa_1 \ldots a_n x_2 \ldots x_i ax_{i+1} \ldots x_n = \varkappa aax_2 \ldots x_n \end{aligned}$$

for any $x_1, \ldots, x_n \in A$. If $n = 1$ then the proof is completed.

Otherwise let $2 \leqslant l = s \leqslant n$ and suppose that (1) is already proved for any $l = 1, \ldots, s-1$. Observing the lines above we conclude $\varkappa ax_1 \ldots x_n = \varkappa a \ldots ax_s \ldots x_n = \varkappa ax_s a \ldots ax_{s+1} \ldots x_n = \varkappa a \ldots ax_{s+1} \ldots x_n$ for any $x_1, \ldots, x_n \in A$. Thus, (1) holds for any $l = 1, \ldots, n$. Putting $l = n$ we obtain $\varkappa ax_1 \ldots x_n = \varkappa a \ldots a = a$ for any $x_1, \ldots, x_n \in A$, whence $a \in C$, a contradiction. Therefore $\varkappa ax_1 \ldots x_n \neq a$ for any $i = 1, \ldots, n$ and for any $x_1, \ldots, x_n \in C \cup \bigcup \{S_l \mid 1 \leqslant l \leqslant n, l \neq i\}$, which completes the proof of the lemma.

PROPOSITION 2. *If $n = 2$ and $|C| \geqslant 3$ or if $n = 3$ and $|C| \geqslant 1$ or if $n \geqslant 4$ then $(A, \varkappa)$ is C-S-maximal.*

Proof. In order to prove this proposition by means of contradiction we assume $(A, \varkappa)$ not to be C-S-maximal. The above lemma then tells us that there exists an element a of A having all properties stated there. Now we distinguish the following cases: 1) $n = 2$, $|C| \geqslant 3$, 2) $n = 3$, $|C| \geqslant 1$, 3) $n \geqslant 4$, n even and 4) $n \geqslant 4$, n odd. In the first case any two constants c, d of $(A, \varkappa)$ for which $\varkappa acs_1 = c$ and $\varkappa ads_1 = d$ must coincide because of $c = \varkappa cda = \varkappa\varkappa acs_1 da = \varkappa acd = \varkappa adc = \varkappa\varkappa ads_1 ca = \varkappa dca = d$. Similarly any two constants e, f of $(A, \varkappa)$ for which $\varkappa aes_1 = \varkappa afs_1 = s_1$ are equal since $e = \varkappa s_1 ea = \varkappa\varkappa afs_1 ea = \varkappa afe = \varkappa aef = \varkappa\varkappa aes_1 fa = \varkappa s_1 fa = f$. Hence we obtain $|C| \leqslant 2$, which contradicts the assumption. Now we turn to the second case. Let $c \in C$. We have $\varkappa acs_1 s_2 = c$; for $\varkappa acs_1 s_2 = s_i$ with some integer i, $1 \leqslant i \leqslant 2$, would imply $s_i = \varkappa acs_1 s_2 = \varkappa acs_2 s_1 = \varkappa\varkappa acs_1 s_2 s_2 s_1 a = \varkappa s_i s_2 s_1 a = s_{3-i}$, which contradicts $|A| > 1$. Suppose $\varkappa as_1 s_2 s_2 = s_1$. From this we conclude $a = \varkappa s_1 acc = \varkappa\varkappa as_1 s_2 s_2 acc = \varkappa aacc = \varkappa acca = \varkappa\varkappa acs_1 s_2 caa = \varkappa ccaa = c$, which contradicts the choice of a. Therefore we have $\varkappa as_1 s_2 s_2 = s_2$, whence $a = \varkappa s_2 caa = \varkappa\varkappa as_1 s_2 s_2 caa = \varkappa acaa = \varkappa\varkappa acs_1 s_2 aaa = \varkappa caaa = c$, which again contradicts the choice of a. In the third case there exists an integer i, $1 \leqslant i \leqslant 2$, such that $\varkappa as_1 s_2 \ldots s_1 s_2 = s_i$, whence $s_i = \varkappa as_1 s_2 \ldots s_1 s_2 = \varkappa as_2 s_1 \ldots s_2 s_1 = \varkappa\varkappa as_1 s_2 \ldots s_1 s_2 s_2 s_1 \ldots s_1 = \varkappa s_i s_2 s_1 \ldots s_1 = s_{3-i}$, which contradicts $|A| > 1$. Finally, let us consider the fourth case. We have $\varkappa a s_1 s_2 s_3 \ldots s_3 = s_3$; for $\varkappa a s_1 s_2 s_3 \ldots s_3 = s_i$ with some integer i, $1 \leqslant i \leqslant 2$, would imply

$$s_i = \varkappa a s_1 s_2 s_3 \ldots s_3 = \varkappa a s_2 s_1 s_3 \ldots s_3 = \varkappa\varkappa a s_1 s_2 s_3 \ldots s_3 s_2 s_1 s_3 \ldots s_3$$
$$= \varkappa s_i s_2 s_1 s_3 \ldots s_3 = s_{3-i},$$

which contradicts $|A| > 1$. Suppose there exists an integer i, $1 \leqslant i \leqslant 2$, such that $\varkappa a s_3 s_1 s_2 \ldots s_1 s_2 = s_i$. From this we conclude

$$s_i = \varkappa a s_3 s_1 s_2 \ldots s_1 s_2 = \varkappa a s_3 s_2 s_1 \ldots s_2 s_1 = \varkappa\varkappa a s_3 s_1 s_2 \ldots s_1 s_2 s_2 s_1 s_3 \ldots s_3$$
$$= \varkappa s_i s_2 s_1 s_3 \ldots s_3 = s_{3-i},$$

which again contradicts $|A| > 1$. Therefore we have $\varkappa a s_3 s_1 s_2 \ldots s_1 s_2 = s_3$, whence

$$a = \varkappa s_3 s_1 a \ldots a = \varkappa\varkappa a s_1 s_2 s_3 \ldots s_3 s_1 a \ldots a = \varkappa a s_1 a \ldots a$$
$$= \varkappa\varkappa a s_3 s_1 s_2 \ldots s_1 s_2 a a s_1 \ldots s_1$$
$$= \varkappa s_3 a a s_1 \ldots s_1 = s_1,$$

which contradicts the choice of a. This completes the proof of the proposition.

Summarizing Proposition 1 and Proposition 2 we obtain

THEOREM 1. *Let $(A, \varkappa)$ be some n-dimensional superassociative system such that for any $i = 1, \ldots, n$ there exists at least one i-th selector in $(A, \varkappa)$. If $n = 2$ and $|C| \geqslant 3$ or if $n = 3$ and $|C| \geqslant 1$ or if $n \geqslant 4$ then $(A, \varkappa)$ is always C-S-maximal. Otherwise it needs not be C-S-maximal.*

COROLLARY. *Let $(A, \varkappa)$ be as in Theorem 1 and additionally assume $(A, \varkappa)$ to be i.g. If $n = 2$ and $|C| \geqslant 3$ or if $n = 3$ and $|C| \geqslant 1$ or if $n \geqslant 4$ then $(A, \varkappa)$ only consists of constants and selectors* (cf. [2]).

We are now going to apply Theorem 1 to an important special case. For this purpose we consider some non-empty set M and denote the set of all n-place functions defined on it by $F_n(M)$. On $F_n(M)$ let us define an $(n+1)$-ary operation $\circ$ in the following way: $\circ$ assigns to each $(n+1)$-tuple $(f_0, \ldots, f_n) \in F_n(M)^{n+1}$ the element $f_0 \circ (f_1, \ldots, f_n) \in F_n(M)$, which is defined by

$$\big(f_0 \circ (f_1, \ldots, f_n)\big)(x_1, \ldots, x_n) := f_0\big(f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)\big)$$

for any $x_1, \ldots, x_n \in M$. Further let p_i denote the ith projection from M^n to M for any $i = 1, \ldots, n$. We note that $(F_n(M), \circ)$ is an n-system, which we shall call the *full n-place function algebra over M*, and that $(p_1, \ldots, p_n)$ is a complete system of selectors for it. Moreover, we immediately see that for this n-system the notion of constant element ith selector and symmetric element coincides with the notion of constant n-place function on M, the ith projection from M^n to M and symmetric n-place function on M, respectively. For any $x \in M$ let $\underline{x}$ denote the constant n-place function on M with x as its value. Finally, we put $\underline{M} := \{\underline{x} | \ x \in M\}$ and $P := \{p_1, \ldots, p_n\}$. Applying Theorem 1 we easily obtain

THEOREM 2. *The full n-place function algebra over M is C-S-maximal iff not* $|M| = n = 2$.

Proof. If $|M| = 1$ then $(F_n(M), \circ)$ is C-S-maximal in some trivial way. Therefore assume $|M| \geqslant 2$. We first consider the case $n = 1$. Now suppose $(F_1(M), \circ)$ not to be C-S-maximal. In view of the above lemma there exists an element $f \in F_1(M)$, which is not the identical mapping of M, such that $(\underline{M} \cup P \cup \{f\}, \circ)$ is an i.g. subalgebra of $(F_1(M), \circ)$ and $f \circ \underline{x} \neq f$ holds for any $x \in M$. From this we conclude $f \circ \underline{x} = \underline{x}$ for any $x \in M$, whence $fx = f(\underline{x}x) = (f \circ \underline{x})x = \underline{x}x = x$ for any $x \in M$, which contradicts the choice of f. Therefore $(F_1(M), \circ)$ must be C-S-maximal. Now we turn to the case $|M| = n = 2$ and write M as $\{a, b\}$ with distinct elements a, b. Let us define an element $f \in F_2(M)$ by means of $f(a, a) := a$ and $f(x, y) := b$ otherwise. We immediately see that $f \notin \underline{M} \cup P$ and that $(\underline{M} \cup P \cup \{f\}, \circ)$ is an i.g. subalgebra of $(F_2(M), \circ)$, which proves the non-C-S-maximality of $(F_n(M), \circ)$ in this case. Finally, we apply Theorem 1 to the remaining cases. This completes the proof of Theorem 2.

References

[1] K. Menger, *Superassociative systems and logical functors*, Math. Ann. 157 (1964), pp. 278–295.

[2] H. Skala, *Irreducibly generated algebras*, Fund. Math. 67 (1970), pp. 31–37.

Accepté par la Rédaction le 20. 10. 1975