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On some functional equations with a restricted domain, II

by

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Abstract. Let $(G, +)$ and $(H, +)$ be two abelian groups and let $c \in H$. The general solutions $f\colon G \to H$ of the equations

$$f(x+y) \neq c \quad \text{implies} \quad f(x+y) = f(x)+f(y)$$

and

$$f(x)+f(y) \neq c \quad \text{implies} \quad f(x+y) = f(x)+f(y)$$

are given (for $c = 0$ we get the functional equations of Mikusiński and Dhombres, respectively). Moreover, we investigate Dhombres' equation almost everywhere and make use of the theorems obtained to give a result regarding the functional equation

$$f(x+y) \neq 0 \text{ and } f(x)+f(y) \neq 0 \quad \text{implies} \quad f(x+y) = f(x)+f(y).$$

The notion "almost everywhere" is introduced in an axiomatic way.

§ 1. This article is a continuation of the investigations contained in our previous paper [6]. The notations and terminology used in the present paper follow those of [6]. In particular, for the notions of a proper linearly invariant set ideal $\mathscr{I}$ (p.l.i. ideal), the associated p.l.i. ideals $\pi(\mathscr{I})$ and $\Omega(\mathscr{I})$, the congruence $(\mathrm{mod}\,\mathscr{I})$, the conjugacy of ideals, and also for the meaning of $(\mathrm{a.e.})_{\mathscr{I}}$ (almost everywhere with respect to $\mathscr{I}$) and some of their properties — the reader is referred to [6]. Moreover, in the whole paper $G = (G, +)$ and $H = (H, +)$ will denote two commutative groups whereas the letter f will always stand for a map from G into H. For every $c \in H$ we put $Z_c := f^{-1}(\{c\})$; except for Lemmas 1 and 2 we shall use the symbol T' for the complement of a set $T \subset G$ with respect to G. If $(K, +)$ is a group and $k_0 \in K$, then $nk_0 := k_0+\ldots+k_0$ (n summands), $-nk_0 := n(-k_0)$ and $\frac{1}{2}k_0 := \{k \in K\colon 2k = k_0\}$; moreover, if $Z \subset K$ is such that $(Z, +)$ is a group, then $Z := (Z, +)$. In the sequel, if (n) is the number of the equation, then $\mathscr{S}(n)$ denotes the family of all solutions of equation (n). Every function f fulfilling the Cauchy functional equation

$$f(x+y) = f(x)+f(y) \tag{1}$$

is said to be additive. Finally, $\mathrm{Hom}(G, H) := \mathscr{S}(1)$.

In [5] (cf. also [1] and [6]) the authors deal with the functional equation of Mikusiński

(2) $f(x+y) \neq 0 \quad \text{implies} \quad f(x+y) = f(x)+f(y)\,.$

J. G. Dhombres and the present author have investigated in [3] and [4], in addition to other equations, the following functional equations:

(3) $f(x)+f(y) \neq 0 \quad \text{implies} \quad f(x+y) = f(x)+f(y)$

and

(4) $f(x+y) \neq 0$ and $f(x)+f(y) \neq 0 \quad \text{implies} \quad f(x+y) = f(x)+f(y)\,.$

The results read as follows:

THEOREM I ([5], Theorem 1). *$\mathscr{S}(2) = \mathrm{Hom}(G, H)$ provided G has no subgroups of index 2. If G does possess subgroups of index 2, then $\mathscr{S}(2)$ is equal to the union of $\mathrm{Hom}(G, H)$ and the family of all functions of the form*

(5) $$f(x) = \begin{cases} 0 & \text{for} \quad x \in Z, \\ d & \text{for} \quad x \in Z', \end{cases}$$

where $d \neq 0$ is an arbitrary element of H and Z is an arbitrary subgroup of G of index 2.

THEOREM II ([3], Theorem 2). *If H does not possess elements of order 2, then $\mathscr{S}(3) = \mathrm{Hom}(G, H)$.*

THEOREM III ([3], Theorem 3). *If $f \in \mathscr{S}(3)$ and all the counter-images Z_c of single points $c \in H$ are members of a p.l.i. set-ideal in G, then $f \in \mathrm{Hom}(G, H)$.*

Theorems I and II have also been proved in the non-abelian case.

Making use of different methods, we are going to solve slightly more general equations:

(2') $f(x+y) \neq c \quad \text{implies} \quad f(x+y) = f(x)+f(y)$

and

(3') $f(x)+f(y) \neq c \quad \text{implies} \quad f(x+y) = f(x)+f(y)\,,$

where c is an arbitrarily given constant from H. In particular, we do not need the assumption on H occurring in Theorem II. Further, we deal with the functional equation

(3a.e.) $f(x)+f(y) \neq 0 \quad \text{implies} \quad f(x+y) = f(x)+f(y) \ (\text{a.e.})_{\Omega(\mathscr{I})}\,,$

i.e. with the case where the validity of equation (3) is postulated not for all pairs $(x, y) \in G^2$ but only for those which belong to the complement of a certain set from the p.l.i. ideal $\Omega(\mathscr{I})$ in G^2 associated with a certain p.l.i. ideal $\mathscr{I}$ in G (see [6]).

Equation (2) with analogous restrictions regarding the domain of its validity was solved in [6].

Using the results concerning equations (3') and (3a.e), we shall improve Theorem III replacing the assumption $\{Z_c\colon c \in H\} \subset \mathscr{I}$ by the less restrictive one $Z_0 \in \mathscr{I}$.

In investigations of the equations postulated almost everywhere we have confined ourselves (here as well as in [6]) to the case $c = 0$. One reason is that this case seems to be the most important. Mikusiński's equation (2) arose from some considerations connected with geometric optics, whereas equation (3) appears in a natural way as the symmetric case to (2). Also zero is a distinguished element of any group. The other reason is that the case $c \neq 0$ leads to a great deal of additional laborious considerations which, however, do not bring anything essentially new to the technique employed, so that the method is sufficiently well illustrated by the case considered.

§ **2.** We start with two lemmas:

LEMMA 1 (on a characterization of subgroups). *Suppose that $K = (K, +)$ is a group (not necessarily commutative). $Z = (Z, +)$ is a subgroup of K if and only if $K \supset Z \neq \emptyset$ and*

(6) $$Z + Z' \subset Z'$$

where $Z' := K \setminus Z$.

Proof [1]. The necessity is obvious. Assume that $\emptyset \neq Z \subset K$ and (6) is satisfied. Take arbitrary $x, y \in Z$ and suppose that $-x+y \in Z'$. Then $y = x+(-x+y) \in Z+Z'$, a contradiction.

LEMMA 2 (on a characterization of subgroups of index 2). *Suppose that $K = (K, +)$ is a group (not necessarily commutative). $Z = (Z, +)$ is a subgroup of index 2 of K if and only if $K \supset Z \neq \emptyset$, $Z' := K \setminus Z \neq \emptyset$ and*

$$Z + Z \subset Z, \qquad Z' + Z' \subset Z\,.$$

Proof. Only the sufficiency requires a motivation. Take a point $a \in Z'$. Then $Z' + a \subset Z$. Suppose that there exists an $x \in Z$ such that $-x \in Z'$. Then $-x \in Z - a$, i.e. $-x = z - a$, $z \in Z$. Hence $a = x + z \in Z + Z \subset Z$, a contradiction. Thus, $Z = -Z$ which together with $Z + Z \subset Z$ implies that Z is a group. Clearly, Z is of index 2, since $Z \cup (Z - a) = G$.

THEOREM 1. *If $2c \neq 0$, then $\mathscr{S}(2') = \mathrm{Hom}(G, H) \cup \{\mathbf{c}\}$, where $\mathbf{c}\colon G \to H$ is the constant function: $\mathbf{c}(x) = c$, $x \in G$. If $2c = 0$, but $c \neq 0$, then $\mathscr{S}(2')$ is the union of $\mathrm{Hom}(G, H) \cup \{\mathbf{c}\}$ and the family of all functions of the form*

$$f(x) = \begin{cases} 0 & \text{for} \quad x \in Z, \\ c & \text{for} \quad x \in Z', \end{cases}$$

where $Z = (Z, +)$ is an arbitrary subgroup of G of index greater than 2. If $c = 0$, then $\mathscr{S}(2')$ is described in Theorem I: $\mathscr{S}(2') = \mathscr{S}(2)$.

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[1] I am indebted to Dr J. Tabor for this short proof. The original version was a little longer.

Proof. Obviously, c is always a solution of (2′). Thus, we may restrict our attention to nonconstant solutions only. Assume f to be a (nonconstant) solution of (2′). We shall distinguish two cases:

(i) $Z_c \cup (Z_c+a) \neq G$ for all $a \in G$,

(ii) non (i).

First, we shall show that

(7) (i) implies $f \in \mathrm{Hom}(G, H)$.

To this end, suppose that case (i) occurs, fix arbitrarily x and y from G and take a $z \in G$ such that $z+x \notin Z_c \cup (Z_c-y)$. Then $f(z+x) \neq c$ as well as $f(x+y+z) \neq c$, whence

$$f(z+x) = f(z)+f(x)$$

and

$$f(x+y+z) = f(x+y)+f(z) = f(x)+f(y+z) = f(x+z)+f(y).$$

Thus

$$f(x+y)-f(x)-f(y) = [f(x)+f(y+z)-f(z)]-f(x)-[f(x)+f(y+z)-f(x+z)] = f(x+z)-f(x)-f(z) = 0,$$

i.e. (7) is proved.

Now, suppose that case (ii) occurs. Thus, there exists an $a \in G$ such that $Z_c \cup (Z_c+a) = G$. Take an $x \in G \setminus Z_c$. Then $x = z+a$, $z \in Z_c$, and

$$c \neq f(x) = f(z+a) = f(z)+f(a).$$

Consequently, we get

(8) $$f(x) \in \{c, c+f(a)\} \quad \text{for all } x \in G$$

and

(9) $$f(a) \neq 0,$$

since f is assumed to be nonconstant. Putting $y = 0$ in (2′), we infer that

$$f(x) \neq c \quad \text{implies} \quad f(0) = 0, \ x \in G,$$

whence, again by the fact that f is nonconstant, we infer that $f(0) = 0$, which together with (8) implies $c = 0$ or $c = -f(a)$. In both cases we get $2c = 0$, since $c = -f(a)$ implies

$$f(a) = -c \in \{c, 0\}$$

on account of (8) and, consequently, $c = -c$ in view of (9). Therefore, condition (ii) gives (8) and $2c = 0$, i.e. $2c \neq 0$ implies (i) and hence the additivity of f on account of (7).

Now, assume that f is nonadditive and $2c = 0$. The nonadditivity condition implies case (ii) and hence, in particular, relation (8). Let us consider two cases:

1° $c \neq 0$. Then $f(a) = -c = c$ and f is of form (5) with $d = c$ and $Z = Z_0$. Take $x \in Z$ and $y \in Z'$. Then, if we had $x+y \in Z$, we would get $0 = f(x+y) \neq c$, whence $0 = f(x+y) = f(x)+f(y) = c$, contrary to our supposition. Thus, condition (6) is fulfilled and hence, by Lemma 1, $\boldsymbol{Z}$ is a subgroup of $\boldsymbol{G}$. Now, it is not hard to check that f is nonadditive if and only if $\boldsymbol{Z}$ is of index greater than 2.

2° $c = 0$. Then (8) implies that f is of form (5) with $d = f(a) \neq 0$ (see (9)) and $Z = Z_0$. To finish the proof it suffices to show that $(Z, +)$ is a subgroup of $\boldsymbol{G}$ of index 2. For, observe that $Z+Z \subset Z$. On the other hand, we have $Z \cup (Z+a) = G$. Thus, taking $x, y \in Z'$, we infer that $x = z_x+a$ and $y = z_y+a$, $z_x, z_y \in Z$, whence $x+y = z+2a$ with $z := z_x+z_y \in Z$. If we had $x+y$ in Z', then

$$0 \neq d = f(x+y) = f(z+2a) = f(z)+f(2a) = f(2a),$$

whence

$$0 \neq d = f(2a) = 2f(a) = 2d,$$

a contradiction. Thus, $Z'+Z' \subset Z$ and it suffices to apply Lemma 2.

A straightforward verfication ensures that each function mentioned in the statement of our theorem yields a solution of (2′). This completes the proof.

LEMMA 3. *Suppose that f is a solution of* (3′), *$f(s)+f(t) = c$ for a certain pair $(s, t) \in G^2$ and*

(10) *for all $x \in G$ we have $f(x) \in \{f(t), c-f(s+t)\}$ or $f(x+s) = f(s)$.*

Then f takes at most three pairwise different values and $T := f(G) \subset \{0, c\} \cup \frac{1}{2}c =: C$.

Proof. Observe that

(11) $$\mathscr{S}(3') \ni f \neq \text{const} \quad \text{implies} \quad f(0) = 0.$$

In fact, putting $y = 0$ in (3′), we infer that $f(x) \neq c-f(0)$ implies $f(x) = f(x)+f(0)$, i.e. $f(0) = 0$.

First, suppose that

(12) $$f(x) \in \{f(t), c-f(s+t)\} \quad \text{for all } x \in G.$$

In such a case f takes at most two different values. If $f(s) = f(t)$, then $\frac{1}{2}c$ is nonempty and $f(s+t) \in \frac{1}{2}c \cup \{c-f(s+t)\}$, whence $f(s+t) \in \frac{1}{2}c$ and, consequently, $T \subset \frac{1}{2}c$. If $f(s) \neq f(t)$, then f is nonconstant and hence, in virtue of (11) and (12), $0 \in \{f(s), f(t)\}$, whence $f(s) = 0$ and $f(t) = c$, or conversely; in both cases $T = \{0, c\}$.

Now, suppose that (12) does not hold, i.e., by assumption (10), there exists an $x \in G$ such that $f(x+s) = f(s)$ and

$$f(x) \notin \{f(t), c-f(s+t)\}.$$

Then, in particular, $f(x)+f(s) \neq f(s)+f(t) = c$, whence $f(s) = f(x+s) = f(x)+f(s)$, i.e. $f(x) = 0$. Hence $T \subset \{0, f(t), c-f(s+t)\}$, $f(t) \neq 0$ (whence $f(s) \neq c$) and

$f(s+t) \neq c$. In particular, f takes at most three pairwise different values. Since $f(s) = c - f(t)$ belongs to T, we infer that

$$(13) \qquad f(s) = 0 \quad \text{or} \quad f(s) = f(t) \quad \text{or} \quad f(s+t) = f(t) \neq f(s) \neq 0\,.$$

In the first case, because of $f(t) = c \neq f(s+t)$ and $f(s+t) \in T$, we get $T \subset C$. The second possibility of (13) implies $f(t) \in \frac{1}{2}c$ and $f(s+t) \in \frac{1}{2}c \cup \{0\}$ and, again, $T \subset C$. The last case of (13) gives

$$T \subset \{0, f(s), f(t)\}\,.$$

If $f(s) \in C$, then so does $f(t)$. So, suppose that $f(s) \neq f(t) = f(s+t)$ and neither $f(s)$ nor $f(t)$ belongs to C. We shall show that such a supposition leads to a contradiction.

First, note that $(Z_0, +)$ must be a group. In fact

$$Z_0 + (Z_{f(s)} \cup Z_{f(t)}) = (Z_0 + Z_{f(s)}) \cup (Z_0 + Z_{f(t)}) \subset Z_{f(s)} \cup Z_{f(t)}$$

and it suffices to apply Lemma 1.

Since $2f(s) \neq c$ and $2f(t) \neq c$, we infer that $2f(s) = f(2s)$ as well as $2f(t) = f(2t)$ belong to T:

$$2f(s) \in \{0, f(t)\} \quad \text{and} \quad 2f(t) \in \{0, f(s)\}\,.$$

Thus the following cases are possible:

1^0 $2f(s) = 2f(t) = 0$,

2^0 $2f(s) = 0$ and $2f(t) = f(s)$,

3^0 $2f(s) = f(t)$ and $2f(t) = 0$,

4^0 $2f(s) = f(t)$ and $2f(t) = f(s)$.

Case 1^0 We must have $c \neq 0$. Hence $2s, 2t \in Z_0$ and $Z_{f(s)} \subset Z_0 - s$ as well as $Z_{f(t)} \subset Z_0 - t$. Since $(Z_0, +)$ is a group, we obtain

$$Z_{f(t)} \subset Z_0 - t = (Z_0 + 2t) - t = Z_0 + t \subset Z_{f(t)}\,,$$

whence $Z_{f(t)} = Z_0 + t$. Likewise, $Z_{f(s)} = Z_0 + s$. Consequently, recalling that $s + t \in Z_{f(t)}$, we get

$$Z_0 + s + t \subset Z_0 + Z_{f(t)} \subset Z_{f(t)} = Z_0 + t\,.$$

Hence $Z_{f(s)} = Z_0 + s \subset Z_0$, a contradiction.

Case 2^0 As in the preceding case we infer that $2s \in Z_0$, whence

$$Z_{f(s)} \subset Z_0 - s = (Z_0 + 2s) - s = Z_0 + s \subset Z_{f(s)}\,,$$

in virtue of the facts that $(Z_0, +)$ is a group and that $f(s) \notin C$. Consequently,

$$Z_{f(s)} = Z_0 - s = Z_0 + s = -(-Z_0 - s) = -(Z_0 - s) = -Z_{f(s)}\,.$$

Since

$$Z_0 \cup Z_{f(s)} \cup Z_{f(t)} = G = -G = (-Z_0) \cup (-Z_{f(s)}) \cup (-Z_{f(t)}) = Z_0 \cup Z_{f(s)} \cup (-Z_{f(t)})\,,$$

we come to the equality

$$Z_{f(t)} = -Z_{f(t)}\,.$$

Now, since $2f(t) = f(s)$, we get $Z_{f(t)} \subset Z_{f(s)} - t$ and $Z_0 + t \subset Z_{f(t)} = -Z_{f(t)} \subset -Z_{f(s)} + t = Z_{f(s)} + t$, i.e. $Z_0 \subset Z_{f(s)}$, a contradiction.

Case 3^0 it is symmetric to 2^0.

Case 4^0 It implies $f(s) = -f(t)$, i.e. $c = 0$. We easily get

$$(14) \qquad Z_{f(s)} + s \subset Z_{f(t)}$$

and

$$(15) \qquad Z_{f(t)} + Z_{f(t)} \subset Z_{f(s)}\,,$$

whence

$$(16) \qquad Z_{f(s)} + Z_{f(t)} + s \subset Z_{f(t)} + Z_{f(t)} \subset Z_{f(s)}\,.$$

(14) and (15) imply also

$$Z_{f(s)} + s + t \subset Z_{f(t)} + t \subset Z_{f(t)} + Z_{f(t)} \subset Z_{f(s)}\,.$$

On the other hand, by (16),

$$Z_{f(s)} + s + t \subset Z_{f(s)} + Z_{f(t)} \subset Z_{f(s)} - s\,.$$

Consequently, $Z_{f(s)} \cap (Z_{f(s)} - s) \neq \emptyset$. This is a contradiction, since if we had $f(x) = f(s)$ and $f(x+s) = f(s)$, we would get $c \neq 2f(s) = f(s) + f(x) = f(x+s) = f(s)$, whence $f(s) = 0$. This completes the proof.

THEOREM 2. *Suppose that f is a solution of (3′). Then the following four cases are the only possible ones*:

(i) $f \in \mathrm{Hom}(\boldsymbol{G}, \boldsymbol{H})$,

(ii) $c^* := \frac{1}{2}c \setminus \{0\} \neq \emptyset$, $f(x) = d \in c^*$ *for all* $x \in G$,

(iii) $c^* \neq \emptyset$, *f is of form* (5), *where Z is such that $(Z, +)$ is a subgroup of* $\boldsymbol{G}$ *and d is an arbitrary element of* c^*,

(iv) $c \in 0^*$, *f is of the form*

$$f(x) = \begin{cases} 0 & \text{for} \quad x \in K, \\ d & \text{for} \quad x \in K_1, \\ -d & \text{for} \quad x \in K_2, \end{cases}$$

where K is such that $(K, +)$ is a subgroup of $\boldsymbol{G}$ of index 3, K_1, K_2 are the cosets of $\boldsymbol{G}$ with respect to K and d is an arbitrary element of $\frac{1}{2}c$.

Conversely, every function f for which one of these four conditions is satisfied yields a solution of (3′).

Proof. Assume f to be a solution of (3′). Obviously, if $f(s) + f(t) \neq c$ for all $s, t \in G$, then f is additive, i.e. we have case (i). Suppose that

$$M := \{(s, t) \in G^2 : f(s) + f(t) = c\} \neq \emptyset\,.$$

We shall distinguish two cases:

Case A. For every $(s,t)\in M$ there exists an $x\in G$ such that

$$f(x)\neq f(t),\quad f(x)\neq c-f(s+t)\quad\text{and}\quad f(x+s)\neq f(s).$$

Then, taking an arbitrary pair $(s,t)\in M$ and a corresponding $x\in G$, we get

$$f(s+t)+f(x)\neq c,\quad f(x+s)+f(t)\neq c\quad\text{and}\quad f(x)+f(s)\neq c.$$

Consequently

$$f(s+t+x)=f(s+t)+f(x),\quad f(s+t+x)=f(x+s)+f(t)\quad\text{and}\quad f(x+s)=f(x)+f(s).$$

Hence

$$f(s+t)-f(s)-f(t)=f(s+t+x)-f(x)+f(x)-f(x+s)+f(x+s)-f(s+t+x)=0,$$

i.e. $f\in\operatorname{Hom}(G,H)$.

Case B. non A. Then there exists a pair $(s,t)\in M$ such that (10) is satisfied. By means of Lemma 3, f takes at most three distinct values and $T:=f(G)\subset\{0,c\}\cup\frac{1}{2}c$. If f is a constant function, then either $f=0$ and we have case (i), or $\frac{1}{2}c\neq\emptyset$ and $f(x)=d\in\frac{1}{2}c$, $d\neq 0$, forall $x\in G$, yielding case (ii). If f is nonconstant, then necessarily $0\in T$ (see (16)). In particular, $Z_0\neq\emptyset$. First, let $c\neq 0$. We shall consider three possibilities:

1^0 $Z_c=\emptyset$. Then $T=\{0,d,e\}$ with $d,e\in\frac{1}{2}c\setminus\{0,c\}$. In the case where $e=d$, f is of form (5) where $d\in c^*$ and $Z=Z_0$; clearly $Z_0+Z_d\subset Z_d$ whence, on account of Lemma 1, $(Z_0,+)$ is a group and we have case (iii). In the case where $e\neq d$, because of $e+d\neq c=2d$, we infer that $e+d\in T$, whence $e=-d$ and $2d=c=2e=-2d$. Thus, $T=\{0,d,-d\}$ and $c=-c$; if we had $2d=c=0$, then we would get $d=-d$, a contradiction. Consequently $c\in 0^*$. Put $K:=Z_0$. Since

$$K+K'=Z_0+(Z_d\cup Z_{-d})=(Z_0+Z_d)\cup(Z_0+Z_{-d})\subset Z_d\cup Z_{-d}=K',$$

we infer that $(K,+)$ is a group. Now take an $x\in Z_d$ and a $y\in Z_{-d}$. On account of the inclusions $Z_d\subset K-y$ and $Z_{-d}\subset K-x$ we get the equality

$$G=-G=K\cup(K+x)\cup(K+y).$$

On the other hand,

$$K_1:=K+x\subset Z_0+Z_d\subset Z_d\quad\text{and}\quad K_2:=K+y\subset Z_0+Z_{-d}\subset Z_{-d},$$

whence we infer that K is of index 3, K_1, K_2 are the cosets of G with respect to K and $f|_{K_1}=d$ as well as $f|_{K_2}=-d$. Thus we have case (iv).

2^0 $Z_c\neq\emptyset$ and $\operatorname{card}T=2$. Then $T=\{0,c\}$ and, because of $2c\neq c$, we get $2c=0$, whence $Z_c+Z_c\subset Z_0$. Since, moreover, $Z_0+Z_0\subset Z_0$, we infer, applying Lemma 2, that $(Z_0,+)$ is a subgroup of index 2 of G. Thus, f is additive.

3^0 $Z_c\neq\emptyset$ and $\operatorname{card}T=3$. Then $T=\{0,c,d\}$ with $2d=c\neq d\neq 0$ and, since $c\neq c+d\in T$, we get $c+d=0$, i.e. $d=-c\neq c$. The following inclusions result immediately from the fact that f is a solution of (3′):

$$Z_0+Z_0\subset Z_0,\tag{17}$$

$$Z_0+Z_{-c}\subset Z_{-c},\tag{18}$$

$$Z_c+Z_c\subset Z_{-c},\tag{19}$$

$$Z_c+Z_{-c}\subset Z_0.\tag{20}$$

Take an $x\in Z_c$ and a $y\in Z_{-c}$. Relation (20) implies that

$$Z_c\subset Z_0-y\tag{21}$$

and

$$Z_{-c}\subset Z_0-x.\tag{22}$$

Suppose that $-Z_0\cap(Z_0-x)\neq\emptyset$. Then, there exists a $z\in G$ such that $f(-z)=0=f(x+z)$, whence

$$c=f(x)=f(-z+x+z)=f(-z)+f(x+z)=0,$$

contrary to our assumption. Consequently, $-Z_0\cap(Z_0-x)=\emptyset$. The equality $-Z_0\cap(Z_0-y)=\emptyset$ may be derived analogously. Hence, by means of (21) and (22), $-Z_0\subset Z_0$, which jointly with (17) states that $(Z_0,+)$ is a group.

Now we shall prove that $(Z_0-x)\cap(Z_0-y)=\emptyset$. In fact, otherwise we would get the existence of a $z\in G$ such that $z=z_x+x=z_y+y$, $z_x,z_y\in Z_0$. This means that $x=z-z_x$ and $f(z)=f(z_y+y)=f(z_y)+f(y)=-c\neq c$. On the other hand, $c=f(x)=f(z-z_x)=f(z)+f(-z_x)=-c$, a contradiction. Thus

$$Z_c=Z_0-y\quad\text{and}\quad Z_{-c}=Z_0-x,$$

whence

$$Z_0+Z_c=Z_0+Z_0-y=Z_0-y=Z_c.$$

Observe that $x+y\in Z_0$ because of (20), and $3x\in Z_0$, since

$$3x\in Z_c+(Z_c+Z_c)\subset Z_c+Z_{-c}\subset Z_0$$

in view of (19) and (20). Therefore

$$Z_{-c}+Z_{-c}=Z_0-x+Z_0-x=Z_0-2x=Z_0-(x+y)-2x=Z_0-3x-y=Z_0-y=Z_c.$$

Thus, f is additive.

Now let $c=0$. Then Lemma 3 implies $T\subset\frac{1}{2}0$, which means that $2f(x)=0$ for all $x\in G$. Thus equation (3′) may equivalently be written in the form

$$f(x)\neq f(y)\quad\text{implies}\quad f(x+y)=f(x)+f(y).\tag{23}$$

Suppose that f takes three pairwise different values: 0, a, b. Then, by virtue of (23),

$$f(x) = a \neq b = f(y) \quad \text{implies} \quad f(x+y) = f(x)+f(y) = a+b \in \{0, a, b\},$$

which is impossible since $a \neq -b = b$, $b \neq 0$ and $a \neq 0$. Consequently, f takes at most two different values, whence either $f(x) = d \in \frac{1}{2}0$ for all $x \in G$ (i.e. we have case (i) or (ii) according as d vanishes or not), or f is of form (5) with $d \in \frac{1}{2}0\setminus\{0\}$ and $Z = Z_0$. In the latter case $(Z_0, +)$ is a group because of $Z_0' = Z_d \supset Z_0+Z_d$ and of Lemma 1. Thus, $f \in \operatorname{Hom}(\boldsymbol{G}, \boldsymbol{H})$ or we have case (iii) according as the index of $(Z_0, +)$ is 2 or greater than 2.

The last part our assertion is obvious.

As a consequence, we get the general solution of equation (3):

THEOREM 3. *If $\boldsymbol{H}$ does not possess elements of order 2, then $\mathscr{S}(3) = \operatorname{Hom}(\boldsymbol{G}, \boldsymbol{H})$. If $\boldsymbol{H}$ has elements of order 2, then $\mathscr{S}(3)$ is equal to the union of $\operatorname{Hom}(\boldsymbol{G}, \boldsymbol{H})$, the family of all constant functions f:*

$$f(x) = d \in \tfrac{1}{2}0, \quad x \in G,$$

and the family of all functions of form (5), where $\boldsymbol{Z} = (Z, +)$ is an arbitrary subgroup of $\boldsymbol{G}$ and $d \neq 0$ is an arbitrary element of order 2 in $\boldsymbol{H}$.

COROLLARY. *If $f \in \mathscr{S}(3)$, then $2f := f+f \in \operatorname{Hom}(\boldsymbol{G}, \boldsymbol{H})$* (cf. Proposition 9 in [4]).

§ 3. In the present section we shall deal with the functional equation (3a.e.). Let us start with the following

LEMMA 4. *Suppose that we are given a p.l.i. ideal $\mathscr{I}$ in $\boldsymbol{G}$ and $p \in H\setminus\{0\}$ is an element of order 3. Put $P := \{0, p, -p\}$ and $\boldsymbol{P} = (P, +)$. Let a function $\varphi\colon G \to P$ satisfy the conditions:*

$$\varphi(x)+\varphi(y) \neq 0 \quad \textit{implies} \quad \varphi(x+y) = \varphi(x)+\varphi(y)$$

for all $(x, y) \in G^2\setminus M$, $M \in \Omega(\mathscr{I})$ and

(24) $$\operatorname{card}\varphi(G\setminus T) = 3 \quad \textit{for all } T \in \mathscr{I}.$$

Then there exists an additive function $\Phi\colon G \to P$ such that $\varphi(x) = \Phi(x)$ (a.e.)$_{\mathscr{I}}$. In particular, φ is $\Omega(\mathscr{I})$-almost additive, i.e., φ satisfies Cauchy's functional equation (1) *almost everywhere with respect to $\Omega(\mathscr{I})$.*

Proof. There exists a set $U(M) \in \mathscr{I}$ such that $V_x(M) := \{y \in G\colon (x, y) \in M\}$ belong to $\mathscr{I}$ whenever $x \in G\setminus U(M)$. Put $Q_q := \varphi^{-1}(\{q\})$, $q \in P$. Assumption (24) says, in particular, that $Q_q \notin \mathscr{I}$ and $Q_q \not\equiv G \pmod{\mathscr{I}}$ for $q \in P$. Write

$$Q := Q_0\setminus(-U(M) \cup U(M)).$$

Evidently, $Q \subset Q_0 \equiv Q \pmod{\mathscr{I}}$. We shall prove that

(25) $$Q = -Q.$$

To this aim, take an $x \in Q$. Let us first note that

(26) $$Q_0 \cup (Q_{-\varphi(-x)}-x) \not\equiv G \pmod{\mathscr{I}}.$$

Actually, if we had

$$\varphi(z) = 0 \quad \text{or} \quad \varphi(z+x) = -\varphi(-x) \quad \text{for all } z \in G\setminus T,\ T \in \mathscr{I},$$

then, taking a $z \in [Q_0 \cup T \cup V_x(M)]'$, we would get

$$-\varphi(-x) = \varphi(z+x) = \varphi(z)+\varphi(x) = \varphi(z),$$

i.e.,

$$\varphi(z) = -\varphi(-x),$$

which means that $\operatorname{card}\varphi([T \cup V_x(M)]') \leqslant 2$. This contradicts (24), since $T \cup V_x(M) \in \mathscr{I}$. Thus (26) is proved and we are able to find a $y \in G$ such that

$$y \notin Q_0 \cup (Q_{-\varphi(-x)}-x) \cup V_x(M) \cup (V_{-x}(M)-x).$$

Consequently

$$(-x, x+y) \notin M, \quad (x, y) \notin M, \quad \varphi(x)+\varphi(y) \neq 0 \quad \text{and} \quad \varphi(-x)+\varphi(x+y) \neq 0.$$

Hence

$$\varphi(x+y) = \varphi(x)+\varphi(y) = \varphi(y) = \varphi(-x+x+y) = \varphi(-x)+\varphi(x+y),$$

which implies $-x \in Q_0$. Evidently, if x does not belong to $-U(M) \cup U(M)$, then neither does $-x$. Thus we get $-x \in Q$, whence $Q \subset -Q$ and relation (25) is proved.

Next, we are going to prove that

(27) $$Q-z \equiv Q \pmod{\mathscr{I}} \quad \text{for all } z \in Q.$$

In fact, if we had $(Q-z)\setminus Q \in \mathscr{I}$ for a $z \in Q$, then we would also get

$$A := ([Q\setminus(V_z(M)+z)]-z)\setminus Q_0 \notin \mathscr{I}.$$

In particular, $A \neq \emptyset$. Taking an $x \in A$, we get

$$\varphi(x) \in \{-p, p\}, \quad x+z \in Q \quad \text{and} \quad x \notin V_z(M).$$

Hence

$$0 = \varphi(x+z) = \varphi(x)+\varphi(z) = \varphi(x) \neq 0,$$

a contradiction. Now, in view of (25), since $(Q-z)\setminus Q \in \mathscr{I}$, we have also $\mathscr{I} \ni (-Q+z)\setminus(-Q) = (Q+z)\setminus Q$ whence $Q\setminus(Q-z) = [(Q+z)\setminus Q]-z \in \mathscr{I}$.

As a consequence of (27) and (25) we obtain also

(28) $$(Q+u) \cap (Q-v) \equiv Q \pmod{\mathscr{I}} \quad \text{for all } u, v \in Q.$$

Let $\boldsymbol{K} = (K, +)$ be the group generated by Q. We shall show that

(29) $$Q \equiv K \pmod{\mathscr{I}}.$$

Since $Q \equiv Q_0 \pmod{\mathscr{I}}$, in order to prove (29) it suffices to show that

(30) $$K \cap Q_p \in \mathscr{I} \quad \text{and} \quad K \cap Q_{-p} \in \mathscr{I}.$$

Obviously, in view of (25), every element of K can be represented (in general not uniquely) as a finite sum of elements from Q:

$$x\in K \quad \text{implies} \quad x = z_1+\ldots+z_{n_x}, \quad z_i\in Q \text{ for } i = 1, \ldots, n_x .$$

Suppose that $K\cap Q_p\notin \mathscr{I}$. Then $(K\cap Q_p)\setminus U(M)\neq\emptyset$ and the definition

$$n := \min\{n_x \colon x\in(K\cap Q_p)\setminus U(M)\}$$

is correct. Evidently, we have $n\geqslant 2$. Take an $x\in(K\cap Q_p)\setminus U(M)$ which admits a representation of the form

$$x = z_1+\ldots+z_n, \quad z_i\in Q,\ i = 1, \ldots, n .$$

By means of (28) we may find an s such that

$$s\in\{[(Q+z_1)\cap(Q-z_2)]\setminus[(V_x(M)+z_1)\cup(U(M)+z_1-x)]\} .$$

Then

$$s-z_1\in Q, \quad z_2+s\in Q, \quad (x, s-z_1)\notin M \quad \text{and} \quad \tilde{x} := x+s-z_1\notin U(M) .$$

Observe that $\varphi(x)+\varphi(s-z_1) = p\neq 0$, whence $\tilde{x}\in Q_p$. On the other hand, $\tilde{x} = (z_2+s)+\ldots+z_n\in K$. Consequently, $\tilde{x}\in(K\cap Q_p)\setminus U(M)$ and $\tilde{x}$ admits an expansion which consists of $n-1$ summands only. This contradicts the minimality of n. The second part of (30) may be derived in a similar way.

The next step is to show that there exist elements $x_p\in Q_p$ and $x_{-p}\in Q_{-p}$ such that

(31) $$Q_p\equiv Q+x_p \pmod{\mathscr{I}} \quad \text{and} \quad Q_{-p}\equiv Q+x_{-p} \pmod{\mathscr{I}} .$$

To this aim, take an arbitrary element $x_p\in Q_p\setminus U(M)$ and observe that $(Q\setminus V_{x_p}(M))+x_p\subset Q_p$, whence

$$(Q+x_p)\setminus Q_p\subset V_{x_p}(M)\in\mathscr{I} .$$

Suppose that the relation $Q_p\setminus(Q+x_p)\in\mathscr{I}$ is not true. Then, since $Q\equiv Q_0 \pmod{\mathscr{I}}$ we have $Q_p\setminus(Q_0+x_p)\notin\mathscr{I}$ or, equivalently,

$$Q_p\cap[(Q_p+x_p)\cup(Q_{-p}+x_p)]\notin\mathscr{I} .$$

However,

$$Q_p+x_p\equiv(Q_p\setminus V_{x_p}(M))+x_p\subset Q_{-p} .$$

Consequently,

$$\mathscr{I}\not\ni Q_p\cap[Q_{-p}\cup(Q_{-p}+x_p)] = Q_p\cap(Q_{-p}+x_p) .$$

Take an $\tilde{x}\in Q_p\setminus[V_{x_p}(M)\cup(U(M)-x_p)\cup U(M)]$. For such an $\tilde{x}$ we have

$$z_{-p} := \tilde{x}+x_p\in Q_{-p}\setminus U(M)$$

and

$$(Q_p+\tilde{x})\cap(Q_{-p}+z_{-p}) = [Q_p\cap(Q_{-p}+x_p)]+\tilde{x}\notin\mathscr{I} .$$

Hence also

$$\emptyset = Q_{-p}\cap Q_p\supset[(Q_p\setminus V_{\tilde{x}}(M))+\tilde{x}]\cap[(Q_{-p}\setminus V_{z_{-p}}(M))+z_{-p}]\notin\mathscr{I} ,$$

a contradiction. Along the same lines one can prove the other part of (31).

Relations (29) and (31) imply the congruence

$$K\cup(K+x_p)\cup(K+x_{-p})\equiv G \pmod{\mathscr{I}} .$$

Recalling Lemma 3 from [6], we come to

$$G = K+[K\cup(K+x_p)\cup(K+x_{-p})] = K\cup(K+x_p)\cup(K+x_{-p}) .$$

One can easily check that every two summands on the right-hand side of the latter relation are disjoint. This means that K is a subgroup of G of index 3. Obviously, the function $\Phi\colon G\to P$ given by the formula

$$\Phi(x) := \begin{cases} 0 & \text{for} \quad x\in K, \\ p & \text{for} \quad x\in K+x_p,\ x\in G, \\ -p & \text{for} \quad x\in K+x_{-p}, \end{cases}$$

is additive and $\varphi(x) = \Phi(x)$ $(\text{a.e.})_{\mathscr{I}}$. Thus our proof is complete.

The result just obtained will be useful in the proof of the next

Lemma 5. *Suppose that we are given a p.l.i. ideal $\mathscr{I}$ in G. If $f\in\mathscr{S}$ (3a.e.) and*

(32) $$\operatorname{card} f(G\setminus T)>2 \quad \textit{for all } T\in\mathscr{I} ,$$

then f is $\Omega(\mathscr{I})$-almost additive.

Proof. Let f satisfy (3) for all pairs $(x, y)\in G^2\setminus M$, where $M\in\Omega(\mathscr{I})$. We introduce the sets $U(M)$ and $V_x(M)$ in the same way as in the preceding lemma. Moreover, put

$$A_x := Z_{-f(x)}\cup V_x(M)\cup U(M)\cup(U(M)-x),$$

$$M_T := M\cup(T\times G)\cup(G\times T)\cup\{(x, y)\in G^2\colon x+y\in T\}, \quad T\in\mathscr{I} ,$$

$$E_0 := \{(x, y)\in G^2\colon y\in Z_{-f(x)}\} \quad \text{and} \quad E := E_0\setminus M_{U(M)} .$$

Clearly,

(33) $$f(x+a) = f(x)+f(a) \quad \text{for all } x\in G \text{ and } a\in G\setminus A_x .$$

Moreover,

(34) $$A_x\equiv Z_{-f(x)} \pmod{\mathscr{I}} \quad \text{for all } x\in G\setminus U(M)$$

and, since $\Omega(\mathscr{I})\ni M\equiv M_T \pmod{\Omega(\mathscr{I})}$ (see Lemmas 1, 2 and Corollary 1 in [6]),

(35) $$M_T\in\Omega(\mathscr{I}) \quad \text{for all } T\in\mathscr{I} .$$

Since $G^2\setminus M_{U(M)}\subset E\cup(G^2\setminus E_0\setminus M_{U(M)})$, we get

(36) $$(s, t)\in E \quad \text{or} \quad f(s+t) = f(s)+f(t) \quad \text{for all } (s, t)\in G^2\setminus M_{U(M)} .$$

Fix a pair $(s,t)\in E$. Observe that

$$(37)\qquad s,t,s+t\notin U(M)\quad\text{and}\quad f(s)=-f(t).$$

Note that $A_s\neq G$. In fact, otherwise (37) and (34) imply $Z_{-f(s)}\equiv G \pmod{\mathscr{I}}$, i.e. $f(x)=-f(s)$ (a.e.)$_{\mathscr{I}}$, contrary to (32). So we may find an $a\in G\setminus A_s$, whence

$$(38)\qquad f(s+a)=f(s)+f(a)$$

in view of (33). Now, we are going to show that

$$(39)\qquad A_t\cup A_a\cup(A_{s+a}-t)\not\equiv G \pmod{\mathscr{I}}.$$

To this end, suppose that (39) does not hold. Hence

$$Z_{-f(t)}\cup Z_{-f(a)}\cup(Z_{-f(s+a)}-t)\equiv G \pmod{\mathscr{I}},$$

since $t\notin U(M)$, $a\notin U(M)$ and $s+a\notin U(M)$ (by (37), the fact that $a\notin A_s$ and the definition of A_s) and since (34) holds. This means that there exists a set $W_1\in\mathscr{I}$ such that

$$(40)\qquad f(x)=-f(t)\quad\text{or}\quad f(x)=-f(a)\quad\text{or}\quad f(x+t)=-f(s+a),$$

for all $x\in G\setminus W_1$. Take an $x\in G\setminus(W_1\cup V_t(M))$. If $f(x)\neq -f(t)$, then

$$f(x)=-f(a)\quad\text{or}\quad -f(s)-f(a)=-f(s+a)=f(x)+f(t),$$

on account of (40), (38) and the fact that $(x,t)\notin M$. Recalling the second part of (37), we infer that the latter alternative reduces to its first part. Thus

$$f(x)\in\{-f(t),-f(a)\}\quad\text{for all } x\notin W_1\cup V_t(M)\in\mathscr{I},$$

which is incompatible with (32). Consequently, (39) is true and, since $s+t\notin U(M)$, we may find a $b\in G\setminus[A_t\cup A_a\cup(A_{s+a}-t)\cup(V_{s+t}(M)-a)]$, whence

$$(41)\qquad \begin{cases} f(b+t)=f(b)+f(t),\quad f(a+b)=f(a)+f(b),\\ f(s+t+a+b)=f(s+a)+f(b+t)=f(s)+f(a)+f(b)+f(t)\\ \qquad\qquad =f(a)+f(b)=f(a+b)\end{cases}$$

and

$$(42)\qquad (s+t,a+b)\notin M,\quad a+b\notin U(M),\quad a+b+s+t\notin U(M),$$

on account of (33), (38) and the second part of (37).

Now, we shall consider two cases:

1^0 $A_{s+t}\cup(A_{a+b}-s-t)\cup A_{s+t+a+b}=G$.

Then, in the same way as in the preceding step, by means of (37), (41), (42) and (34) we obtain the alternative

$$f(x)=-f(s+t)\quad\text{or}\quad f(x+s+t)=-f(a+b)\quad\text{or}$$
$$f(x)=-f(s+t+a+b)=-f(a+b),$$

for all $x\in G\setminus W_2$, $W_2\in\mathscr{I}$. Suppose that $x\in G\setminus(W_2\cup V_{s+t}(M))$, $f(x)\neq -f(s+t)$ and $f(x)\neq -f(a+b)$. Then

$$-f(a+b)=f(x+s+t)=f(x)+f(s+t),$$

whence the relation

$$(43)\qquad f(x)\in\{-f(s+t),-f(a+b),-f(a+b)-f(s+t)\}\ \text{(a.e.)}_{\mathscr{I}}$$

follows. If we had $f(a+b)+f(s+t)\neq 0$, then, by (41) and (42), we would obtain

$$f(a+b)=f(a+b+s+t)=f(a+b)+f(s+t),$$

i.e. $f(s+t)=0$, which compared with (43) leads to

$$f(x)\in\{0,-f(a+b)\}\ \text{(a.e.)}_{\mathscr{I}},$$

contrary to (32). Consequently, $f(s+t)=-f(a+b)=:p$ and (43) assumes the form

$$f(x)\in\{0,p,-p\}=:P\quad\text{for all } x\in G\setminus T_0,\ T_0\in\mathscr{I}.$$

(32) excludes the possibility $p=-p$. Therefore $2p\neq 0$. Hence $3p=0$; this results from the fact that $2p\in P$. Consequently, $(P,+)$ is a group. Put

$$\varphi(x):=\begin{cases}0 & \text{for}\quad x\in T_0,\\ f(x) & \text{for}\quad x\in T_0'.\end{cases}$$

Evidently, φ maps G into P, φ satisfies (24) in virtue of (32) and

$$(44)\qquad f(x)=\varphi(x)\ \text{(a.e.)}_{\mathscr{I}}.$$

Moreover,

$$\varphi(x)+\varphi(y)\neq 0\quad\text{implies}\quad \varphi(x+y)=\varphi(x)+\varphi(y)\ \text{for all } (x,y)\notin M_{T_0}\in\Omega(\mathscr{I}).$$

Thus, φ satisfies the assumptions of Lemma 4 and hence φ is $\Omega(\mathscr{I})$-almost additive. On account of (44) so is also f, which proves our assertion in the case under consideration.

2^0 $A_{s+t}\cup(A_{a+b}-s-t)\cup A_{s+t+a+b}\neq G$.

Then we may find a $c\in G\setminus[A_{s+t}\cup(A_{a+b}-s-t)\cup A_{s+t+a+b}]$, whence by (33)

$$f(s+t+c)=f(s+t)+f(c),$$
$$f(a+b+c+s+t)=f(c+s+t)+f(a+b),$$
$$f(a+b+c+s+t)=f(a+b+s+t)+f(c).$$

Hence we get in view of (38) and (41)

$$\begin{aligned} f(s+t)-f(s)-f(t) &= [f(s+t+c)-f(c)]-[f(s+a)-f(a)]-[f(b+t)-f(b)]\\ &= f(a+b+c+s+t)-f(a+b)-f(c)-\\ &\quad -f(a+b+s+t)-f(a)-f(b)=0.\end{aligned}$$

This proves that relation (36) reduces to

$$f(s+t) = f(s)+f(t) \quad \text{for all } (s,t) \in G^2\setminus M_{U(M)},$$

which states that f is $\Omega(\mathscr{I})$-almost additive, in view of (35) and the fact that $U(M) \in \mathscr{I}$. This completes the proof.

THEOREM 4. *Suppose that we are given a p.l.i. ideal $\mathscr{I}$ in G and f satisfies* (3a.e.). *Then there exists a function $F\colon G \to H$, $F \in \mathscr{S}(3)$, such that $f(x) = F(x)$ (a.e.)$_{\mathscr{I}}$. In the case where f satisfies* (32) *the corresponding function F is unique.*

Proof. Let the symbols: M, $U(M)$ and $V_x(M)$ have the same meaning as in the last two lemmas. The proof is divided into three cases:

1^0 There exists a set $T \in \mathscr{I}$ such that $\operatorname{card} f(G\setminus T) = 1$. Then $f(x) = d \in H$ for all $x \in T'$ and one can easily check that $d \in \frac{1}{2}0$; on the other hand, the function $F\colon G \to H$ given by the formula $F(x) = d$, $x \in G$, belongs to $\mathscr{S}(3)$ (cf. Theorem 3) and we have $f(x) = F(x)$ (a.e.)$_{\mathscr{I}}$.

2^0 There exists a set $T \in \mathscr{I}$ such that $\operatorname{card} f(G\setminus T) = 2$. Then $f(x) \in \{d, e\} \subset H$ for all $x \in T'$, $d \neq e$. We may also assume that neither Z_d nor Z_e belongs to $\mathscr{I}$; otherwise the situation reduces to the previous one. First we shall show that the possibility $e = -d$ is excluded. To this end, suppose that $f(x) \in \{d, -d\}$ for all $x \in T'$. Clearly $2d \neq 0$, since otherwise f would be "almost constant". Equation (3a.e.) implies that $0 \neq 2d \in \{d, -d\}$, whence $3d = 0$. Put $S\colon = -(U(M) \cup T) \cup U(M)$. We shall prove that

(45) $$-(Z_d\setminus S) \subset Z_{-d} \quad \text{and} \quad Z_{-d}\setminus S \subset -Z_d.$$

For, take an $x \in Z_d\setminus S$. Then $-x \notin U(M) \cup T$ and if we had $-x \in Z_d$, then, taking a

$$y \in (Z_d - x)\setminus[(V_{-x}(M)-x) \cup U(M) \cup (U(M)-x)]$$

and a

$$z \in (Z_d+x)\setminus[(V_x(M)+x) \cup V_y(M) \cup (V_{x+y}(M)+x)],$$

we would get

$$f(x+y) = f(z-x) = d,$$

$$(-x, x+y) \notin M, \quad (x, z-x) \notin M, \quad (y,z) \notin M \quad \text{and} \quad (x+y, z-x) \notin M,$$

whence

$$f(y) = f(x+y)+f(-x) = d+d = -d,$$
$$f(z) = f(z-x)+f(x) = d+d = -d.$$

Consequently, $d = -d-d = f(y)+f(z) = f(y+z)$. On the other hand,

$$d = -d-d = f(x+y)+f(z-x)+f(-x)+f(x) = f(y+z)-d,$$

i.e. $f(y+z) = 2d = -d$. This contradiction proves that the first of the inclusions (45) is true. The other may be derived in a similar way. Relation (45) says, in particular, that

(46) $$Z_{-d} \equiv Z_d \pmod{\mathscr{I}}.$$

Now, take an arbitrary $x \in G\setminus(U(M) \cup T)$. If $x \in Z_d$, then

$$Z_d \equiv Z_d\setminus V_x(M) \subset Z_{-d}-x.$$

Since $Z_{-d} \cup Z_d \equiv G \pmod{\mathscr{I}}$, we obtain $Z_{-d} \cup (Z_{-d}+x) \equiv G \pmod{\mathscr{I}}$, whence in view of (46)

(47) $$Z_d \cup (Z_d+x) \equiv G \pmod{\mathscr{I}}.$$

Similarly, if $x \in Z_{-d}$, then $Z_{-d} \equiv Z_{-d}\setminus V_x(M) \subset Z_d - x$, which again leads to (47). Thus congruence (47) is satisfied for all $x \notin U(M) \cup T =: S_0$. Hence

$$Z_{-d} \cap (Z_d+x) \equiv Z_{-d} \pmod{\mathscr{I}}, \quad x \in G\setminus S_0,$$

and, as a consequence, we get

$$(Z_d+x) \cap (Z_d+y) \notin \mathscr{I} \quad \text{for all } x, y \in G\setminus S_0.$$

Note that for every $z \in G$ one can find an $s \in G\setminus[S_0 \cup (S_0 - z)]$, whence, putting $x = z+s$ and $y = s$, we obtain $z = x-y$, $x, y \notin S_0$. Thus

$$Z_d \cap (Z_d+z) \notin \mathscr{I} \quad \text{for all } z \in G.$$

In particular, taking a $z \in Z_d\setminus U(M)$, we get

$$\mathscr{I} \not\ni Z_p \cap (Z_d+z) \equiv Z_d \cap [(Z_d\setminus V_z(M))+z] \subset Z_d \cap Z_{-d} = \emptyset.$$

This proves that $e = -d$ is impossible. Consequently, $d+e \neq 0$, which leads to $d+e \in \{d, e\}$. Without loss of generality we may assume $e = 0$, i.e. $f(x) \in \{0, d\}$ for all $x \in G\setminus T$. Evidently, we have $2d = 0$.

If we had $-Z_0 \cap Z_d \notin \mathscr{I}$, then, taking an $x \in (-Z_0 \cap Z_d)\setminus(-U(M) \cup U(M))$ and a $y \in Z_0\setminus[V_x(M) \cup (V_{-x}(M)-x)]$, we would get $f(x)+f(y) = d \neq 0$, whence $x+y \in Z_d$. Then

$$d = f(x+y)+f(-x) = f(y) = 0,$$

a contradiction. Thus, $W\colon = (-Z_0 \cap Z_d) \cup T \cup (-T) \in \mathscr{I}$, whence

$$Z\colon = Z_0\setminus(-W \cup W) \equiv Z_0 \pmod{\mathscr{I}}$$

and $-Z = -Z_0\setminus(-W \cup W) \subset Z$, which gives $Z = -Z$. Let $K = (K, +)$ denote the group generated by Z. Making use of the method applied in the proof of Lemma 4, one can show that

$$K \equiv Z_0 \pmod{\mathscr{I}}.$$

Define a function $F\colon G \to H$ by the formula

$$F(x) = \begin{cases} 0 & \text{for } x \in K, \\ d & \text{for } x \in K'. \end{cases}$$

F is a solution of (3) (cf. Theorem 3) and $f(x) = F(x)$ (a.e.)$_{\mathscr{I}}$.

3^0 Neither of the cases 1^0 and 2^0 occurs. Then f satisfies (32) and we may apply Lemma 5. Thus f is $\Omega(\mathscr{I})$-almost additive. On account of de Bruijn's result contained in [2] we infer that (with the aid of our terminology) there exists exactly one function $F\in \mathrm{Hom}(G, H)$ such that $f(x) = F(x)$ $(\mathrm{a.e.})_{\mathscr{I}}$. Thus, our proof has been completed.

Remark 1. F need not be unique provided condition (32) is not satisfied. This can be seen from the following

EXAMPLE 1. Put

$$f(x) = \begin{cases} \alpha(x) & \text{for } x\in T, \\ d\in \tfrac{1}{2}0 & \text{for } x\in G\setminus T, \end{cases}$$

where $0\in T\in \mathscr{I}$ and $\alpha\colon T\to H$ is an arbitrary function. Then one can take $F(x) = d$, $x\in G$, or

$$F(x) = \begin{cases} 0 & \text{for } x = 0, \\ d & \text{for } x\in G\setminus\{0\}. \end{cases}$$

A not so trivial, but less general situation is shown by the following

EXAMPLE 2. Take $G = H = R\setminus\{0\}$ and $\boldsymbol{G} = \boldsymbol{H} = (R\setminus\{0\}, \cdot)$ the multiplicative group of all nonzero real numbers, $\mathscr{I}$-the p.l.i. ideal of all at most countable subsets of $R\setminus\{0\}$, and

$$f(x) = \begin{cases} \alpha(x) & \text{for } x\in Q\setminus\{0\}, \\ -1 & \text{for } x\in R\setminus Q, \end{cases}$$

where Q stands for rationals and $\alpha\colon Q\to R\setminus\{0\}$ is an arbitrary function. Then, similarly, one can take $F(x) = -1$ for $x\in R\setminus\{0\}$, or

$$F(x) = \begin{cases} 1 & \text{for } x\in Q\setminus\{0\}, \\ -1 & \text{for } x\in R\setminus Q. \end{cases}$$

The result obtained in the latter theorem may also be formulated in a slightly more general form:

THEOREM 5. *Suppose that we are given two conjugate p.l.i. ideals $\mathscr{I}_1$ and $\mathscr{I}_2$ in $\boldsymbol{G}$ and $\boldsymbol{G}^2$, respectively. If f satisfies* (3) *almost everywhere with respect to $\mathscr{I}_2$, then there exists a solution $F\colon G\to H$ of* (3) *such that $f(x) = F(x)$ $(\mathrm{a.e.})_{\mathscr{I}_1}$. F is unique provided f satisfies* (32) *with $\mathscr{I} = \mathscr{I}_1$.*

Proof. It suffices to extend the p.l.i. ideal $\mathscr{I}_2$ to $\Omega(\mathscr{I}_1)$ (cf. Corollary 1 in [6]) and to make use of Theorem 4.

In the case where $\mathscr{I}_2 = \pi(\mathscr{I}_1)$ the above result may be strengthened. The corresponding theorems will be preceded by two lemmas:

LEMMA 6. *Suppose that $\mathscr{I}$ is a p.l.i. ideal in $\boldsymbol{G}$ and $(K, +)$ is a subgroup of $\boldsymbol{G}$ such that $K'\in\mathscr{I}$. Then $K = G$.*

Proof. Since $K\equiv G\ (\mathrm{mod}\,\mathscr{I})$ and $K = K+K$, it suffices to apply Lemma 3 from [6].

LEMMA 7. *Suppose that is a p.l.i. ideal in $\boldsymbol{G}$ and $(K, +)$ is a subgroup of $\boldsymbol{G}$ with $K\notin\mathscr{I}$. Then, for all $W\in\mathscr{I}$, we have*

$$(K\setminus W)+(K'\setminus W) = K'. \tag{48}$$

Proof. The case $K = G$ is trivial. So we may assume $K\neq G$. It suffices to prove that for a given $W\in\mathscr{I}$ we have $K'\subset(K\setminus W)+(K'\setminus W)$. Take a $z\in K' = K+K'$. Then $z = x+y$, $x\in K$, $y\in K'$. First, let us note that

$$(K-y)\cap(W'-y)\cap(K'+x)\cap(x-W')\notin\mathscr{I}. \tag{49}$$

In fact, since evidently $(W'-y)\cap(x-W')\equiv G\ (\mathrm{mod}\,\mathscr{I})$, it suffices only to observe that $K-y\subset K'$ and $K'+x = K'$, whence

$$(K-y)\cap(K'+x) = K-y\notin\mathscr{I}.$$

(49) allows one to find an $s\in G$ such that

$$s+y\in K,\quad s+y\notin W,\quad s-x\in K' \quad\text{and}\quad s-x\in -W',$$

or, equivalently,

$$s+y\in K\setminus W \quad\text{and}\quad x-s\in K'\setminus W$$

because of $K' = -K'$. Hence

$$z = x+y = (x-s)+(s+y)\in(K'\setminus W)+(K\setminus W),$$

which ends the proof.

To simplify some further statements let us adopt the following

DEFINITION. Suppose that we are given a p.l.i. ideal $\mathscr{I}$ in $\boldsymbol{G}$. A function $g\colon G\to H$ such that $g(x) = d\in\frac{1}{2}0$ $(\mathrm{a.e.})_{\mathscr{I}}$ is said to be a *d-function*. Moreover,

$$\mathscr{D} := \bigcup_{d\in\frac{1}{2}0}\{g\colon G\to H\colon g \text{ is a } d\text{-function}\}.$$

THEOREM 6. *Let $\mathscr{I}$ denote a p.l.i. ideal in $\boldsymbol{G}$. Suppose that $f\notin\mathscr{D}$ and f satisfies* (3) *for all $x, y\in G\setminus W$, $W\in\mathscr{I}$. Then*

$$f(x) = \begin{cases} \alpha(x) & \text{for } x\in T, \\ F(x) & \text{for } x\in T', \end{cases} \tag{50}$$

where F is an arbitrary member of $\mathscr{S}(3)\setminus\mathscr{D}$, T is a subset of $W\cap F^{-1}(\{0\})$ and $\alpha\colon T\to H$ is an arbitrary function. Conversely, every such function f satisfies (3) *for all $x, y\in G\setminus W$.*

Proof. Suppose that

$$f(x)+f(y)\neq 0 \quad\text{implies}\quad f(x+y) = f(x)+f(y)$$

for all $(x, y)\in G^2\setminus[(G\times W)\cup(W\times G)]$, where $W\in\mathscr{I}$. This means that (3) holds almost everywhere with respect to $\pi(\mathscr{I})$. Theorem 5 ensures that there exists a function $F\colon G\to H$, $F\in\mathscr{S}(3)$, such that

$$T := \{x\in G\colon f(x)\neq F(x)\}\in\mathscr{I}.$$

Evidently, we must have $F \notin \mathscr{D}$. Consequently (cf. Theorem 3) F is either additive or

(51) $$F(x) = \begin{cases} 0 & \text{for} \quad x \in K \notin \mathscr{I}, \\ d & \text{for} \quad x \in K', \end{cases}$$

where $(K, +)$ is a subgroup of G (of index greater than 2) and $d \in \frac{1}{2}0 \setminus \{0\}$. Now, we are going to prove that in both cases we obtain

$$T \subset W \cap F^{-1}(\{0\}) .$$

Assume first that F is additive and that $T \setminus W \neq \emptyset$. Take $x \in T \setminus W$ and $y \in G \setminus E_x$, where

(52) $$E_x := T \cup W \cup (T - x) .$$

Then

$$f(x) + F(y) \neq 0 \quad \text{implies} \quad F(x+y) = f(x) + F(y) ,$$

whence

$$F(y) \neq -f(x) \quad \text{implies} \quad F(x) = f(x) .$$

Since $x \in T$, we get $F(y) = -f(x)$ for all $y \notin E_x \in \mathscr{I}$. An additive and "almost constant" function vanishes identically. Hence F should be a 0-function, contrary to our hypothesis. Thus $T \subset W$. Now take an $x \in T$ and an $s \in G \setminus [W \cup (x - W)]$. Then, if we had

$$F(x) = F(x-s) + F(s) = f(x-s) + f(s) \neq 0 ,$$

we would get $F(x) = f(x)$, which is impossible because of $x \in T$. Consequently, $T \subset W \cap F^{-1}(\{0\})$.

Now assume that F is of form (51). Suppose that $(T \setminus W) \cap K \neq \emptyset$ and take an $s \in (T \setminus W) \cap K$ and a $t \in K \setminus E_s$ (cf. (52)). Then

$$f(s) \neq 0 \quad \text{implies} \quad F(s+t) = f(s) ,$$

a contradiction, since $f(s) \neq F(s) = 0$ and $s + t \in K$. Thus $T \setminus W \subset K'$. Now, if we had $T \setminus W \neq \emptyset$, then, taking $x \in T \setminus W \subset K'$ and $y \in K \setminus E_x$, we would get $F(x) + F(y) = d \neq 0$ and $f(y) = F(y) = 0$, whence, since $x \notin W$, $y \notin W$ and $x + y \notin T$,

$$f(x) \neq 0 \quad \text{implies} \quad F(x) = F(x) + F(y) = F(x+y) = f(x+y) = f(x) + f(y) = f(x) ,$$

which is incompatible with $x \in T$. Thus $f|_{T \setminus W} = 0$. Taking a $u \in T \setminus W$ and a $v \in G \setminus E_u$, we come to

$$f(v) \neq 0 \quad \text{implies} \quad f(u+v) = f(v) ,$$

whence, since $v \notin T$ and $u + v \notin T$,

$$F(v) \neq 0 \quad \text{implies} \quad F(u+v) = F(v) .$$

Hence the inclusion $K' \setminus E_u \subset K' - u$ follows. Consequently

$$K - u \subset K \cup E_u .$$

Since $K \notin \mathscr{I}$ and $E_u \in \mathscr{I}$, we infer that $K \cap (K - u) \neq \emptyset$, whence $u \in K$, a contradiction. This proves that $T \subset W$.

Now take $x \in K \setminus W$ and $y \in K' \setminus W$ ($K' \notin \mathscr{I}$; otherwise F would be a 0-function by Lemma 6). In particular, $x, y \in T'$ and we get

$$f(x) + f(y) = F(x) + F(y) = d \neq 0 ,$$

whence

$$f(x+y) = F(x+y) ,$$

i.e.,

$$(K \setminus W) + (K' \setminus W) \subset T' .$$

Applying Lemma 7 (relation (48)), we obtain $K' \subset T'$ or equivalently $T \subset K = F^{-1}(\{0\})$.

To check that a function f given by (50) (with $F \in \mathscr{S}(3) \setminus \mathscr{D}$, $T \subset W \cap F^{-1}(\{0\})$ and $\alpha \colon T \to H$ an arbitrary function) satisfies (3) for all $x, y \in G \setminus W$, it suffices to consider the case where $x, y \in T'$ and $x + y \in T$, only. In thus case $F(x+y) = 0$, which implies

$$0 = F(x) + F(y) = f(x) + f(y) .$$

This completes the proof.

Regarding d-functions we have the following:

THEOREM 7. *Suppose that we are given a p.l.i. ideal $\mathscr{I}$ in G and a $d \in H$ such that $2d = 0$. If f is a d-function, i.e.*

$$T = \{x \in G \colon f(x) \neq d\} \in \mathscr{I} ,$$

and f satisfies (3) *for all $x, y \in G \setminus W$, $W \in \mathscr{I}$, then either*

(a) *$d = 0$, $T \subset W$ and*

$$f(x) = \begin{cases} \alpha(x) & \text{for} \quad x \in T, \\ 0 & \text{for} \quad x \in T', \end{cases}$$

where $\alpha \colon T \to H \setminus \{0\}$ is an arbitrary function, or

(b) *$d \neq 0$, T satisfies the condition*

(53) $$(T \setminus W) + (T' \setminus W) \subset T'$$

and

$$f(x) = \begin{cases} 0 & \text{for} \quad x \in T \setminus W, \\ \beta(x) & \text{for} \quad x \in T \cap W, \\ d & \text{for} \quad x \in T', \end{cases}$$

where $\beta \colon T \cap W \to H \setminus \{d\}$ is an arbitrary function. Conversely, each of the functions f described in (a) *and* (b) *is a d-function fulfilling* (3) *for all $(x, y) \in G \setminus W$ provided $T \in \mathscr{I}$.*

Proof. First assume $d = 0$. If we had $T \setminus W \neq \emptyset$, then, taking an $x \in T \setminus W$ and a $y \in G \setminus E_x$ (cf. (52)), we would get $f(x) \neq 0$, $f(y) = f(x+y) = 0$ and, since $x, y \notin W$, $0 = f(x+y) = f(x) + f(y) \neq 0$, a contradiction. Thus $T \subset W$.

Now let $d \neq 0$. Take an $x \in T \setminus W$ and a $y \in G \setminus E_x$. Then we have $f(x)+f(y) = f(x)+d \neq 2d = 0$ and $f(x+y) = d$, whence $d = f(x+y) = f(x)+d$, and finally $f(x) = 0$. This means that $f|_{T\setminus W} = 0$. Now take an $x \in T \setminus W$ and a $y \in T' \setminus W$. Then $f(x)+f(y) = d \neq 0$, whence $d = f(x)+f(y) = f(x+y)$, i.e. $x+y \in T'$. Thus condition (53) holds.

The last part of our assertion is obvious.

Remark 2. Condition (53) is trivially satisfied whenever $T \subset W$ or $T \in \mathscr{I}$ is such that $(T, +)$ is a group. However, it may happen that none of these two conditions is satisfied and relation (53) does hold. This readily seen from the following

EXAMPLE 3. Suppose that G does not possess elements of finite orders and $\mathscr{I}$ is a p.l.i. ideal of finite subsets of G, and take

$$T = \{b, 2b\}, \qquad W = \{0, 2b\},$$

where $b \neq 0$ is an arbitrary element of G. Then

$$(T\setminus W)+(T'\setminus W) = b+(G\setminus\{0, b, 2b\}) = G\setminus\{b, 2b, 3b\} \subset T',$$

i.e., (53) is satisfied. The d-function

$$f(x) = \begin{cases} 0 & \text{for} \quad x = b, \\ c & \text{for} \quad x = 2b, \\ d & \text{for} \quad x \in G\setminus\{b, 2b\}, \end{cases}$$

where $c \in H$ is an arbitrary constant and $d \in \frac{1}{2}0\setminus\{0\}$, satisfies (3) for all $x, y \in G\setminus\{0, 2b\}$.

§ 4. Here we are going to strengthen the result from [3] regarding equation (4). On the other hand, we shall simultaneously illustrate the applicability of results of the kind presented in Theorem 4. Namely, we have the following

THEOREM 8. *Suppose that f is a solution of* (4) *and no finite union of sets of the form* $(Z_0-x) \cup (y-Z_0)$, $x, y \in G$, *coincides with* G. *Then* $f \in \mathrm{Hom}(G, H)$ *provided* H *does not possess elements of order* 2. *If there exist elements of order* 2 *in* H, *then* $f \in \mathrm{Hom}(G, H)$ *or*

$$f(x) = \begin{cases} 0 & \text{for} \quad x \in Z_0, \\ d \in \frac{1}{2}0\setminus\{0\} & \text{for} \quad x \in Z_0'. \end{cases} \tag{54}$$

Proof. Let $\mathscr{I}$ denote the linearly invariant set ideal generated by Z_0, i.e. the family of all sets of the form

$$\bigcup_{i=1}^{n} \bigcup_{j=1}^{m} [(Z_0-x_i) \cup (y_j-Z_0)],$$

where n, m are arbitrary positive integers, $x_i, y_j \in G$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, and all their subsets. Our assumption on Z_0 simply states that $\mathscr{I}$ is proper. Consequently, $\mathscr{I}$ yields a p.l.i. ideal in G. Obviously, $Z_0 \in \mathscr{I}$. Write

$$M := \{(x, y) \in G^2 \colon x+y \in Z_0\}.$$

By Lemma 2 from [6], $M \in \Omega(\mathscr{I})$. Now equation (4) may be written in the form

$$f(x)+f(y) \neq 0 \quad \text{implies} \quad f(x+y) = f(x)+f(y) \quad \text{for all } (x, y) \in G^2\setminus M$$

and we may apply Theorem 4: there exists a function $F\colon G \to H$, $F \in \mathscr{S}(3)$, such that $f(x) = F(x)\,(\text{a.e.})_{\mathscr{I}}$. By means of Theorem 3 and the fact that $Z_0 \in \mathscr{I}$ we infer that f is a d-function with $d \neq 0$ or F is additive. Assume that

$$S := \{x \in G \colon f(x) \neq d\} \in \mathscr{I}$$

for a certain $d \in \frac{1}{2}0\setminus\{0\}$. Taking an $x \in S$ and a $y \in G\setminus[S \cup (S-x)]$, we come to

$$f(x+y) = d \neq 0 \quad \text{and} \quad f(x)+f(y) = f(x)+d \neq 2d = 0.$$

Hence

$$d = f(x+y) = f(x)+f(y) = f(x)+d,$$

i.e. $S \subset Z_0$. Thus f is of form (54). Conversely, every function of that form yields a solution of (4) [1].

Now assume that $F \in \mathrm{Hom}(G, H)$. Put $T := \{x \in G \colon f(x) \neq F(x)\}$. Clearly, $T \in \mathscr{I}$. Take an $x \in G$. We have

$$Z_{F(x)-f(x)} \not\equiv (\mathrm{mod}\, \mathscr{I})$$

(otherwise $f(t) = \text{const}\ (\text{a.e.})_{\mathscr{I}}$, whence $F(t) = \text{const}\ (\text{a.e.})_{\mathscr{I}}$, which implies, in view of the additivity of F, that $f(t) = 0\,(\text{a.e.})_{\mathscr{I}}$, i.e. $Z_0 \equiv G\,(\mathrm{mod}\,\mathscr{I})$, a contradiction). Choose an

$$s \in G\setminus[Z_{F(x)-f(x)} \cup Z_0 \cup T \cup (T+x)].$$

Then $f(s) \neq 0$ and

$$f(x)+f(s-x) = f(x)+F(s-x) = f(x)+F(s)-F(x) = f(s)-(F(x)-f(x)) \neq 0,$$

whence

$$F(s) = f(s) = f(x)+f(s-x) = F(s)+f(x)-F(x).$$

i.e. $f(x) = F(x)$ and our proof is completed.

References

[1] K. Baron and R. Ger, *On Mikusiński–Pexider functional equation*, Colloq. Math. 28 (1973), pp. 307–312.

[2] N. G. de Bruijn, *On almost additive functions*, ibidem 15 (1966), pp. 59–63.

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[1] With no restrictions on Z_0.

[3] J. G. Dhombres and R. Ger, *Équations de Cauchy conditionelles*, Comptes Rendus Acad. Sc. Paris 280 (1975), pp. 513–515.

[4] — — *Conditional Cauchy equations*, Glasnik Matematički (to appear).

[5] L. Dubikajtis, C. Ferens, R. Ger and M. Kuczma, *On Mikusiński's functional equation*, Ann. Polon. Math. 28 (1973), pp. 39–47.

[6] R. Ger, *On some functional equations with a restricted domain*, Fund. Math. 89 (1975), pp. 131–149.

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C-S-maximal superassociative systems

by

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Abstract. Let $(A, \varkappa)$ be an n-dimensional superassociative system, C the set of its constants and S the set of its selectors. Further assume, that for any $i = 1, \ldots, n$ there exists at least one ith selector in $(A, \varkappa)$. The problem of determining all the pairs $(n, |C|)$, for which $(C \cup S, \varkappa)$ is already a maximal irreducibly generated subalgebra of $(A, \varkappa)$ is solved.

This paper is devoted to the study of certain superassociative systems. The notion of superassociativity was introduced by K. Menger, who was the first to point out the importance of considering such algebras (cf. [1]).

Let n be some positive integer and for each set X denote its cardinality by $|X|$. Now we define an *n-dimensional superassociative system* or an *n-system* — as we shall call it briefly — to be an algebra $(A, \varkappa)$ of type $n+1$ such that the equality

$$\varkappa\varkappa x_0 \ldots x_n y_1 \ldots y_n = \varkappa x_0 \varkappa x_1 y_1 \ldots y_n \ldots \varkappa x_n y_1 \ldots y_n$$

holds for any $x_0, \ldots, x_n, y_1, \ldots, y_n \in A$. $(A, \varkappa)$ is called *trivial*, if $|A| \leqslant 1$. A subalgebra of $(A, \varkappa)$ is an algebra (B, λ) of type $n+1$ such that B is a subset of A and $\lambda x_0 \ldots x_n = \varkappa x_0 \ldots x_n$ for any $x_0, \ldots, x_n \in B$. By a *constant* of $(A, \varkappa)$ we mean some element c of A, for which $\varkappa c x_1 \ldots x_n = c$ for any $x_1, \ldots, x_n \in A$, and denote the set of all constants by C. An element $s_i \in A$, $1 \leqslant i \leqslant n$, is called an ith *selector* of $(A, \varkappa)$ provided that $\varkappa s_i x_1 \ldots x_n = x_i$ for any $x_1, \ldots, x_n \in A$. Let S_i denote the set of all ith selectors of $(A, \varkappa)$. We put $S := S_1 \cup \ldots \cup S_n$ and call the elements of S selectors of $(A, \varkappa)$. Further we define an n-tuple $(s_1, \ldots, s_n) \in A^n$ to be a *complete system of selectors* for $(A, \varkappa)$ provided that 1) s_i is an ith selector of $(A, \varkappa)$ for any $i = 1, \ldots, n$ and 2) the equality $\varkappa x s_1 \ldots s_n = x$ holds for any $x \in A$. An element a of A is called *symmetric*, if the equation $\varkappa a x_1 \ldots x_n = \varkappa a x_{\pi(1)} \ldots x_{\pi(n)}$ holds for any $x_1, \ldots, x_n \in A$ and for any permutation π of the set $\{1, \ldots, n\}$. An *irreducibly generated* (*i.g.*) n-system is an n-system $(A, \varkappa)$ such that we have $\varkappa x_0 \ldots x_n \in \{x_0, \ldots, x_n\}$ for any $x_0, \ldots, x_n \in A$. I. g. n-systems were also considered by H. Skala (cf. [2]). $C \cup S$ obviously induces an i.g. subalgebra of $(A, \varkappa)$. Applying Zorn's Lemma to this special case we see that there exist maximal i.g. subalgebras of $(A, \varkappa)$. Now there is the question, whether $(C \cup S, \varkappa)$ is already a maximal i.g. subalgebra of $(A, \varkappa)$ or not. If the first comes true, we shall call $(A, \varkappa)$ *C-S-maximal*. Obviously each