Some functors related to polynomial theory

by

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Abstract. This paper is concerned with the study of functors $\tilde{\Gamma}^m$, $\mathring{\Gamma}^m$ which represent such forms f of degree m (in the sense of N. Roby) that f vanishes as a polynomial mapping (resp. that the associated m-linear form Pf is zero). The main results (Theorems 5.3 and 6.4) characterize such modules X that $\mathring{\Gamma}^m(X) = 0$ (resp. $\tilde{\Gamma}^m(X) = 0$). Sections 7 and 8 yield some structure theorems for $\tilde{\Gamma}^m(X)$

0. Conventions and notation. In this paper all rings and algebras are commutative and have the unit element 1; all modules are unitary. We use the following notation:

R-Mod = the category of all R-modules,

R-Alg = the category of all (commutative) R-algebras,

$\mathrm{Max}(R)$ = the set of all maximal ideals in the ring R,

$|X|$ = the cardinality of the set X.

We abbreviate $\otimes = \otimes_R$, $\mathrm{Hom} = \mathrm{Hom}_R$ if there is no danger of confusion.

1. Polynomials and polynomial mappings. We recall some definitions contained in [2].

For any R-module X consider the functor $X\otimes\_ : R\text{-Alg} \to \text{Sets}$. Any natural transformation $f = (f_A)\colon X\otimes\_ \to Y\otimes\_$ is called a *polynomial* on (X, Y) (a "loi polynome sur le couple (X, Y)" in [2]). In a natural way we form the functor $\mathscr{P}\colon R\text{-Mod}^0 \times R\text{-Mod} \to R\text{-Mod}$ where $\mathscr{P}(X, Y)$ consists of all polynomials on (X, Y). We also have the natural transformation $\nu\colon \mathscr{P} \to \mathrm{Map}$ defined by $\nu(f) = f_R$. The mapping f_R, where $f \in \mathscr{P}(X, Y)$, is called the *polynomial mapping induced by the polynomial* f. If $X = R^n$ and $Y = R$, then $\mathscr{P}(X, Y) = R[T_1, \ldots, T_n]$ and ν is the standard homomorphism which carries $F \in R[T_1, \ldots, T_n]$ to $\bar{F}\colon R^n \to R$ where $\bar{F}(r_1, \ldots, r_n) = F(r_1, \ldots, r_n)$.

A polynomial $f \in \mathscr{P}(X, Y)$ is called a *form of degree* m iff $f_A(za) = f_A(z)a^m$ for all $A \in \mathrm{Ob}\, R\text{-Alg}$, $a \in A$, $z \in X\otimes A$. Restricting our considerations to forms of degree m, we obtain the functor $\mathscr{P}^m\colon R\text{-Mod}^0 \times R\text{-Mod} \to R\text{-Mod}$ and the natural transformation $\nu^m\colon \mathscr{P}^m \to \mathrm{Map}$. Write $\tilde{\mathscr{P}}^m = \mathrm{Ker}\, \nu^m$, $\mathrm{Hom}^m = \mathrm{Im}\, \nu^m$. It is known from [2] that for all R-modules X, Y

$$\mathscr{P}^0(X, Y) = \mathrm{Hom}^0(X, Y) = Y,$$
$$\mathscr{P}^1(X, Y) = \mathrm{Hom}^1(X, Y) = \mathrm{Hom}(X, Y),$$
$$\mathscr{P}^2(X, Y) = \mathrm{Hom}^2(X, Y) = \mathrm{Quad}(X, Y).$$

This means that $\bar{\mathscr{P}}^0 = \bar{\mathscr{P}}^1 = \bar{\mathscr{P}}^2 = 0$. In the general case $\bar{\mathscr{P}}^m \neq 0$ for $m \geqslant 3$. The purpose of this paper is the study of these functors.

2. Representability and the alternate definition of $\mathscr{P}^m$. For any R-module X consider the graded divided power R-algebra $\Gamma(X) = \bigoplus \Gamma^m(X)$ (see [2] p. 248). Write

$$\gamma_{m_1 \ldots m_k}(x_1, \ldots, x_k) = x_1^{[m_1]} \ldots x_k^{[m_k]} \in \Gamma(X).$$

It is easy to prove that $\Gamma^m(X)$ is the R-module which is given by the generators

$$\gamma_{m_1 \ldots m_k}(x_1, \ldots, x_k), \quad k \geqslant 1, \ \Sigma m_i = m, \ x_i \in X$$

and the relations

$$(2.1)\quad \gamma_{m_1 \ldots m_k}(x_1, \ldots, x_k) = \begin{cases} \gamma_{m_{s(1)} \ldots m_{s(k)}}(x_{s(1)}, \ldots, x_{s(k)}) & \text{if } s \text{ is a permutation},\\ r^{m_1}\gamma_{m_1 \ldots m_k}(x, x_2, \ldots, x_k) & \text{if } x_1 = rx,\\ \gamma_{m_1 \ldots m_k}(0, x_2, \ldots, x_k) & \text{if } m_1 = 0,\\ (m_1, m_2)\gamma_{m_1+m_2, m_3 \ldots m_k}(x, x_3, \ldots, x_k) & \text{if } x_1 = x_2 = x,\\ \sum_{i+j=m_1} \gamma_{ijm_2 \ldots m_k}(x, y, x_2, \ldots, x_k) & \text{if } x_1 = x+y, \end{cases}$$

where $(i, j) = \binom{i+j}{i} = \binom{i+j}{j}$. Recall the following

THEOREM 2.1 ([2], p. 266). *The functor $\mathscr{P}^m(X, -)$ is represented by $\Gamma^m(X)$. More precisely, any $\varphi \in \mathrm{Hom}(\Gamma^m(X), Y)$ corresponds to $f \in \mathscr{P}^m(X, Y)$ defined by*

$$f_A(x_1 \otimes a_1 + \ldots + x_k \otimes a_k) = \sum_{m_1 + \ldots + m_k = m} \varphi(\gamma_{m_1 \ldots m_k}(x_1, \ldots, x_k)) \otimes a_1^{m_1} \ldots a_k^{m_k}.$$

Let us define the functor $\mathscr{F}^m$: $R\text{-Mod}^0 \times R\text{-Mod} \to R\text{-Mod}$ as follows:

(a) Any $F \in \mathscr{F}^m(X, Y)$ is a system $F = (F_{m_1 \ldots m_k}, k \geqslant 1, m_i \geqslant 0, \Sigma m_i = m)$ of mappings $F_{m_1 \ldots m_k}$: $X^k \to Y$ satisfying relations analogous to (2.1)

(b) $(rF + sG)_{m_1 \ldots m_k} = rF_{m_1 \ldots m_k} + sG_{m_1 \ldots m_k}$, $r, s \in R$,

(c) $(\mathscr{F}^m(f, g)F)_{m_1 \ldots m_k} = g \circ F_{m_1 \ldots m_k} \circ f^k$.

In particular, $\gamma = (\gamma_{m_1 \ldots m_k}) \in \mathscr{F}^m(X, \Gamma^m(X))$. Theorem 2.1 shows that $\mathscr{P}^m \approx \mathscr{F}^m$. More precisely, $F \in \mathscr{F}^m(X, Y)$ corresponds to such an $f \in \mathscr{P}^m(X, Y)$ that

$$f_A(x_1 \otimes a_1 + \ldots + x_k \otimes a_k) = \sum_{m_1 + \ldots + m_k = m} F_{m_1 \ldots m_k}(x_1, \ldots, x_k) \otimes a_1^{m_1} \ldots a_k^{m_k}.$$

In this correspondence $f_R = F_m$ and hence ν^m: $\mathscr{F}^m \approx \mathscr{P}^m \to \mathrm{Hom}^m$ carries F to F_m.

3. Defects and the associated multilinear form. Consider the R-homomorphisms Δ^k: $\mathrm{Map}(X, Y) \to \mathrm{Map}(X^k, Y)$, $k \geqslant 0$, defined by the formula

$$(\Delta^k f)(x_1, \ldots, x_k) = \sum_{H \subset \{1, 2, \ldots, k\}} (-1)^{k-|H|} f\Big(\sum_{i \in H} x_i\Big).$$

An easy verification shows that

$$(3.1)\qquad f(x_1 + \ldots + x_n) = \sum_{k=0}^{n} \sum_{1 \leqslant j_1 < \ldots < j_k \leqslant n} (\Delta^k f)(x_{j_1}, \ldots, x_{j_k}).$$

LEMMA 3.1. *If $F \in \mathscr{F}^m(X, Y)$, then for any $k > 0$*

$$\Delta^k F_m = \sum_{m_i > 0} F_{m_1 \ldots m_k}.$$

Proof. It is easy to verify the formula

$$(\Delta^{k+1} f)(x_0, \ldots, x_k)$$
$$= (\Delta^k f)(x_0 + x_1, x_2, \ldots, x_k) - (\Delta^k f)(x_1, \ldots, x_k) - (\Delta^k f)(x_0, x_2, \ldots, x_k).$$

Moreover, Relations (2.1) show that

$$F_{m_1 \ldots m_k}(x_0 + x_1, x_2, \ldots, x_k) = \sum_{i+j=m_1} F_{ijm_2 \ldots m_k}(x_0, \ldots, x_k)$$

and

$$F_{0m_1 \ldots m_k}(x_0, \ldots, x_k) = F_{m_1 \ldots m_k}(x_1, \ldots, x_k).$$

Induction on k completes the proof.

In particular, $\Delta^k F_m = 0$ for $k > m$ and $\Delta^m F_m = F_{1 \ldots 1}$ for $m > 0$. Relations (2.1) show that $F_{1 \ldots 1} \in \mathrm{Sym}^m(X, Y)$, i.e., $F_{1 \ldots 1}$ is m-linear and symmetric. This mapping is called the *m-linear form associated with F*. Let $f \in \mathscr{P}^m(X, Y)$. Then we have the associated form $\Delta^m f_R$ which is denoted by Pf (cf. [2], p. 234).

Put $\mathring{\mathscr{P}}^m(X, Y) = \{f \in \mathscr{P}^m(X, Y) |\ Pf = 0\}$. Then we have the commutative diagram with exact rows

$$(3.2)\qquad \begin{array}{ccccccc} 0 \to & \bar{\mathscr{P}}^m(X, Y) & \hookrightarrow & \mathscr{P}^m(X, Y) & \xrightarrow{\nu^m} & \mathrm{Hom}^m(X, Y) & \to 0 \\ & \downarrow & & \| & & \downarrow \Delta^m & \\ 0 \to & \mathring{\mathscr{P}}^m(X, Y) & \hookrightarrow & \mathscr{P}^m(X, Y) & \xrightarrow{P} & \mathrm{Sym}^m(X, Y) & \end{array}$$

in which all homomorphisms are natural.

4. Representability of $\mathring{\mathscr{P}}^m$, $\bar{\mathscr{P}}^m$, Hom^m. The natural transformations of representable functors

$$\nu^m: \mathscr{P}^m(X, -) \to \mathrm{Map}(X, -), \qquad P: \mathscr{P}^m(X, -) \to \mathrm{Sym}^m(X, -)$$

induce the R-homomorphisms

$$F(X) \to \Gamma^m(X), \qquad x \mapsto \gamma_m(x),$$
$$S^m(X) \to \Gamma^m(X), \qquad x_1 \vee \ldots \vee x_m \mapsto \gamma_{1 \ldots 1}(x_1, \ldots, x_m).$$

Since Hom is left exact, it follows that the cokernels of these homomorphisms represent the functors $\tilde{\mathscr{P}}^m(X, -)$ and $\mathring{\mathscr{P}}^m(X, -)$. More precisely:

COROLLARY 4.1. *The functor $\tilde{\mathscr{P}}^m(X, -)$ is represented by*

$$\tilde{\Gamma}^m(X) = \frac{\Gamma^m(X)}{R\{\gamma_m(x) \mid x \in X\}}.$$

The functor $\mathring{\mathscr{P}}^m(X, -)$ is represented by

$$\mathring{\Gamma}^m(X) = \frac{\Gamma^m(X)}{R\{\gamma_{1\ldots1}(x_1, \ldots, x_m) \mid x_1, \ldots, x_m \in X\}}.$$

Let us write $\bar{\Gamma}^m(X) = R\{\gamma_m(x) \mid x \in X\} \subset \Gamma^m(X)$. The connection of $\bar{\Gamma}^m$ and Hom^m is described in

COROLLARY 4.2. *The following conditions are equivalent:*

(i) $\mathrm{Hom}^m(X, -)$ *is representable,*

(ii) $\mathrm{Hom}^m(X, -)$ *is represented by* $\bar{\Gamma}^m(X)$,

(iii) *the exact sequence*

$$0 \to \bar{\Gamma}^m(X) \to \Gamma^m(X) \to \tilde{\Gamma}^m(X) \to 0 \tag{4.1}$$

splits.

Proof. It suffices to prove (i)⇒(iii). Let $\mathrm{Hom}^m(X, -)$ be represented by the R-module M. Then the exact sequence

$$0 \to \mathrm{Hom}(\tilde{\Gamma}^m(X), -) \to \mathrm{Hom}(\Gamma^m(X), -) \to \mathrm{Hom}(M, -) \to 0$$

induces the commutative diagram with an exact row

$$\begin{array}{ccccccc} M & \to & \Gamma^m(X) & \to & \tilde{\Gamma}^m(X) & \to & 0 \\ & \diagdown\!\!\diagdown & \downarrow & & & & \\ & & M & & & & \end{array}$$

where $\Gamma^m(X) \to M$ is an element of $\mathrm{Hom}(\Gamma^m(X), M)$ which corresponds to the identity in $\mathrm{Hom}(M, M)$. This completes the proof.

Using standard methods of homological algebra, we obtain another connection:

COROLLARY 4.3. (4.1) *induces the exact sequence*

$$0 \to \mathrm{Hom}^m(X, -) \xrightarrow{\varepsilon} \mathrm{Hom}(\bar{\Gamma}^m(X), -) \xrightarrow{d} \mathrm{Ext}^1(\tilde{\Gamma}^m(X), -) \to \mathrm{Ext}^1(\Gamma^m(X), -) \to \ldots$$

If X is projective, then $\Gamma^m(X)$ is also projective and hence the sequence

$$0 \to \mathrm{Hom}^m(X, -) \xrightarrow{\varepsilon} \mathrm{Hom}(\bar{\Gamma}^m(X), -) \xrightarrow{d} \mathrm{Ext}^1(\tilde{\Gamma}^m(X), -) \to 0$$

is exact.

EXAMPLE 4.4. Let $R = Z$, $X = R^2 = Re_1 \oplus Re_1$. Then the elements

$$\gamma_{3,0} = \gamma_3(e_1), \quad \gamma_{2,1} = \gamma_{2,1}(e_1, e_2), \quad \gamma_{1,2} = \gamma_{1,2}(e_1, e_2), \quad \gamma_{0,3} = \gamma_3(e_2)$$

form a basis of $\Gamma^3(X)$. (For the proof see [2], p. 272.) $\bar{\Gamma}^3(X)$ is generated by

$$\gamma_3(r_1 e_1 + r_2 e_2) = r_1^3 \gamma_{3,0} + r_1^2 r_2 \gamma_{2,1} + r_1 r_2^2 \gamma_{1,2} + r_2^3 \gamma_{0,3}, \qquad r_1, r_2 \in R.$$

Putting $(r_1, r_2) = (1, 0), (0, 1), (1, 1), (1, -1)$, we get

$$\gamma_{3,0}, \gamma_{0,3}, \gamma_{1,2} + \gamma_{2,1}, \gamma_{1,2} - \gamma_{2,1} \in \bar{\Gamma}^3(X).$$

It is easy to see that these elements are generators of $\bar{\Gamma}^3(X)$. Hence $\tilde{\Gamma}^3(X) \approx Z_2$ and, by Corollary 4.2, $\mathrm{Hom}^3(X, -)$ is not representable. Corollary 4.3 gives the exact sequence

$$0 \to \mathrm{Hom}^3(X, -) \xrightarrow{\varepsilon} \mathrm{Hom}(\bar{\Gamma}^3(X), -) \xrightarrow{d} Z_2 \otimes \_ \to 0.$$

Let us exchange $\mathrm{Hom}^m(X, Y)$ for $\mathrm{Hom}(\bar{\Gamma}^m(X), Y)$ in (3.2). Then we obtain the induced commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & S^m(X) & \to & \Gamma^m(X) & \to & \mathring{\Gamma}^m(X) & \to & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \to & \bar{\Gamma}^m(X) & \to & \Gamma^m(X) & \to & \tilde{\Gamma}^m(X) & \to & 0 \end{array} \tag{4.2}$$

in which all homomorphisms are natural. It is known that Γ^m, S^m commute with direct limits (for Γ^m see [2], p. 277). Using standard methods, we can prove

COROLLARY 4.5. *All functors in* (4.2) *commute with direct limits.*

5. The functor $\mathring{\Gamma}^m$. Let us denote by $\mathring{\gamma}_{m_1\ldots m_k}(x_1, \ldots, x_k)$ the class of $\gamma_{m_1\ldots m_k}(x_1, \ldots, x_k)$ in $\mathring{\Gamma}^m(X)$.

LEMMA 5.1. *Any ring homomorphism $R \to A$ induces the natural A-isomorphism*

$$\mathring{\Gamma}^m_R(X) \otimes A \approx \mathring{\Gamma}^m_A(X \otimes A).$$

Proof. Consider the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & K_1 & \hookrightarrow & \Gamma^m_R(X) \otimes A & \to & \mathring{\Gamma}^m_R(X) \otimes A & \to & 0 \\ & & & & \approx \downarrow f & & & & \\ 0 & \to & K_2 & \hookrightarrow & \Gamma^m_A(X \otimes A) & \to & \mathring{\Gamma}^m_A(X \otimes A) & \to & 0 \end{array}$$

where f denotes the standard isomorphism (see [2], p. 262) which carries $\gamma_{m_1\ldots m_k}(x_1, \ldots, x_k) \otimes 1$ to $\gamma_{m_1\ldots m_k}(x_1 \otimes 1, \ldots, x_k \otimes 1)$. Since

$$K_1 = A\{\gamma_{1\ldots1}(x_1, \ldots, x_m) \otimes 1\},$$

$$K_2 = A\{\gamma_{1\ldots1}(x_1 \otimes 1, \ldots, x_m \otimes 1)\},$$

it follows that $K_1 \approx K_2$ by f. Now it suffices to complete the above diagram.

LEMMA 5.2. *Let X be a free R-module and let $\{e_1, \ldots, e_N\}$ be a basis of X. Then*

$$\mathring{\Gamma}^m(X) = \bigoplus R\mathring{\gamma}_{m_1\ldots m_N}(e_1, \ldots, e_N)$$

and

$$R\mathring{\gamma}_{m_1\ldots m_N}(e_1, \ldots, e_N) \approx \frac{R}{m_1!\ldots m_N!R}.$$

Proof. From [2], p. 272, it follows that $\{\gamma_{m_1\ldots m_N}(e_1, \ldots, e_N) |\ m_1+\ldots+m_N = m\}$ is a basis of $\Gamma^m(X)$. The image of $S^m(X)$ in $\Gamma^m(X)$ is generated by

$$\gamma_{1\ldots 1}(\underbrace{e_1, \ldots, e_1}_{m_1}, \ldots, \underbrace{e_N, \ldots, e_N}_{m_N}) = m_1!\ldots m_N!\gamma_{m_1\ldots m_N}(e_1, \ldots, e_N).$$

This completes the proof.

From the above lemmas we get the following

THEOREM 5.3. *Let X be a finitely generated R-module. Then the following conditions are equivalent*:

(i) $\mathring{\Gamma}^m_R(X) = 0$,

(ii) $\mathring{\Gamma}^m_{R/I}(X/IX) = 0$ *for any* $I \in \mathrm{Max}(R)$,

(iii) *for any* $I \in \mathrm{Max}(R)$ *either* $X = IX$ *or* $m! \notin I$.

Proof. The equivalence (i)⇔(ii) follows from the Nakayama Lemma and Lemma 5.1. For the proof of (ii)⇔(iii) observe that, in the case of a linear space over a field, $\mathring{\Gamma}^m(X) = 0$ iff $X = 0$ or $m! \neq 0$ by Lemma 5.2. This completes the proof.

COROLLARY 5.4. *The following conditions are equivalent*:

(i) $\mathring{\Gamma}^m_R = 0$,

(ii) $\mathring{\Gamma}^m_R(R) = 0$,

(iii) $m!$ *is invertible in* R,

(iv) $S^m \to \Gamma^m$ *is bijective.*

Proof. (ii)⇒(iii) follows from Theorem 5.3. (iii)⇒(iv) follows from [2], p. 256.

In the next section we prove similar results for $\tilde{\Gamma}^m$.

6. The functor $\tilde{\Gamma}^m$. Let us denote by $\tilde{\gamma}_{m_1\ldots m_k}(x_1, \ldots, x_k)$ the class of $\gamma_{m_1\ldots m_k}(x_1, \ldots, x_k)$ in $\tilde{\Gamma}^m(X)$. We first prove

THEOREM 6.1. *Any ring homomorphism $R \to A$ induces the natural A-epimorphism*

$$g\colon \tilde{\Gamma}^m_R(X)\otimes A \to \tilde{\Gamma}^m_A(X\otimes A)$$

which carries $\tilde{\gamma}_{m_1\ldots m_k}(x_1, \ldots, x_k)\otimes 1$ to $\tilde{\gamma}_{m_1\ldots m_k}(x_1\otimes 1, \ldots, x_k\otimes 1)$. Moreover, if any element of $X\otimes A$ has a form $x\otimes a$ where $x \in X$ and $a \in A$, then g is bijective.

Proof. Observe that the standard isomorphism f (see the proof to Lemma 5.1) carries the submodule

$$K = A\{\gamma_m(x)\otimes 1 |\ x \in X\} \subset \Gamma^m_R(X)\otimes A$$

into $\overline{\Gamma}^m_A(X\otimes A)$. Hence we obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & K & \hookrightarrow & \Gamma^m_R(X)\otimes A & \to & \tilde{\Gamma}^m_R(X)\otimes A & \to 0 \\ & \downarrow & & \approx\downarrow f & & \downarrow g & \\ 0 \to & \overline{\Gamma}^m_A(X\otimes A) & \hookrightarrow & \Gamma^m_A(X\otimes A) & \to & \tilde{\Gamma}^m_A(X\otimes A) & \to 0. \end{array}$$

The snake lemma (see [1], Proposition 2.10) shows that g is surjective and

$$\mathrm{Ker}\, g \approx \frac{\overline{\Gamma}^m_A(X\otimes A)}{A\{\gamma_m(x\otimes 1) |\ x \in X\}}.$$

This completes the proof.

COROLLARY 6.2. $\tilde{\Gamma}^m_{R_S}(X_S) \approx \tilde{\Gamma}^m_R(X)_S$ *for any multiplicative* $S \subset R$,

$$\tilde{\Gamma}^m_{R/I}(X/IX) \approx \tilde{\Gamma}^m_R(x)/I\tilde{\Gamma}^m_R(x) \quad \textit{for any ideal } I \subset R$$

and

$$\tilde{\Gamma}^m_{R\times R'}(X\times X') \approx \tilde{\Gamma}^m_R(X)\times\tilde{\Gamma}^m_{R'}(X').$$

Now we study the conditions under which $\tilde{\Gamma}^m$ vanishes. We first prove

LEMMA. 6.3. *If K is a field then the following conditions are equivalent*:

(i) $\tilde{\Gamma}^m_K(K^N) = 0$,

(ii) $\tilde{\mathscr{P}}^m_K(K^N, K) = 0$,

(iii) *every form of degree m in $K[T_1, \ldots, T_N]$ which vanishes as a mapping is zero,*

(iv) $N \leqslant 1$ *or* $m \leqslant |K|$.

Proof. Equivalences (i)⇔(ii)⇔(iii) are evident. If $N > 1$ and $m > |K|$ then the non-zero form $T_1^{m-|K|-1}(T_1^{|K|}T_2 - T_1T_2^{|K|})$ vanishes as the mapping $K^N \to K$.

Conversely, we prove (iv)⇒(iii). The case $N \leqslant 1$ is evident. Let $m \leqslant |K|$ and let $F \in K[T_1, \ldots, T_N]$ be a form of degree m. The condition $F(0, \ldots, 1, \ldots, 0) = 0$ implies that all degrees of T_i in F are $< |K|$. If such a polynomial vanishes as a mapping, then it is zero (for the proof apply induction on N). This completes the proof.

In the general case we have

THEOREM 6.4. *If X is a finitely generated R-module, then the following conditions are equivalent*:

(i) $\tilde{\Gamma}^m_R(X) = 0$,

(ii) $\tilde{\Gamma}^m_{R/I}(X/IX) = 0$ *for any* $I \in \mathrm{Max}(R)$,

(iii) *for any* $I \in \mathrm{Max}(R)$ *either* $\dim_{R/I} X/IX \leqslant 1$ *of* $m \leqslant |R/I|$.

Proof. (i)⇔(ii) by the Nakayama Lemma and Corollary 6.2.

(ii)⇔(iii) by Lemma 6.3.

The above theorem and Corollary 4.5 immediately imply

COROLLARY 6.5. *The following conditions are equivalent*:

(i) $\tilde{\Gamma}_R^m = 0$,

(ii) $\tilde{\Gamma}_R^m(R^2) = 0$,

(iii) $m \leqslant d(R)$,

where $d(R) = \inf\{|R/I|;\ I \in \operatorname{Max}(R)\}$. *In particular* $\tilde{\Gamma}_R^m = 0$ *for all* m *iff* $d(R)$ *is infinite.*

EXAMPLE 6.6. Let X be a finitely generated Z-module. Let N (resp. N_p for any prime p) denote the number of those summands in the canonical decomposition of X which are isomorphic to Z (resp. to Z_{p^n} for some n). Then Theorem 6.4 shows that $\tilde{\Gamma}_Z^m(X) = 0$ iff $N + N_p \leqslant 1$ for any prime $p < m$.

7. Modules $\tilde{\Gamma}^m(X)$ over integral domains. In this section, R denotes an infinite integral domain.

Observe that $\tilde{\Gamma}_R^m(X)_{(0)} = \tilde{\Gamma}_{R_{(0)}}^m(X_{(0)}) = 0$ by Corollary 6.2 and 6.5. Hence we get

COROLLARY 7.1. *All $\tilde{\Gamma}_R^m(X)$ are torsion R-modules. In particular, if X is finitely generated, then* $\operatorname{Ann}(\tilde{\Gamma}_A^m(X)) \neq 0$.

Note also two consequences of the above fact:

COROLLARY 7.2. $v^m\colon \mathscr{P}_R^m(-, Y) \to \operatorname{Hom}_R^m(-, Y)$ *is bijective for any natural m and any torsion-free R-module Y.*

(Compare also [2], Proposition I.8.)

COROLLARY 7.3. *If X is a finitely generated projective R-module, then the following conditions are equivalent*:

(i) $\operatorname{Hom}_R^m(X, -)$ *is representable*,

(ii) $\tilde{\mathscr{P}}_R^m(X, -) = 0$,

(iii) $\operatorname{rank} X \leqslant 1$ *or* $m \leqslant d(R)$.

Proof. (i)$\Leftrightarrow$(ii) follows from Corollaries 4.1 and 4.2 since $\Gamma^m(X)$ is projective and $\tilde{\Gamma}^m(X)$ is a torsion R-module.

(ii)$\Leftrightarrow$(iii) follows from Theorem 6.4.

If R is a Dedekind domain, then we can prove some structural theorems.

THEOREM 7.4. *Let R be a Dedekind domain and let X be a finitely generated R-module. If*

$$\operatorname{Ann}(\tilde{\Gamma}_R^m(X)) = \prod_{P \in \operatorname{Max}(R)} P^{n_P}$$

and $k_P \geqslant n_P$ for all $P \in \operatorname{Max}(R)$, then there exists a natural R-isomorphism

$$\tilde{\Gamma}_R^m(X) \approx \bigoplus_{P \in \operatorname{Max}(R)} \tilde{\Gamma}_{R/P^{k_P}}^m(X/P^{k_P}X)$$

induced by $X \to X/P^{k_P}X$.

Proof. Let $n_P = 0$ for $P \neq P_1, \ldots, P_s$ and $k_i \geqslant n_{P_i}$. Then

$$I = P_1^{k_1} \ldots P_s^{k_s} \subset \operatorname{Ann}(\tilde{\Gamma}_R^m(X)).$$

Since $P_i^{k_i} + P_j^{k_j} = R$ for all $i \neq j$, it follows that

$$R/I \approx \prod_{i=1}^{s} R/P_i^{k_i}$$

(see [1], Proposition 1.10). Hence by Corollary 6.2

$$\tilde{\Gamma}_R^m(X) = \frac{\tilde{\Gamma}_R^m(X)}{I\tilde{\Gamma}_R^m(X)} \approx \tilde{\Gamma}_{R/I}^m\left(\frac{X}{IX}\right) \approx \bigoplus_{i=1}^{s} \tilde{\Gamma}_{R/P_i^{k_i}}^m\left(\frac{X}{P_i^{k_i}X}\right).$$

If $P \neq P_1, \ldots, P_s$, then $I + P^k = R$ for any natural k. Hence

$$\tilde{\Gamma}_{R/P^k}^m\left(\frac{X}{P^kX}\right) = \frac{\tilde{\Gamma}_R^m(X)}{P^k\tilde{\Gamma}_R^m(X)} = 0.$$

COROLLARY 7.5. *If R is a Dedekind domain, then there exists a natural R-isomorphism*

$$\tilde{\Gamma}_R^m(X) \approx \bigoplus_{P \in \operatorname{Max}(R)} \tilde{\Gamma}_{R_P}^m(X_P)$$

induced by $X \to X_P$.

Proof. If X is finitely generated, then we apply the above theorem to X and X_P for $P \in \operatorname{Max}(R)$. Next we apply Corollary 4.5.

Analogously we can prove

COROLLARY 7.6.

$$\tilde{\Gamma}_Z^m(X) \approx \bigoplus_{p} \tilde{\Gamma}_{Z_p}^m(X \otimes_Z Z_p)$$

where Z_p denotes the ring of p-adic integers.

8. The structure of $\tilde{\Gamma}^m(R^N)$. For any natural m, k write

$$\Gamma^{mk} = R\{\gamma_{m_1 \ldots m_k}(e_1, \ldots, e_k) \mid m_j > 0,\ \textstyle\sum m_j = m\} \subset \Gamma^m(R^k),$$

$$\bar{\Gamma}^{mk} = R\{(\Delta^k \gamma_m)(r_1 e_1, \ldots, r_k e_k) \mid r_j \in R\} \subset \bar{\Gamma}^m(R^k),$$

where $e_1, \ldots, e_k$ form the standard basis of R^k. Lemma 3.1 shows that $\bar{\Gamma}^{mk} \subset \Gamma^{mk}$. Hence we can define $\tilde{\Gamma}^{mk} = \Gamma^{mk}/\bar{\Gamma}^{mk}$. It is easy to see that $\tilde{\Gamma}^{mk} = 0$ for $k = 1$ and $k \geqslant m$.

THEOREM 8.1. *If e_i, $i \in I$, form a basis of X and I is ordered by $<$, then for any natural m*

$$\Gamma^m(X) = \bigoplus_k \bigoplus_{i_1 < \ldots < i_k} R\{\gamma_{m_1 \ldots m_k}(e_{i_1}, \ldots, e_{i_k}) \mid m_j > 0,\ \textstyle\sum m_j = m\},$$

$$\bar{\Gamma}^m(X) = \bigoplus_k \bigoplus_{i_1 < \ldots < i_k} R\{(\Delta^k \gamma_m)(r_1 e_{i_1}, \ldots, r_k e_{i_k}) \mid r_j \in R\},$$

$$\tilde{\Gamma}^m(X) = \bigoplus_k \bigoplus_{i_1 < \ldots < i_k} R\{\tilde{\gamma}_{m_1 \ldots m_k}(e_{i_1}, \ldots, e_{i_k}) \mid m_j > 0,\ \textstyle\sum m_j = m\}.$$

Moreover, the above summands are isomorphic with Γ^{mk}, $\bar{\Gamma}^{mk}$, $\tilde{\Gamma}^{mk}$, respectively.

Proof. The first decomposition is evident since $\gamma_{m_1\dots m_k}(e_{i_1}, \dots, e_{i_k})$ form a basis of $\Gamma^m(X)$ (see [2], p. 272). The second follows from formula (3.1) applied to $\gamma_m(\sum r_i e_i)$. The third we obtain by division. The evident isomorphism

$$R\{\gamma_{m_1\dots m_k}(e_{i_1}, \dots, e_{i_k})|\ m_j>0,\ \sum m_j = m\} \approx \Gamma^{mk}$$

induces the next two isomorphisms.

COROLLARY 8.2.

$$\overline{\Gamma}^m(R^N) \approx \bigoplus_{k=1}^{N} \binom{N}{k} \overline{\Gamma}^{mk}, \quad \tilde{\Gamma}^m(R^N) \approx \bigoplus_{k=1}^{N} \binom{N}{k} \tilde{\Gamma}^{mk} = \bigoplus_{k=2}^{\min(N,m-1)} \binom{N}{k} \tilde{\Gamma}^{mk}.$$

EXAMPLE 8.3. Example 4.4 shows that $\tilde{\Gamma}_Z^{3,2} = \tilde{\Gamma}_Z^3(Z^2) = Z_2$. Hence

$$\tilde{\Gamma}_Z^3(Z^N) = Z_2 \oplus \dots \oplus Z_2 \quad \left(\binom{N}{2}\text{-times}\right).$$

We get another application in the case where $R = K$ is a finite field. Write

$$M_{mk} = \{(m_1, \dots, m_k)|\ m_i>0,\ \sum m_i = m\},$$
$$N_{mk} = \{(m_1, \dots, m_{k-1})|\ m_i\geqslant 0,\ \sum m_i\leqslant m-k\},$$
$$A_{mk}^q = \{(m_1, \dots, m_{k-1})|\ 0\leqslant m_i\leqslant q-2,\ \sum m_i\leqslant m-k\}.$$

THEOREM 8.4. *If* $|K| = q$ *then* $\dim \overline{\Gamma}^{mk} = |A_{mk}^q|$.

Proof. We can assume that $1<k<m$. Write

$$E(x, (m)) = x_1^{m_1} \dots x_k^{m_k} \quad \text{for} \quad x = (x_1, \dots, x_k) \in K^k,\ (m) = (m_1, \dots, m_k) \in N^k.$$

Then $\gamma_{(m)}(e_1, \dots, e_k)$, $(m) \in M_{mk}$ form a basis of Γ^{mk} and $\overline{\Gamma}^{mk}$ is generated by $\sum E(x, (m))\gamma_{(m)}(e_1, \dots, e_k)$, $x \in K^k$. Hence

$$\begin{aligned}\dim\overline{\Gamma}^{mk} &= \operatorname{rank}(E(x, (m))), \quad x \in K^k, \quad (m) \in M_{mk}\\ &= \operatorname{rank}(E(x, (m))), \quad x \in (K^*)^k, \quad (m) \in M_{mk}\\ &= \operatorname{rank}(E(x, (m))), \quad x \in (K^*)^{k-1}, \quad (m) \in N_{mk}\end{aligned}$$

where $K^* = K\setminus\{0\}$. Since $r^i = r^j$ for any $r \in K^*$ and $i \equiv j \pmod{q-1}$, we can assume that $0\leqslant m_i\leqslant q-2$. This means that

$$\dim\overline{\Gamma}^{mk} = \operatorname{rank}(E(x, (m))), \quad x \in (K^*)^{k-1}, (m) \in A_{mk}^q.$$

It suffices to prove that the columns of the above matrix are linearly independent. Observe that these columns are contained in any analogous matrix constructed for $m'>m$. Hence we can assume that $m\geqslant k+(k-1)(q-2)$. In this case, we have the quadratic matrix since $A_{mk}^q = \{(m_1, \dots, m_{k-1})|\ 0\leqslant m_i\leqslant q-2\}$. We must prove that its determinant is non-zero. This condition is equivalent to the following:

(*) If $F = \sum_{m_i\leqslant q-2} r_{m_1\dots m_{k-1}} T_1^{m_1} \dots T_{k-1}^{m_{k-1}} \in K[T_1, \dots, T_{k-1}]$ vanishes on $(K^*)^{k-1}$ then $F = 0$.

The above property can easily be proved by induction on k.

The above theorem and Corollary 8.2 show that

$$\begin{aligned}\dim\overline{\Gamma}^m(K^N) &= \sum_{k=0}^{N-1} \binom{N}{k+1} |A_{m,k+1}^q| = \sum_{k=0}^{N-1}\sum_{s=0}^{N-1} \binom{s}{k} |A_{m,k+1}^q|\\ &= \sum_{s=0}^{N-1}\sum_{k=0}^{s} \binom{s}{k} |\{(m_1, \dots, m_k)|\ 1\leqslant m_i\leqslant q-1,\ \sum m_i\leqslant m-1\}|\\ &= \sum_{s=0}^{N-1} |\{(m_1, \dots, m_s)|\ 0\leqslant m_i\leqslant q-1,\ \sum m_i\leqslant m-1\}| = |B_{mN}^q|,\end{aligned}$$

where $\binom{p}{q} = 0$ for $p<q$, and

$$B_{mN}^q = \{(m_1, \dots, m_s)|\ 0\leqslant m_i\leqslant q-1,\ \sum m_i\leqslant m-1,\ s = 0, 1, \dots, N-1\}.$$

COROLLARY 8.5. *If* $|K| = q$ *then* $\dim\overline{\Gamma}^m(K^N) = |B_{mN}^q|$. *In particular:*

(1) *If* $m-1\geqslant(N-1)(q-1)$ *then*

$$\dim\overline{\Gamma}^m(K^N) = 1+q+\dots+q^{N-1} = \frac{q^N-1}{q-1},$$

$$\dim\tilde{\Gamma}^m(K^N) = \binom{m+N-1}{N-1} - \frac{q^N-1}{q-1}.$$

(2) $$\dim\tilde{\Gamma}^m(K^2) = \max(0, m-q).$$

References

[1] M. Atiyah and I. MacDonald, *Introduction to Commutative Algebra*, Addison–Wesley 1969.

[2] N. Roby, *Lois polynômes et lois formelles en théorie des modules*, Ann. Éc. Norm. Sup. 80 (1963), pp. 213–348.

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