The span and the width of continua

by

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Abstract. The concept of the span is compared with Burgess’ notion of the width. Some basic properties of the span and the width are derived from an investigation of sets of distances associated with continuous functions.

Two metric properties of continua have been investigated in connection with some topological phenomena, and each of them can be expressed by means of a numerical quantity designed to measure the “non-arc-likeness” of a given continuum. The concept of the width was introduced by C. E. Burgess [1] in the theory of tree-like continua, and the author [7] defined the span of a metric space without any restrictions imposed upon its structure. The present paper resulted from an attempt to compare these two notions. Although, in general, the span and the width are not even for very primitive objects such as simple triods, some comparisons can be made and the similarities do exist (see Sections 3–5). At the end of the paper two results are obtained (see 5.3 and 5.4) concerning the set of distances between points that belong to point-inverses under continuous mappings.

1. Definitions and preliminaries. If $X$ is a non-empty metric space, we define the span $\sigma(X)$ of $X$ to be the least upper bound of the set of real numbers $a$ which satisfy the following condition: there exists a connected space $C$ and continuous mappings $f_1, f_2 : C \to X$ such that $f_2(C) = f_1(C)$ and $a < \text{dist}(f_1(c), f_2(c))$ for $c \in C$. Equivalently (see [7], p. 209), the span $\sigma(X)$ is the least upper bound of numbers $a$ for which there exist connected subsets $C_a$ of the product $X \times X$ such that $p_1(C_a) = p_2(C_a)$ and $a < \text{dist}(x, y)$ for $(x, y) \in C_a$, where $p_1$ and $p_2$ denote the projections of $X \times X$ onto $X$, i.e. $p_1(x, y) = x$ and $p_2(x, y) = y$ for $x, y \in X$. We note that, for compact spaces $X$, the sets $C_a$ in the latter definition of $\sigma(X)$ can be assumed to be closed in $X \times X$.

The definition of the width is a bit more involved in that it requires rather special notation. If $A, B \subseteq X$ are non-empty subsets of the metric space $X$, we denote by $\delta(A)$ and $\varrho(A, B)$ the diameter of $A$ and the distance between $A$ and $B$, respectively, i.e.

$$\delta(A) = \sup \{\text{dist}(a, a') : a, a' \in A\}, \quad \varrho(A, B) = \inf \{\text{dist}(a, b) : a \in A, b \in B\}.$$
If $C$ is a collection of non-empty subsets of $X$, we denote by $|C|$ the union of all the sets which belong to $C$, and we put

$$\text{mesh}(C) = \sup \{\delta(A) : A \in C\}.$$

Furthermore, we say $C$ is a chain provided $C$ is finite and the nerve of $C$, if non-degenerate, is an arc (see [5], p. 318). For any collection $C$ of non-empty subsets of $X$, Chain$(C)$ will denote the family of all the chains which are contained in $C$. For finite collections $C$ of non-empty subsets of $X$, we define real numbers $w(C)$ by the formula

$$w(C) = \min \{\max \{q(A, C') : A \in C\} : C' \in \mathrm{Chain}(C)\}.$$

We also consider finite open covers of $X$ by which we mean finite collections $C$ of non-empty open subsets of $X$ such that $|C| = X$. Now, if $X$ is a non-empty compact metric space, we define the width $w(X)$ of $X$ to be the least upper bound of the set of real numbers $a$ which satisfy the following condition: for each $\epsilon > 0$, there exists a finite open cover $C$ of $X$ such that mesh$(C) < \epsilon$ and $a \leq w(C)$. Thus $w(X)$ is a well-defined real number and $0 \leq w(X) < \delta(X)$.

Let $X$ be a non-empty continuum, i.e., connected compact metric space. We say $X$ is a tree provided $X$ is homeomorphic to a connected one-dimensional polyhedron which contains no simple closed curve. A continuum $X$ is called tree-like (or arc-like) provided, for each $\epsilon > 0$, there exists a finite open cover $C$ of $X$ such that mesh$(C) < \epsilon$ and the nerve of $C$, if non-degenerate, is a tree (or an arc, respectively). The concept of the width of a tree-like continuum was defined earlier in [1] using slightly different considerations. Before we see that, restricted to such a continuum, the two definitions coincide, we need to prove some auxiliary propositions. By a refinement of a finite open cover $C$ we mean, as usual, any finite open cover whose elements are contained in elements of $C$.

1.1. If $C$ is a finite open cover of a metric space and $D$ is a refinement of $C$ such that the nerve of $C$ is a tree and the nerve of $D$ is connected, then

$$w(D) \leq w(C) + 2 \text{mesh}(C).$$

Proof. Let $X$ denote the space and let $C' \in \text{Chain}(C)$. The proof of 1.1 will be complete if we show the existence of a set $A \in C$ such that

$$w(D) \leq w(A, |C'|) + 2 \text{mesh}(C). \quad (1)$$

The following two cases are to be distinguished:

Case 1. $C$ consists of at least 2 elements. Then $C$ has exactly 2 elements and let $C = (A, B)$. Thus $A \cap B \neq \emptyset$ and $A \cup B = |C| = X$; inequality (1) follows from

$$w(D) \leq \delta(A) + \delta(B) \leq \delta(A) + \delta(B) \leq 2 \text{mesh}(C).$$

Case 2. $C$ consists of more than 2 elements. Then the chain $C'$ is contained in a chain $C'' \subset C$ which has at least 3 elements, say $C'' = \{C_1, \ldots, C_m\}$, where $m \geq 3$ and $C_1 \cap C_j \neq \emptyset$ if and only if $|i-j| \leq 1$ for $i, j = 1, \ldots, m$. Thus $C_1 \cap C_m = \emptyset$ and the non-empty sets $C_1, C_m$ intersect some non-empty sets belonging to the refinement $D$ of $C$. Since the nerve of $D$ is connected, those elements of $D$ can be joined together by means of a chain $D' \subset D$, so that we can write $D' = \{D_1, \ldots, D_n\}$, where $C_1 \cap D_i \neq \emptyset = C_2 \cap D_i$ and $D_i \cap D_j \neq \emptyset$ if and only if $|i-j| \leq 1$ for $i, j = 1, \ldots, n$. As a result, we obtain

$$D_1 \neq C_1 \cap D_1 \neq C_1 \cap |D'|,$$

and we claim that $C_1 \cap |D'| \neq \emptyset$, too, for $i = 2, \ldots, m-1$. Indeed, since the nerve of $C$ is a tree, all the elements of $C \cap |C'|$ (i.e., $m-1$) can be grouped in two collections $C_1$ and $C_2$ such that $C_1 \in C_1, C_2 \in C_2$ and $|C_1| \cap |C_2| = \emptyset$. Supposing $C_1 \cap |D'| = \emptyset$, we would get

$$|D'| = |C_1| \cup |C_2|,$$

and since each element of $D'$ is contained in an element of $C$, the collection $D'$ would split into two collections composed of subsets of $|C_1|$ and $|C_2|$, respectively. One of these collections would contain $D_1$, and the other one would contain $D_2$. This, however, is impossible, $D'$ being a chain; therefore $C_1 \cap |D'| \neq \emptyset$ for $i = 1, \ldots, m$. It follows that, for each point $y \in |C'|$, there exists an integer $i = 1, \ldots, m$ and a point $y' \in |D'|$ such that $y, y' \in C_i$, and thus

$$\text{dist}(y, y') \leq \delta(C_i) \leq \text{mesh}(C).$$

On the other hand, $D' \in \text{Chain}(D)$ implies the existence of a set $B \in D$ satisfying the inequality

$$w(D) \leq w(B, |D'|),$$

and since $D$ is a refinement of $C$, there is a set $A \in C$ such that $B \supseteq A$. Let us select a point $x' \in B$ and observe that $|C' \cap |C'|| = C'$. We conclude that, given arbitrary points $x \in A$ and $y \in |C'|$, one has $\text{dist}(x, x') \leq \delta(A) \leq \text{mesh}(C)$ as well as

$$w(D) \leq w(B, |D'|) \leq \text{dist}(x', y') \leq \text{dist}(x', x) + \text{dist}(x, y) + \text{dist}(y', y') \leq \text{dist}(y, x') + 2 \text{mesh}(C),$$

whence (1) and the proof of 1.1 is completed.

1.2. If $X$ is a tree-like continuum and $C_1, C_2, \ldots$ is a sequence of finite open covers of $X$ such that the nerve of $C_n$ is a tree $(n = 1, 2, \ldots)$ and

$$\lim_{n \to \infty} \text{mesh}(C_n) = 0,$$

then $\lim_{n \to \infty} w(C_n) = w(X)$.

Proof. Let $\gamma > 0$ be a number arbitrarily chosen. If the inequality

$$\text{mesh}(C_n) < w(X) + \gamma$$

for all $n$, then $w(C_n) < w(X) + \gamma$. This proves the first part of the theorem. The other part follows from the inequality

$$w(X) + \gamma \geq w(C_n) + \gamma \geq w(X) + \gamma.$$

were violated by infinitely many positive integers \( n \), the number \( x = \lambda(X) + \gamma \) would have the property of possessing finite open covers \( C_n \) of \( X \) with small mesh \( \lambda(C_n) \), by (2), and such that \( \pi \leq \lambda(C) \). According to the definition of the width \( w(X) \), we would then have \( \pi \leq \lambda(X) \) which contradicts the assumption that \( \gamma > 0 \). Consequently, there exists a positive integer \( n_1 \) such that (3) holds for \( n \geq n_1 \).

By virtue of (2), there also exists a positive integer \( n_2 \) such that mesh \( \lambda(C_0) < \frac{\gamma}{\sqrt{3}} \) for \( n \geq n_2 \). Let \( n \geq n_1 + n_2 \) be fixed for a while. Since \( X \) is compact, there is a number \( \epsilon > 0 \) such that each finite open cover \( C \) of \( X \) with mesh \( \lambda(C) < \epsilon \) is a refinement of \( C_0 \).

But since \( w(X) \) is the width of \( X \), a cover \( C \) of this type can be found satisfying the inequality

\[
w(X) < \left(1 + \frac{1}{2} \right) \lambda(C),
\]

and the connectedness of \( X \) implies the connectedness of the nerve of \( C \). By 1.1, we obtain

\[
w(C) \leq w(C) + 3 \lambda(C) < w(C) + \frac{3 \gamma}{2},
\]

whence \( w(X) - \frac{3 \gamma}{2} < w(C) \). Combining the latter inequality with inequality (3), we see that

\[
\left| w(C) - w(X) \right| < \gamma
\]

provided \( n \geq n_1 \) and \( n \geq n_2 \). Thus 1.2 is proved.

Remark: It follows from 1.2 that, for tree-like continua, the width as defined in this paper is equal to that invented by Burgess [1]. On the other hand, the condition that the nerves of \( C_n \) are trees surely implies, by (2), that the continuum \( X \) from 1.2 is tree-like. It will be shown (see 3.7) that this condition cannot be omitted in 1.2.

2. The width of dendroids. Each arcwise connected tree-like continuum is said to be a dendroid. Clearly, all trees as well as all dendrites are dendroids. By a simple trick we mean a tree which is the union of three arcs having a common end-point and being pairwise disjoint except at that point. Since chains are related to arcs in a natural way, there should also exist a relationship between the concept of the width, as defined in Section 1, and another concept that expresses a similar idea by means of arcs instead of chains. To have such a relationship, however, one should assume the existence of sufficiently many arcs in the space, and the class of dendroids seems to be an appropriate range for this purpose. Before studying them, we prove two propositions of a more general character. For any metric space \( X \), \( \text{Arc}(X) \) will denote the collection of all the arcs which are contained in \( X \) and of all the non-empty degenerate subsets of \( X \). If \( a \) is a degenerate set and \( B \) is a non-empty set, we denote the distance between them by \( \lambda(a, B) \) rather than by \( \lambda([a], B) \).

2.1. If \( X \) is a metric space, \( a \in \text{Arc}(X) \) and \( \gamma > 0 \), then there exists a number \( \epsilon > 0 \) such that each finite open cover \( C \) of \( X \) with mesh \( \lambda(C) < \epsilon \) has a chain \( C' \in \text{Chain}(C) \) satisfying the condition

\[
\sup \{ \lambda([a], [C']): a \in A \} < \gamma.
\]
According to the definition of \( w(C_0) \), an element \( C_0 \in C_0 \) must exist such that
\[
w(C_0) \leq q(G_0, |C_0|) + \gamma,
\]
whence \( w(X) = w(C_0) + \gamma \leq q(G_0, |C_0|) + \gamma \) and, taking a point \( x_0 \in G_0 \), we get
\[
w(X) = q(G_0, |C_0|) + \gamma \leq q(x_0, |C_0|) + \gamma.
\]
Since to each point \( a \in A_0 \) there corresponds a point \( y \in [C_0] \) fulfilling (5), we obtain
\[
q(x_0, |C_0|) \leq \text{dist}(x_0, y) \leq \text{dist}(x_0, a) + \text{dist}(a, y) \leq \text{dist}(x_0, a) + 2\gamma,
\]
which implies that \( q(x_0, |C_0|) \leq q(x_0, A_0) + 2\gamma \), whence
\[
w(X) = q(x_0, |C_0|) + \gamma \leq q(x_0, A_0) + 3\gamma.
\]
By (4), the width of \( X \) thus satisfies the inequality \( w(X) \leq r + 4\gamma \), which completes the proof of 2.2.

2.3. Theorem. If \( X \) is a dendroid, then
\[
w(X) = \inf \{ \sup \{ q(x, A) : x \in X \} : A \in \text{Arc}(X) \}.
\]

Proof. Keeping a notation of the preceding proof, let \( r \) be the number which is the right-hand side of this inequality and let \( \gamma > 0 \) be an arbitrary number. If the inequality
\[
w(C) = w(X) + \gamma
\]
did not hold for some finite open covers \( C \) of \( X \) with \( w(C) \) as small as one wants, the number \( a = w(X) + \gamma \) would satisfy the condition from the definition of the width \( w(X) \) and, consequently, we would have \( a = w(X) \) which is not the case as \( \gamma > 0 \). Therefore there exists a number \( \varepsilon_0 = 0 \) such that (6) holds if \( \text{mesh}(C) < \varepsilon_0 \).

The dendroid \( X \) being a tree-like continuum, there exists a finite open cover \( C_0 \) of \( X \) such that
\[
\text{mesh}(C_0) < \varepsilon_0,
\]
and the nerve of \( C_0 \) is either degenerate or a tree. Then, by (6), we have \( w(C_0) = w(X) + \gamma \). It follows from the definition of \( w(C_0) \) that there exists a chain \( C_0 \in \text{Chain}(C_0) \) such that
\[
\text{Max} \{ q(G, |C_0|) : G \in C_0 \} = w(C_0) = w(X) + \gamma,
\]
and we claim that an \( A_0 \in \text{Arc}(X) \) can be picked up so that
\[
\sup \{ q(x, A_0) : x \in |C_0| \} \leq \gamma.
\]

Indeed, if \( C_0 \) is a degenerate chain \( \{ C \} \), it is enough to put \( A_0 = \{ a_0 \} \), where \( a_0 \in C \) and then \( q(x, A_0) = \text{dist}(x, a_0) \leq \text{dist}(x, C) \leq \text{mesh}(C_0) < \gamma \) for each \( x \in |C_0| \).

If \( C_0 \) is non-degenerate, we have \( C_0 = \{ C_1, ..., C_k \} \), where \( k \geq 2 \) and \( C_i \cap C_j = \emptyset \) if and only if \( |i - j| \leq 1 \) for \( i, j = 1, ..., k \). Let us select two distinct points \( p \in C_1 \) and \( q \in C_k \). Since \( X \) is arcwise connected, there exists an arc \( A_0 = X \) joining \( p \) and \( q \).

Thus \( C_i \cap A_0 \neq \emptyset \) and \( C_j \cap A_0 \neq \emptyset \). We show that \( C_i \cap A_0 \neq \emptyset \) for \( i = 1, ..., k \).

If it were not so, \( k \geq 3 \) and there would be an integer \( b_i = 2, ..., k-1 \) such that \( A_0 \cap (C_i 
ion C_0) = \emptyset \). But the nerve of \( C_0 \) being a tree, all the elements of \( C_i \setminus C_0 \) would group in two collections \( C_1 \) and \( C_2 \) such that \( C_1 \in C_1, C_2 \in C_2 \) and \( |C_1| \setminus |C_2| = \emptyset \).

The arc \( A_0 \) would then be contained in the union of two disjoint open sets \( |C_1| \) and \( |C_2| \) each of them intersecting \( A_0 \), which contradicts the connectedness of \( A_0 \).

Hence \( A_0 \) meets all the sets \( C_i, ..., C_k \). Given a point \( x \in |C_0| \), there is an integer \( j = 1, ..., k \) such that \( x \in C_j \), and (6) follows from the inequalities
\[
q(x, A_0) \leq q(x, C_j, A_0) \leq \text{dist}(C_j) \leq \text{mesh}(C_0) < \gamma.
\]

By the definition of the number \( r \), there exists a point \( x_0 \in X \) such that
\[
r - \gamma < q(x_0, A_0) \leq q(x_0, C_j, A_0) \leq \text{dist}(C_j) < \text{mesh}(C_0) < \gamma.
\]

This implies that \( r - 4\gamma < w(X) + \gamma \) and, by (7), \( r - 5\gamma < w(X) \) and, by 2.2, the proof of 2.3 is complete.

2.4. If \( X = A_0 \cup A_1 \cup A_2 \) is a simple triod, where \( A_0 \) are arcs having a common end-point \( \sigma \) and \( A_1 \cap A_1 = \{ \sigma \} \) for \( i = 0, 1, 2 \) and the subscripts of \( A_1 \) taken mod 3, then
\[
w(X) = \min \{ \sup \{ q(x, A_1 +, A_1) : x \in A_1 \} : i = 0, 1, 2 \}.
\]

Proof. Let \( m \) stand for this Min Max. It follows from (8) that \( w(X) = m \).

If \( A \in \text{Arc}(X) \), there is a subscript \( i = 0, 1, 2 \) such that \( A \in \text{Ar}(X) \), and
\[
q(x, A_1 +, A_1 +) \leq q(x, A)
\]
for each point \( x \in X \). Consequently, we get the inequalities
\[
m \leq \max \{ q(x, A_1 +, A_1 +) : x \in A_1 \} \leq \sup \{ q(x, A) : x \in X \},
\]
and 2.3 implies \( m \leq w(X) \). Thus \( w(X) = m \).

3. Some continuity properties. The span and the width, when treated as real-valued functions defined on collections of tree-like continua, seem to behave alike as far as their continuity is concerned. If \( X \) is a non-empty subset of a metric space \( Z \) and \( \varepsilon > 0 \), we call the set \( \{ x \in Z : q(x, X) < \varepsilon \} \) the \( \varepsilon \)-neighbourhood of \( X \) in \( Z \).

3.1. Let \( X \) be a non-empty compact set contained in a metric space \( Z \). If \( \beta \) is a real number and, for \( n = 1, 2, ..., \exists \) a subset \( Z_n \) of the \( (1/n) \)-neighbourhood of \( X \) in \( Z \) with \( \beta < \varepsilon(Z_n) \), then \( \beta < \varepsilon(X) \) (compare [7], p. 211).
Proof. Denote by $p_1, p_2: Z \times Z \to Z$ the standard projections of the product $Z \times Z$ onto $Z$. Let $\gamma > 0$ be an arbitrary real number. Since $\beta - \gamma \leq \sigma(Z)$, it follows from the definition of the span that there exists a non-empty connected set $C_n \subseteq Z \times Z$ such that $p_1(C_n) = p_2(C_n)$ and $\beta - \gamma \leq \text{dist}(x, y)$ for $(x, y) \in C_n$ ($n = 1, 2, \ldots$). Let us select points $c_n \in C_n$ and observe that the set $Z \times C_n$, hence also the set $c_n$, is contained in the $(\sqrt{2/n})$-neighbourhood of $X \times X$ in $Z \times Z$. Consequently, there exist points $z_n \in X \times X$ such that $\text{dist}(z_n, c_n) < \sqrt{2/n}$ for $n = 1, 2, \ldots$. By the compactness of $X \times X$, a subsequence $z_{n_k}, z_{n_k}, \ldots$ ($n_k < n_{k+1}$) converges to a point $z \in X \times X$, whence also

$$\lim_{k \to \infty} c_{n_k} = z.$$ \hspace{1cm} (9)

We define $D = \bigcup_{n=1}^{\infty} C_n$ (see [5], p. 337). Thus each point of $D$ is the limit of a sequence of points of $C_{n_1} \cup C_{n_2} \cup \ldots$. It follows that $\beta - \gamma \leq \text{dist}(x, y)$ for $(x, y) \in D$. Moreover, given a point $d \in D$, there exist points $u_d \in C_{n_d}$ (where $l_d < l_{d+1}$) such that the sequence $u_{n_d}, u_{n_{d+1}}, \ldots$ converges to $d$. The point $u_d$ belongs to the $(\sqrt{2/n_d})$-neighbourhood of $X \times X$, whence $d \in X \times X$, so that $D \subseteq X \times X$. On the other hand, since $p_1(C_n) = p_2(C_n)$, there are points $c'_n \in C_{n_d}$ such that $p_1(u_d) = p_2(c'_n)$. We conclude, as we did before for the points $c_n$, that a subsequence $c'_{n_k}, c'_{n_k}, \ldots$ ($l_d < l_{d+1}$) converges to a point $z' \in X \times X$. By the definition of $D$, we have $z' \in D$, and

$$p_1(D) = \lim_{k \to \infty} p_1(c_{n_k}) = \lim_{k \to \infty} p_2(c'_{n_k}) = p_2(z'),$$

which implies that $p_1(D) \subseteq p_2(D)$. A symmetric argument shows that $p_2(D) \subseteq p_1(D)$, whence $p_1(D) = p_2(D)$.

Now, we claim that the set $D$ is connected. Suppose it is not, and notice that $z \in D$, by (9). Then there exist two disjoint open sets $U$ and $V$ of $Z \times Z$ such that $D \subseteq U \cup V$, $z \in U$ and $D \cap V = \emptyset$. Let $d \in D \cap V$ be a point. As we have seen, there are points $u_d \in C_{n_d}$ where $l_d < l_{d+1}$ which converge to $d'$, so that $u_d \in V$ for $m > n$, where $m$ is a positive integer. By (9), $c_{n_d} \in U$ for $m > n_d$. The connected set $C_{n_d}$ intersects both $U$ and $V$ provided $m > n_d$ and $m > n_d$. Therefore, almost all of the sets $C_{n_d}$ contain points $c'_{n_d} \notin U \cup V$ which, again, must have a subsequence converging to a point $z'' \in D$. This implies that $z'' \notin U \cup V$, contrary to the inclusion $D \subseteq U \cup V$. Hence $D$ is a connected set.

Setting $\alpha = \beta - \gamma$ and $C_\alpha = D$ in the definition of $\sigma(X)$, we obtain $\beta - \gamma \leq \alpha(X)$. Since $\gamma$ was an arbitrarily chosen positive number, the inequality $\beta \leq \alpha(X)$ follows and the proof of 3.1 is completed.

If $X$ is a non-empty subset of a metric space $Z$ and $\varepsilon > 0$, we say that a continuous mapping $f: X \to Z$ is an $\varepsilon$-translation provided $\text{dist}(x, f(x)) < \varepsilon$ for $x \in X$. Thus if $f: X \to Z$ is an $\varepsilon$-translation, then the set $f(X)$ is contained in the $\varepsilon$-neighbourhood of $X$ in $Z$. It follows directly from the triangle inequality for distances that if $f: X \to Z$ is an $\varepsilon$-translation, then

$$\text{dist}(x, y) - \text{dist}(f(x), f(y)) < 2\varepsilon \quad (x, y \in X).$$ \hspace{1cm} (10)

3.2. Let $X$ be a non-empty set contained in a metric space $Z$. If $\beta$ is a real number and, for $n = 1, 2, \ldots$, there exists a $(1/n)$-translation $f_n: X \to Z$ such that $\sigma(f_n(X)) \leq \beta$, then $\sigma(X) \leq \beta$.

Proof. Assume $x$ is a real number and $C \subseteq X \times X$ is a connected set such that $p_1(C) = p_2(C)$ and $x \leq \text{dist}(x, y)$ for $(x, y) \in C$, where $p_1$ and $p_2$ are the standard projections of $Z \times Z$ onto $Z$. The set

$$C_\alpha = \{(f_n(x), f_n(y)) : (x, y) \in C\}$$

is connected and $C_\alpha \subseteq f_n(X) \times f_n(X)$. Moreover,

$$p_1(C_\alpha) = f_n[p_1(C)] = f_n[p_2(C)] = p_2(C_\alpha)$$

and it follows from (10) that, for any point $(f_n(x), f_n(y))$ of $C_\alpha$, we have

$$x - 2/n \leq \text{dist}(x, y) - 2/n \leq \text{dist}(f_n(x), f_n(y)),$$

whence $x - 2/n \leq \sigma(f_n(X))$. Consequently, the inequality $x - 2/n \leq \beta$ holds for $n = 1, 2, \ldots$. This means that $x \leq \beta$ and, as a result, the least upper bound $\sigma(X)$ of such numbers $\alpha$ also satisfies the inequality $\sigma(X) \leq \beta$.

3.3. Lemma. Let $X$ be a non-empty subset of a bounded metric space $Z$ and let $f: X \to Z$ be an $\varepsilon$-translation. If $C$ is a finite open cover of $f(X)$ and $C' = \{f^{-1}(G) : G \in C\}$, then

$$\text{mesh}(C') - \text{mesh}(C) < 2\varepsilon, \quad |w(C) - w(C')| < 2\varepsilon.$$

Proof. By (10), for any pair of non-empty sets $A, B \in f(X)$, we get the inequalities

$$|\delta(A) - \delta(f^{-1}(A))| < 2\varepsilon, \quad |e(A, B) - e(f^{-1}(A), f^{-1}(B))| < 2\varepsilon,$$

which imply the two inequalities required in 3.3, respectively. It suffices to notice that some sets $G_1, \ldots, G_n$ of $C$ form a chain if and only if the sets $f^{-1}(G_1), \ldots, f^{-1}(G_n)$ form a chain $(C')$ and

$$f^{-1}(G_1) \cup \ldots \cup G_n = f^{-1}(G_1) \cup \ldots \cup f^{-1}(G_n).$$

3.4. Let $X$ be a non-empty compact set contained in a metric space $Z$. If $\beta$ is a real number and, for $n = 1, 2, \ldots$, there exists a $(1/n)$-translation $f_n: X \to Z$ such that $\beta \leq w(f_n(X))$, then $\beta \leq w(X)$.

Proof. Since $\beta - 1/n \leq w(f_n(X))$ ($n = 1, 2, \ldots$), it follows from the definition of the width that there exists a finite open cover $C_n$ of $f_n(X)$ such that

$$\text{mesh}(C_n) < 1/n, \quad \beta - 1/n \leq w(C_n).$$

We take the finite open cover $C_n = \{f_n^{-1}(G) : G \in C_n\}$ of $X$ ($n = 1, 2, \ldots$) which, according to 3.3, fulfills the conditions

$$\text{mesh}(C_n) < \text{mesh}(C_n) + 2/n < 3/n$$ \hspace{1cm} (11)
and $\beta - 3/n \leq w(C_n) - 2/n \leq w(C'_n)$. Setting $\alpha = \beta - 3/n$ and $C = C_n$ ($n \geq 3$) in the definition of $w(X)$, we obtain $\beta - 3/n \leq w(X)$, by (11). The inequality $\beta \leq w(X)$ now follows.

3.5. Let $X$ be a tree-like continuum contained in a metric space $Z$. If $\beta$ is a real number and, for $n = 1, 2, \ldots$, there exists a $(1/n)$-translation $f_n: X \to Z$ such that $Z_n = f_n(X)$ is a tree-like continuum with $w(Z_n) \leq \beta$, then $w(X) \leq \beta$.

Proof. We can assume $X$ is non-degenerate; otherwise $w(X) = 0$. Thus all but a finite number of the continua $Z_n$ also are non-degenerate and, without loss of generality, we can as well assume that each $Z_n$ is a non-degenerate tree-like continuum ($n = 1, 2, \ldots$). According to 1.4, there exists a finite open cover $C_n$ of $Z_n$ ($n = 1, 2, \ldots$) such that the nerve of $C_n$ is a tree and

$$\text{mesh}(C_n) < 1/n, \quad |w(C_n) - w(Z_n)| < 1/n.$$  

The finite open cover $C_n$ of $X$ from the proof of 3.4 fulfills condition (11), by 3.3. Furthermore, for $n = 1, 2, \ldots$, we have

$$w(C'_n) \leq w(C_n) + 2/n < w(Z_n) + 3/n \leq \beta + 3/n,$$

and the nerve of $C'_n$ is obviously the same as the nerve of $C_n$; thus it is a tree. Since (11) implies (2) with $C_n$ replaced by $C'_n$, we can apply 1.2 again to conclude from (12) that

$$w(X) = \lim_{n \to \infty} w(C'_n) \leq w(C_n) + (\beta + 3/n) = \beta.$$  

Remark. Observe that the tree-likeness of $Z_n$ in 3.5 implies, by 3.3, the tree-likeness of the continuum $X$ itself. However, even when the continuum $X$ is assumed to be tree-like, the conclusion of 3.5 is no longer true when the tree-likeness of the continua $Z_n$ is dropped.

We provide an example to explain this possibility (see 3.7).

3.6. Corollary. If $X$ is a tree-like continuum contained in a metric space and, for $n = 1, 2, \ldots$, a tree-like continuum $X_n$ is the image of $X$ under an $(1/n)$-translation such that

$$\lim_{n \to \infty} w_n = 0,$$

then $\lim_{n \to \infty} \sigma(X_n) = \sigma(X) \text{ and } \lim_{n \to \infty} w(X_n) = w(X).$

3.7. Example. There exists a simple triad $T$ on the plane $R^2$, $(1/n)$-translations $f_n: T \to R^2$ and finite open covers $C_n$ of $T$ ($n = 1, 2, \ldots$) such that

(i) $w(T) = 1$,  
(ii) $w[f_n(T)] = 0$ for $n = 1, 2, \ldots$,  
(iii) $\lim_{n \to \infty} \text{mesh}(C_n) = \lim_{n \to \infty} w(C_n) = 0$.

Proof. Let $S_0$, $S_1$, and $S_2$ be the straight-line segments joining the origin with the points $(0, 1)$, $(1, 0)$, and $(0, -1)$, respectively. Then $T = S_0 \cup S_1 \cup S_2$ is a simple triad and (i) follows from 2.4. Let $f_n: T \to R^2$ be a $(1/n)$-translation such that $f_n(T)$ is topologically a disk ($n = 1, 2, \ldots$). We get (ii), by 2.2, which means that if $D$ is

a finite open cover of the disk $f_n(T)$ with mesh $D$ sufficiently small (depending on $n$), then $w(D)$ is small, too. Thus there exists, for $n = 1, 2, \ldots$, a finite open cover $D_n$ of $f_n(T)$ such that

$$\text{mesh}(D_n) < 1/n, \quad w(D_n) < 1/n.$$  

We define $C_n = \{f_n^{-1}(G): G \in D_n\}$ ($n = 1, 2, \ldots$). It follows from 3.3 that $C_n$ satisfies condition (iii).

4. Simple triods and trees. The results of this section indicate that the problem of finding a relationship between the span and the width of tree-like continua reduces partially to another problem, rather combinatorial in its nature, namely, that of connecting the width of a tree with the widths of simple triods contained in it.

4.1. If $T$ is a simple triod, then $w(T) \leq \sigma(T)$.

Proof. By 2.4, this is a consequence of a lemma proved in [8].

Remark. The inequality in 4.1 cannot be replaced by the equality (see 4.5). Also, the assumption that $T$ is a simple triod is essential in 4.1. To support the latter statement, we study, in 4.2 below, some properties of an example of a tree which has been constructed in [9].

4.2. Example. There exists a tree $T$ in the 3-space $R^3$ such that

(i) $w(T) = 1$,  
(ii) $\sigma(T) = \frac{1}{2}$, and  
(iii) $T = A_0 \cup A_1 \cup A_2 \cup A_3$, where $A_i$ are arcs having a common end-point $v$ and $A_i \cap A_j = \{v\}$ for $i \neq j$ ($i, j = 0, 1, 2, 3$); thus $T$ is a simple "4-od".

Proof. The space $R^3$ with the ordinary Pythagorean distance will be used. Given two points $p, q \in R^3$, we denote by $pq$ the straight-line segment having $p$ and $q$ as the end-points. For $i = 1, 2, 3$, we take the points

$$p_i = (\frac{1}{2} \cos \frac{i}{2} \pi, \frac{1}{2} \sin \frac{i}{2} \pi, 0), \quad q_i = (\frac{1}{2} \cos \frac{i}{2} \pi + 1, \frac{1}{2} \sin \frac{i}{2} \pi + 1, 0),$$

and let $q_0 = (0, 0, 1)$ and $v = (0, 0, 0)$. We define $A_0 = q_0v$ and $A_i = p_iq_i \cup \bar{p}_iq_i$ ($i = 1, 2, 3$). Then the union $T$ of these arcs satisfies condition (iii). We also denote

$$B = A_1 \cup A_2 \cup A_3,$$

so that $B$ is a simple triod and $T = A_0 \cup B$. Clearly, $\text{dist}(x, v) \leq 1$ for $x \in B$. Since $v \in A_0$, we get

$$\text{Sup}(q(x, A_0): x \in T) \leq 1,$$

which implies that $w(T) \leq 1$, by 2.2. Now, let $A \in \text{Arc}(T)$. If $A \subset B$, we have

$$1 = \text{dist}(q_0, v) = \text{dist}(q_0, B) \leq \text{dist}(q_0, A),$$

whence

$$\text{Sup}(q(x, A): x \in T) \leq 1.$$
If \( A \neq B \), there is a subscripts \( j = 1, 2, 3 \) such that \( A \subset A_1 \cup A_j \). The set \( A_0 \cup A_j \) is contained in the cylindrical section

\[
S = \{(r \cos \theta, r \sin \theta, t) : 0 \leq r \leq 1, \frac{\pi}{2} \leq \theta \leq \pi, 0 \leq t \leq 1 \}
\]

and one of the points \( q_k \), namely the point

\[
g = (\cos \frac{3}{4} \pi (j+2), \sin \frac{3}{4} \pi (j+2), 0)
\]

has the distance \( g(S, g) = 1 \). Since \( A \subset A_0 \cup A_j \subset S \), we obtain \( g(S, g) \leq g(A, A) \), whence (13) holds again. By 2.3, this implies the inequality \( 1 \leq w(T) \), and so (i) is proved.

The set \( T^* = \overline{q_1 q_2} \cup \overline{q_2 q_3} \) is a simple triad contained in \( T \), and \( w(T^*) = \frac{1}{2} \), by 2.4. It follows from 4.1 that \( \frac{1}{2} \leq w(T^*) \leq w(T) \), so it suffices to prove the proof of (ii), that we have to show that \( w(T) \leq \frac{1}{2} \). Suppose, on the contrary, that \( w(T) > \frac{1}{2} \).

Then there exists a number \( a_0 > \frac{1}{2} \), a continuum \( C \) and two continuous mappings \( f_1, f_2 : C \to T \) such that \( f_1(C) \subset f_2(C) \) and \( a_0 \leq \text{dist}(f_1(c), f_2(c)) \) for \( c \in C \). The vertex \( v \) is a cut-point of \( T \). If \( v \notin f_1 \), the continuum \( f_1(C) \) would be contained in one of the sets \( A \) (\( i = 0, 1, 2, 3 \)), which is impossible since the span of any arc is zero. Thus \( v \notin f_1 \). We denote

\[
V_1 = f_1^{-1}(v), \quad X_1 = f_1^{-1}(A_0), \quad Y_1 = f_1^{-1}(B) \quad (i = 1, 2).
\]

Let \( \leq \) be the ordering of the arc \( A \) from \( v \) to \( q_v \), that is, \( (0, 0, 0) \leq (0, 0, r) \) if and only if \( r \leq r' \). The sets

\[
P_1 = \{ c \in X_1 \cap X_2 : f_2(c) \notin f_1(c) \}, \quad P_2 = \{ c \in X_1 \cap X_2 : f_2(c) \notin f_1(c) \}
\]

are compact subsets of \( C \) whose union is \( Q_1 \) and \( Q_2 \). They are also disjoint since \( f_1(c) \neq f_2(c) \) for \( c \in C \). Hence

\[
X_1 \cap X_2 = P_1 \cup P_2, \quad P_1 \cap P_2 = \emptyset, \quad V_1 \cap X_1 \cap X_2 = P_1 \quad (i = 1, 2).
\]

We claim that a decomposition similar to (15) also exists for the set \( Y_1 \) \& \( Y_2 \). More precisely, we are going to prove that there exist compact sets \( Q_1 \) and \( Q_2 \) satisfying the conditions

\[
Y_1 \cap Y_2 = Q_1 \cup Q_2, \quad Q_1 \cap Q_2 = \emptyset, \quad V_1 \cap Y_1 \cap Y_2 = Q_1 \quad (i = 1, 2),
\]

or, in other words, that the set \( Y_1 \cap Y_2 \) is connected between \( V_1 \cap Y_1 \cap Y_2 \) and \( V_2 \cap Y_1 \cap Y_2 \). This will be achieved when we show that the set \( Y_1 \cap Y_2 \) is not connected between any two points belonging to these sets (see [6], p. 168).

Let \( q_i \in V_1 \cap Y_1 \cap Y_2 \quad (i = 1, 2) \) be points arbitrarily selected. By (14), we have \( f_i(q_i) = v = f_i(q_2) \), whence \( f_2(q_1) \neq v = f_1(q_1) \). Also by (14), the points \( f_1(q_1) \) and \( f_2(q_2) \) are in \( B \), so that each of them is in one of the sets \( A \) (\( i = 1, 2, 3 \)), and there exists a subscript \( k = 1, 2, 3 \) such that the arc \( A_k \) contains neither \( f_1(q_1) \) nor \( f_2(q_2) \). Let \( i照亮 \) be the last remaining from the \( 1, 2, 3 \) arranged so that the arcs \( A_i \) and \( A_m \) are obtained from \( A_k \) by the counter-clockwise rotation through the angles \( \frac{3}{4} \pi \) and \( \frac{3}{4} \pi \), respectively. In other words, \( (k, l, m) \) is either \((1, 2, 3)\) or \((2, 3, 1)\) or \((3, 1, 2)\). We consider a retraction \( g \) : \( B \to A_0 \cup A_0 \), which is defined in the following way. The mapping \( g \) maps \( A_0 \) into \( A_0 \), \( g(p) = v \), \( g(q_0) = p_0 \), and \( g \) is linear on both segments \( \overline{p_0 q_0} \) and \( \overline{p_0 q} \) whose union is \( A_0 \). Consequently, we have

\[
g^{-1}(x) = \begin{cases} y & \text{for } x \in (A_0 \cup A_0) \\ \overline{p_0 q_0} & \text{for } x = v, \overline{p_0 q} & \text{for } x \in \overline{p_0 q_0} \setminus \{v\} \end{cases}
\]

where \( y(x) \) is the point of the segment \( \overline{p_0 q_0} \) such that

\[
\overline{p_0 y(x)} = \overline{p_0 v}, \overline{p_0 y(x)} = \overline{p_0 v},
\]

Next, we need to show that

\[
\delta(x, y(x)) \leq \frac{1}{2} \quad (x \in A_0 \cup A_0).
\]

Since \( \delta(p, v) = \delta(q_1, v) = \frac{1}{2} \), inequality (19) holds for \( x \in (A_0 \cup A_0) \). Notice that \( \delta(x, y(x)) \leq \frac{1}{2} \) for such \( x \in \overline{p_0 q_0} \setminus \{v\} \). To this end, let us denote \( \lambda = \text{dist}(x, v) \), and observe that \( \delta(p_0, q_1) = \frac{1}{4} \) and \( \delta(v, p_0) = \frac{1}{4} \). Hence \( 0 \leq \lambda \leq \frac{1}{4} \) and

\[
\delta(p_0, y(x)) = \delta(p_0, v) = \frac{1}{2}
\]

by (18). Also, it follows from the definition of \( l \) and \( m \) that \( p_0 \) is the mid-point of the segment \( \overline{p_0 q_0} \). Thus the point \( x \) belongs to \( \overline{p_0 q_0} \) and

\[
\delta(q_1, y(x)) = \delta(q_1, v) = \text{dist}(v, x) = 1 - \lambda
\]

On the other hand, we get

\[
\delta(q_1, y(x)) = \delta(q_1, v) - \delta(p_0, y(x)) = \frac{1}{2} \]

and \( \cos \theta = \frac{1}{2} \), where \( \theta \) is the angle between the segments \( p_0q_0 \) and \( q_0 + \overline{p_0 q} \). As a result, we obtain

\[
\text{dist}(x, y(x)) = (1 - \lambda)^2 + 7(1 - \lambda)^2 - 2 - \frac{1}{8}(1 - \lambda) \leq \frac{1}{4}
\]

whence \( \text{dist}(x, y(x)) \leq \frac{1}{4} \), and, by (17), the proof of (19) is complete.

Let \( \leq \) be the ordering of the arc \( A_0 \cup A_0 \) from \( q_0 \) to \( q_0 \), the end-points of this arc. The sets

\[
Q_1 = \{ c \in Y_1 \cap Y_2 : f_1(c) \cap f_2(c) \}, \quad Q_2 = \{ c \in Y_1 \cap Y_2 : f_1(c) \cap f_2(c) \}
\]

are compact subsets of \( C \) whose union is \( Y_1 \cap Y_2 \), by (14). Since \( \frac{1}{4} \leq \delta \leq \frac{3}{4} \pi \), it follows from (19) that \( \delta(f_1(c)) \neq \delta(f_2(c)) \) for
\[g(f(d_1)) = g(v) = v \quad (i = 1, 2), \]
and because \(g(f(d_i)) = g(v)\), we conclude that
\[g(f(d_1)) = v \iff g(f(d_2)), \quad g(f(d_2)) = v \iff g(f(d_1)),\]
which means that \(d_i \in Q_i\) (\(i = 1, 2\)). The decomposition of \(Y_1 \cap Y_2\) into \(Q_i\)'s and \(Q_2\) then establishes the non-connectedness of \(Y_1 \cap Y_2\) between \(d_1\) and \(d_2\). Consequently, we have also proved the existence of compact sets \(Q_1\) and \(Q_2\) which satisfy (16).

We now distinguish two cases to prove that
\[(20) \quad X_1 \cap Y_2 \neq \emptyset \neq X_2 \cap Y_1.\]

Let \(u = (0, 0, \ell)\).

Case 1. \(w \in f_1(C)\). Since \(f_1(C) = f_2(C)\), there exist points \(c_i \in C\) such that \(f_1(c_i) = u (i = 1, 2)\). Let \(u \in A_0\), so that \(c_i \in X_{i} (i = 1, 2)\), by (14). Each point of \(T\) whose distance from \(u\) exceeds \(\ell/2\) belongs to \(B\). It follows from the inequalities
\[\frac{1}{2} < c_i \leq \dist (f_1(c_i), f_1(c_2)) \quad (i = 1, 2),\]
that \(f_2(c_1), f_2(c_2) \in B\), whence \(c_1 \in Y_2\) and \(c_2 \in Y_1\), by (14). We get \(c_1 \in X_1 \cap Y_2\) and \(c_2 \in X_2 \cap Y_1\).

Case 2. \(w \notin f_1(C)\). We know that \(v \in f_1(C)\). Let \(v\) be the last point of the segment \(A_0\) which belongs to \(f_1(C)\), in the ordering \(e_0\). The set \(A_0 \cap f_1(C)\) is connected, \(f_1(C)\) being a continuum. Thus, in this case, we have \(w \in C\). Consequently, by the definition of \(w\), each point of \(f_1(C)\) whose distance from \(w\) exceeds \(\ell/2\) must belong to \(B\). Since \(f_2(C) = f_2(C)\), there exist points \(c_i \in C\) such that \(f_1(c_i) = w\) \((i = 1, 2)\). As in Case 1, we get \(c_1 \in X_1 \cap Y_2\) and \(c_2 \in X_2 \cap Y_1\), which completes the proof of (20).

The sets
\[M = P_1 \cup Q_1 \cup (X_1 \cap Y_2), \quad N = P_2 \cup Q_2 \cup (X_2 \cap Y_1)\]
are compact and non-empty, by (20). Also, we have

\[
C = C \cap C = f_1^{-1}(T) \cup f_2^{-1}(T) = f_1^{-1}(A_0 \cup B) \cup f_2^{-1}(A_0 \cup B) \\
= (X_1 \cap Y_1) \cap (X_2 \cap Y_2) = (X_1 \cap X_2) \cup (X_2 \cap Y_1) \cup (X_1 \cap Y_2) \cup (Y_1 \cap Y_2) \\
= P_1 \cup P_2 \cup (X_1 \cap Y_2) \cup (X_2 \cap Y_1) \cup Q_1 \cup Q_2 = M \cup N,
\]
by (14), (15) and (16). Since \(A_0 \cap B = \{0\}\), it follows from (14) that \(X_i \cap Y_i = Y_i\) \((i = 1, 2)\). Hence

\[
M \cap N = (P_1 \cap Q_1) \cup (P_1 \cap X_1 \cap Y_2) \cup (Q_2 \cap P_2) \cup (Q_2 \cap X_1 \cap Y_2) \cup \\
(P_2 \cap X_2 \cap Y_1) \cup (Q_1 \cap X_2 \cap Y_1) \cup (X_1 \cap X_2 \cap Y_1 \cap Y_2) \\
\in (P_1 \cap X_2 \cap Y_2) \cup (P_1 \cap X_2 \cap Y_2) \cup (Q_2 \cap Y_1 \cap X_2) \cup (Q_2 \cap X_1 \cap Y_2) \cup \\
(P_2 \cap Y_1 \cap X_2) \cup (Q_1 \cap Y_2 \cap X_2) \cup (V_1 \cap V_2) \\
= (P_1 \cap V_2) \cup (Q_2 \cap V_1) \cup (P_2 \cap V_1) \cup (Q_1 \cap V_2) \cup (V_1 \cap V_2) = \emptyset,
\]
by (15) and (16). This contradicts the assumption that \(C\) is a continuum. Condition (ii) is then proved, and so is 4.2.

4.3. Let \(T\) be a tree which is not an arc. Let \(v_1, \ldots, v_k\) be all the branch-points of \(T\), and let \(n_i\) denote the ramification order of \(T\) at \(v_i\), i.e. \(n_i\) is the number of components of \(T \setminus \{v_i\}\) \((i = 1, \ldots, k)\). Denote
\[m(T) = \left[\sum_{i=1}^{k} n_i \right] - (k + 1).\]

Then there exists a simple triod \(T' \subset T\) such that \(m(T') \leq m(T)\) (see [9], p. 8).

Let us define separately \(m(A) = 1\) for each arc \(A\). Observe that if \(T' \subset T\) are trees, then \(m(T') \leq m(T)\). The next result follows from 4.1 and 4.3.

4.4. Corollary. If \(T\) is a tree, then \(m(T) \leq \sigma(T)\).

Remarks. For some trees, the inequality in 4.4 provides a sharp estimation of the span. Indeed, the 4-od \(T\) from 4.2 has \(m(T) = (4 - 2)^{-1} = \frac{1}{2}\) and \(\sigma(T) = \frac{1}{2} = m(T) \leq w(T)\). This estimation, however, is not the best one for all trees. We cite an interesting problem that seems to be important here. It is the following unsettled conjecture of Frances O. McDonald: is it true that each tree \(T\) which is not an arc contains a simple triod \(T'\) such that \(w(T') \leq 2w(T)\)? If the answer were "yes", we would get, by 4.1, the inequality \(w(T) \leq \frac{1}{2}w(T)\) for all trees. Up to now, McDonald's conjecture has been proved for \(T\) being "n-odds" and \(n = 4, 5, 6\) (see [9], p. 13). Thus, for all the 5-odds and 6-odds, it already provides a better estimation of the span than that given by 4.4.

4.5. Example. For each \(r > 0\), there exists a simple triod \(T'\) on the plane \(R^2\) such that \(\sigma(T') = 1\) and \(w(T') < r\) for each simple triod \(T' \subset T\).

Proof. There exists an asteroid-like continuum \(X \subset R^2\) such that \(\sigma(X) > 0\) (see [3], pp. 100 and 106). Moreover, \(X\) is "trioid-like" in the sense that there are finite open covers of \(X\) with the mesh arbitrarily small and with the nerve being a simple triod. These covers can be constructed by means of open disks on the plane, so that their nerves are embeddable in the union of the disks (ibidem, see also [4], p. 76). It follows (compare 5.1 below) that there exists \(n = 1, 2, \ldots, (1/n)\)-translaiton \(f_n: X \rightarrow R^2\) such that \(f_n(X)\) is a simple triod. Let \(\gamma = \frac{1}{n} \sigma(X)\). We have
\[
\lim_{n \rightarrow \infty} \frac{1}{n} \sigma(X) = \sigma(X),
\]
by 3.6, and therefore there is a positive integer \(n_0\) such that \(\frac{1}{n} \sigma(X) \leq \sigma(T_n)\) for \(n \geq n_0\).
We claim that there also exists an integer \(n_1 \geq n_0\) such that \(w(T') < \gamma\) for each simple
trio $T' \subset T_n$. If it were not so, each $T_n$ (where $n > n_0$) would contain a simple trio $T_n'$ with $y \in w(T_n')$. Hence, by 2.4, there would exist arcs $A_{n_0} = T_n' \subset T_n$ ($i = 0, 1, 2$) such that $A_{n_0} \cap A_{n_1} \cap A_{n_2} \neq \emptyset$ and none of the arcs $A_{n_0}, A_{n_1}, A_{n_2}$ is contained in the $\gamma$-neighbourhood of the union of the other two. All these arcs are contained in a bounded subset of the plane. Without loss of generality, we can assume that the three sequences of the arcs $A_n$ ($i = 0, 1, 2$, respectively) converge to some three continua $C_i \subset X$ (when $n \to \infty$). Also then $C_0 \cap C_1 \cap C_2 \neq \emptyset$, and none of the continua $C_0, C_1, C_2$ is contained in the union of the other two, which contradicts the fact that $X$ is triodic (see [10], p. 443). The existence of $n_1$ is thus proved.

Let $p \in R^2 \setminus T_n$. We define an embedding $h: T_n \to R^2$ by taking $h(x)$ to be the point of the ray $\tilde{p}x$ ($x \in T_n$) such that

$\text{dist}(p, h(x)) = \text{dist}(p, x)/\sigma(T_n)$,

whence $\text{dist}(h(x), h(y)) = \text{dist}(x, y)/\sigma(T_n)$ for $x, y \in T_n$. The set $T = h(T_n)$ is a simple trio with the span $\sigma(T) = \sigma(T_n)/\sigma(T_n) = 1$. If $T' \subset T$ is a simple trio, then

$w(T') = \text{dist}(T', T)/\sigma(T_n) = \text{dist}(T', T)/\sigma(T_n)$.

5. The span of certain tree-like continua. The following well-known lemma establishes a relationship between finite covers, their $\varepsilon$-translations, and the span of continua.

5.1. Lemma. Let $X$ be a non-empty subset of the Hilbert space $R^\varepsilon$. If $\varepsilon > 0$ and $C$ is a finite open cover of $X$ such that $\text{mesh}(C) < \varepsilon$, then there exists a polyhedron $P \subset R^\varepsilon$ contained in the $\varepsilon$-neighbourhood of $X$ in $R^\varepsilon$ and an $\varepsilon$-translation $\tilde{x}: X \to R^\varepsilon$ such that $P$ is topologically the nerve of $C$ and $\sigma(\tilde{x}) = P$. Moreover, if the nerve of $C$ has dimension $n$, then $R_n$ can be replaced by the Euclidean $(2n+1)$-space $R^{2n+1}$ (see [5], pp. 319, 324 and 330).

Let $\Pi$ be a collection of polyhedra. A compact metric space $X$ is called $\Pi$-like provided, for each $\varepsilon > 0$, there exists a finite open cover $C$ of $X$ such that $\text{mesh}(C) < \varepsilon$ and the nerve of $C$ is a $\Pi$-like polyhedron to $\Pi$.

Our next proposition involves the McDonald coefficient $m(\Pi)$ as defined in 4.3.

5.2. Let $\Pi$ be a finite collection of trees and let $X$ be a $\Pi$-like continuum. Denote

$m(\Pi) = \text{Min} \{m(T): T \in \Pi\}$.

Then $m(\Pi)w(X) \leq \sigma(X)$.

Proof. We can assume $X$ is non-degenerate; otherwise 5.2 states a trivial fact. Since then $X$ is one-dimensional, it is embeddable in $R^2$. Let us also assume that $X \subset R^2$, and, moreover, that the space $R^2$ is remetrized so that the metric in $R^2$ is an extension of the given metric in $X$ (see [2], p. 353). By 5.1, there exist trees $T_n' \subset R^2$ and $\varepsilon_n$-translations $f_n: X \to R^2$ such that $s_n \to 0$, $f_n(X) \to \Pi_n$ ($n = 1, 2, \ldots$). Since $X$ is one-dimensional, all but a finite number of the sets $T_n' = f_n(X)$ are non-degenerate. Consequently, they are trees and the inclusions $T_n' \subset T_n$ imply the inequalities

$m(T) \leq m(T) \leq m(T_n')$.

It follows from 3.6 and 4.4 that

$m(\Pi)w(X) = m(\Pi)\lim_{n \to \infty} w(T_n') = \lim_{n \to \infty} m(\Pi)w(T_n')$.

$\leq \lim_{n \to \infty} m(T_n')w(\varepsilon_n) = \lim_{n \to \infty} \sigma(T_n') = \sigma(X)$.

A continuum $X$ is called $\alpha$-unicoherent provided the common part of any two continua whose union is $X$ is connected. We say that a continuum $X$ is $P$-unicoherent if there exists a collection $\Pi$ of unicoherent connected polyhedra such that $X$ is $\Pi$-like. If $\tilde{f}$ is a mapping of a metric space $X$, we denote

$A_\tilde{f} = \{ \text{dist}(x, \tilde{f}(x)), x \in X \}$.

5.3. Theorem. If $f: X \to Y$ is a continuous mapping of a $P$-unicoherent continuum $X$ into an arc-like continuum $Y$, then

$[0, \sigma(X)] \subset A_\tilde{f}$.

Proof. Without loss of generality, let us assume that $X$ is a subset of the Hilbert space $R^\varepsilon$ and that the metric in $R^\varepsilon$ is an extension of the given metric in $X$ (see [2], p. 353). Also, since $Y$ is one-dimensional (or degenerate), we can assume that $Y \subset R^2$.

We are given a collection $\Pi$ of unicoherent connected polyhedra such that $X$ is $\Pi$-like. By 5.1, there exist polyhedra $P_n \subset R^\varepsilon$ contained in the $\varepsilon_n$-neighbourhoods of $X$ in $R^\varepsilon$, respectively, and $\varepsilon_n$-translations $f_n: X \to R^\varepsilon$ such that $s_n \to 0$, $f_n(X) \to \Pi_n$ ($m = 1, 2, \ldots$). Similarly, there exist arcs $A_\varepsilon \subset R^2$ and $\varepsilon_n$-translations $g_n: X \to R^2$ such that $s_n \to 0$, $g_n(X) \to \Pi_n$ ($n = 1, 2, \ldots$). Let $\Phi_\varepsilon: R_\varepsilon \to A_\varepsilon$ ($m = 1, 2, \ldots$) be a continuous extension of the mapping $g_\varepsilon: X \to A_\varepsilon$ (see [6], p. 332).

Since $P_n$ is a unicoherent locally connected continuum, the mapping $h = \Phi_\varepsilon(P_n): P_n \to A_\varepsilon$ fulfills the condition

$[0, \sigma(P_n)] \subset A_\varepsilon$.

(see [8], p. 207). Given any number $r \in [0, \sigma(X)]$, there exists a number $s_\varepsilon \in [0, \sigma(P_n)]$ such that

$|r - s_\varepsilon| \leq \sigma(X) - \sigma(P_n)$,

and thus we also have $s_\varepsilon \in A_\varepsilon$. Consequently, there exist points $p_n, p_n' \in P_n$ with $d(P_n, p_n') = s_\varepsilon$, and $h(p_n) = h(p_n')$, whence

$|x - \text{dist}(p_n, p_n')| \leq |\sigma(X) - \sigma(P_n)|$ (m, n = 1, 2, ...)

and

$\Phi_\varepsilon(p_n) = \Phi_\varepsilon(p_n')$ (m, n = 1, 2, ...).

Let $n$ be fixed for this part of the proof. Since the polyhedron $P_n$ is contained in the $\varepsilon_n$-neighbourhood of $X$ in $R^2$, there exist points $x_n, x_n' \in X$ such that

$\text{dist}(p_n, x_n) = \varepsilon_n, \text{dist}(p_n', x_n') = \varepsilon_n$ (m = 1, 2, ...),

where $x_n, x_n' \in X$. Since

$[0, \sigma(X)] \subset A_\varepsilon$,
whence
\[ \lim_{n \to \infty} \text{dist}(p_{m_n}, x_{m_n}) = \lim_{n \to \infty} \text{dist}(p'_{m_n}, x'_{m_n}) = 0, \]
according to the assumption made about the sequence \(x_1, x_2, \ldots\) By the compactness
of \(X\), there exists a sequence \(m_1 < m_2 < \ldots\) of positive integers such that
\[ \lim_{n \to \infty} x_{m_n} = x, \quad \lim_{n \to \infty} x'_{m_n} = x', \]
where \(x, x' \in X\). (Actually, the sequence \(m_1, m_2, \ldots\) may depend on \(n\), but this is
irrelevant here, \(n\) being fixed.) Thus we also have
\[ \lim_{n \to \infty} p_{m_n} = x, \quad \lim_{n \to \infty} p'_{m_n} = x', \]
and therefore
\[ g_n f(x_n) = \Phi_n(x_n) = \lim_{n \to \infty} \Phi_n(p_{m_n}) = \lim_{n \to \infty} \Phi_n(p'_{m_n}) = \Phi_n(x') = g_n f(x'), \]
by (22). Moreover, it follows from 3.1 that
\[ \lim_{n \to \infty} \sup \sigma(p_{m_n}) \leq \sigma(X), \]
and 3.2 implies
\[ \sigma(X) \leq \inf \sigma(f_{m_n}(X)) \leq \lim_{n \to \infty} \inf \sigma(P_{m_n}), \]
since \(f_{m_n}: X \to R^\alpha \) is an \(a_m\)-translation and \(f_{m_n}(X) \subset P_{m_n}\) \((i = 1, 2, \ldots)\). As a result, we obtain
\[ \lim_{n \to \infty} \sigma(p_{m_n}) = \sigma(X), \]
and condition (21) implies that
\[ \text{dist}(x_n, x') = \lim_{n \to \infty} \text{dist}(p_{m_n}, p'_{m_n}) = \alpha. \]
Again, \(X\) being compact, there exists a sequence \(n_1 < n_2 < \ldots\) of positive integers
such that
\[ \lim_{j \to \infty} x_j = x, \quad \lim_{j \to \infty} x'_j = x', \]
where \(x, x' \in X\). Consequently, we get
\[ \text{dist}(x, x') = \lim_{j \to \infty} \text{dist}(x_j, x'_j) = \alpha \]
and \(g_{a_j} f(x_j) = g_{a_j} f(x'_j)\) for \(j = 1, 2, \ldots\). But the mapping \(g_{a_j}\) is an \(a_m\)-translation, whence
\[ \text{dist}(f(x_j), f(x'_j)) \leq 2a_{n_j} \quad (j = 1, 2, \ldots), \]
by (90). The latter inequality yields
\[ f(x) = \lim_{j \to \infty} f(x_j) = \lim_{j \to \infty} f(x'_j) = f(x)' \]