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The span and the width of continua

by

A. Lelek (Detroit, Mich.)

Abstract. The concept of the span is compared with Burgess' notion of the width. Some basic properties of them are established. The results are applied to an investigation of sets of distances associated with continuous functions.

Two metric properties of continua have been investigated in connection with some topological phenomena, and each of them can be expressed by means of a numerical quantity designed to measure the "non-arc-likeness" of a given continuum. The concept of the width was introduced by C. E. Burgess [1] in the theory of tree-like continua, and the author [7] defined the span of a metric space without any restrictions imposed upon its structure. The present paper resulted from an attempt to compare these two notions. Although, in general, the span and the width are not equal even for very primitive objects such as simple triods, some comparisons can be made and the similarities do exist (see Sections 3–5). At the end of the paper two results are obtained (see 5.3 and 5.4) concerning the set of distances between points that belong to point-inverses under continuous mappings.

1. Definitions and preliminaries. If X is a non-empty metric space, we define the *span* $\sigma(X)$ of X to be the least upper bound of the set of real numbers α which satisfy the following condition: there exists a connected space C and continuous mappings $f_1, f_2\colon C\to X$ such that $f_1(C) = f_2(C)$ and $\alpha\leqslant \operatorname{dist}[f_1(c), f_2(c)]$ for $c\in C$. Equivalently (see [7], p. 209), the span $\sigma(X)$ is the least upper bound of numbers α for which there exist connected subsets C_α of the product $X\times X$ such that $p_1(C_\alpha) = p_2(C_\alpha)$ and $\alpha\leqslant\operatorname{dist}(x, y)$ for $(x, y)\in C_\alpha$, where p_1 and p_2 denote the projections of $X\times X$ onto X, i.e. $p_1(x, y) = x$ and $p_2(x, y) = y$ for $x, y\in X$. We note that, for compact spaces X, the sets C_α in the latter definition of $\sigma(X)$ can be assumed to be closed in $X\times X$.

The definition of the width is a bit more involved in that it requires rather special notation. If $A, B\subset X$ are non-empty subsets of the metric space X, we denote by $\delta(A)$ and $\varrho(A, B)$ the diameter of A and the distance between A and B, respectively, i.e.

$$\delta(A) = \operatorname{Sup}\{\operatorname{dist}(a, a')\colon a, a'\in A\}, \quad \varrho(A, B) = \operatorname{Inf}\{\operatorname{dist}(a, b)\colon a\in A, b\in B\}.$$

If C is a collection of non-empty subsets of X, we denote by $|C|$ the union of all the sets which belong to C, and we put

$$\operatorname{mesh}(C) = \operatorname{Sup}\{\delta(A)\colon A \in C\}.$$

Furthermore, we say C is a *chain* provided C is finite and the nerve of C, if non-degenerate, is an arc (see [5], p. 318). For any collection C of non-empty subsets of X, $\operatorname{Chain}(C)$ will denote the family of all the chains which are contained in C. For finite collections C of non-empty subsets of X, we define real numbers $w(C)$ by the formula

$$w(C) = \operatorname{Min}\{\operatorname{Max}\{\varrho(A, |C'|)\colon A \in C\}\colon C' \in \operatorname{Chain}(C)\}.$$

We also consider finite open covers of X by which we mean finite collections C of non-empty open subsets of X such that $|C| = X$. Now, if X is a non-empty compact metric space, we define the *width* $w(X)$ of X to be the least upper bound of the set of real numbers α which satisfy the following condition: for each $\varepsilon>0$, there exists a finite open cover C of X such that $\operatorname{mesh}(C)<\varepsilon$ and $\alpha \leqslant w(C)$. Thus $w(X)$ is a well-defined real number and $0 \leqslant w(X) \leqslant \delta(X)$.

Let X be a non-empty continuum, i.e. connected compact metric space. We say X is a *tree* provided X is homeomorphic to a connected one-dimensional polyhedron which contains no simple closed curve. A continuum X is called *tree-like* (or *arc-like*) provided, for each $\varepsilon>0$, there exists a finite open cover C of X such that $\operatorname{mesh}(C)<\varepsilon$ and the nerve of C, if non-degenerate, is a tree (or an arc, respectively). The concept of the width of a tree-like continuum was defined earlier in [1] using slightly different considerations. Before we see that, restricted to such a continuum, the two definitions coincide, we need to prove some auxiliary propositions. By a *refinement* of a finite open cover C we mean, as usual, any finite open cover whose elements are contained in elements of C.

1.1. *If C is a finite open cover of a metric space and D is a refinement of C such that the nerve of C is a tree and the nerve of D is connected, then*

$$w(D) \leqslant w(C) + 2\operatorname{mesh}(C).$$

Proof. Let X denote the space and let $C' \in \operatorname{Chain}(C)$. The proof of 1.1 will be complete if we show the existence of a set $A \in C$ such that

$$w(D) \leqslant \varrho(A, |C'|) + 2\operatorname{mesh}(C). \tag{1}$$

The following two cases are to be distinguished:

Case 1. C consists of at most 2 elements. Then C has exactly 2 elements and let $C = \{A, B\}$. Thus $A \cap B \neq \emptyset$ and $A \cup B = |C| = X$; inequality (1) follows from

$$w(D) \leqslant \delta(X) = \delta(A \cup B) \leqslant \delta(A) + \delta(B) \leqslant 2\operatorname{mesh}(C).$$

Case 2. C consists of more than 2 elements. Then the chain C' is contained in a chain $C'' \subset C$ which has at least 3 elements, say $C'' = \{C_1, \ldots, C_m\}$, where $m \geqslant 3$ and $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$ for $i, j = 1, \ldots, m$. Thus $C_1 \cap C_m = \emptyset$ and the non-empty sets C_1, C_m intersect some non-empty sets belonging to the refinement D of C. Since the nerve of D is connected, those elements of D can be joined together by means of a chain $D' \subset D$, so that we can write $D' = \{D_1, \ldots, D_n\}$, where $C_1 \cap D_1 \neq \emptyset \neq C_m \cap D_n$ and $D_i \cap D_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$ for $i, j = 1, \ldots, n$. As a result, we obtain

$$\emptyset \neq C_1 \cap D_1 \subset C_1 \cap |D'|,$$
$$\emptyset \neq C_m \cap D_n \subset C_m \cap |D'|,$$

and we claim that $C_i \cap |D'| \neq \emptyset$, too, for $i = 2, \ldots, m-1$. Indeed, since the nerve of C is a tree, all the elements of $C \setminus \{C_i\}$ $(i = 2, \ldots, m-1)$ can be grouped in two collections C_1 and C_2 such that $C_1 \in C_1$, $C_m \in C_2$ and $|C_1| \cap |C_2| = \emptyset$. Supposing $C_i \cap |D'| = \emptyset$, we would get

$$|D'| \subset |C \setminus \{C_i\}| = |C_1| \cup |C_2|,$$

and since each element of D' is contained in an element of C, the collection D' would split into two collections composed of subsets of $|C_1|$ and $|C_2|$, respectively. One of these collections would contain D_1 and the other one would contain D_n. This, however, is impossible, D' being a chain; therefore $C_i \cap |D'| \neq \emptyset$ for $i = 1, \ldots, m$. It follows that, for each point $y \in |C''|$, there exists an integer $i = 1, \ldots, m$ and a point $y' \in |D'|$ such that $y, y' \in C_i$, and thus

$$\operatorname{dist}(y, y') \leqslant \delta(C_i) \leqslant \operatorname{mesh}(C).$$

On the other hand, $D' \in \operatorname{Chain}(D)$ implies the existence of a set $B \in D$ satisfying the inequality

$$w(D) \leqslant \varrho(B, |D'|),$$

and since D is a refinement of C, there is a set $A \in C$ such that $B \subset A$. Let us select a point $x' \in B$ and observe that $|C'| \subset |C''|$. We conclude that, given arbitrary points $x \in A$ and $y \in |C'|$, one has $\operatorname{dist}(x, x') \leqslant \delta(A) \leqslant \operatorname{mesh}(C)$ as well as

$$\begin{aligned} w(D) &\leqslant \varrho(B, |D'|) \leqslant \operatorname{dist}(x', y') \leqslant \operatorname{dist}(x', x) + \operatorname{dist}(x, y) + \operatorname{dist}(y, y') \\ &\leqslant \operatorname{dist}(x, y) + 2\operatorname{mesh}(C), \end{aligned}$$

whence (1) and the proof of 1.1 is completed.

1.2. *If X is a tree-like continuum and $C_1, C_2, \ldots$ is a sequence of finite open covers of X such that the nerve of C_n is a tree $(n = 1, 2, \ldots)$ and*

$$\lim_{n\to\infty} \operatorname{mesh}(C_n) = 0, \tag{2}$$

then $\lim_{n\to\infty} w(C_n) = w(X)$.

Proof. Let $\gamma>0$ be a number arbitrarily chosen. If the inequality

$$w(C_n) < w(X) + \gamma \tag{3}$$

were violated by infinitely many positive integers n, the number $\alpha = w(X)+\gamma$ would have the property of possessing finite open covers C_n of X with small mesh(C_n), by (2), and such that $\alpha \leqslant w(C_n)$. According to the definition of the width $w(X)$, we would then have $\alpha \leqslant w(X)$ which contradicts the assumption that $\gamma > 0$. Consequently, there exists a positive integer n_1 such that (3) holds for $n \geqslant n_1$.

By virtue of (2), there also exists a positive integer n_2 such that mesh$(C_n) < \frac{1}{3}\gamma$ for $n \geqslant n_2$. Let $n \geqslant n_2$ be fixed for a while. Since X is compact, there is a number $\varepsilon > 0$ such that each finite open cover C of X with mesh$(C) < \varepsilon$ is a refinement of C_n. But since $w(X)$ is the width of X, a cover C of this type can be found satisfying the inequality

$$w(X) - \tfrac{1}{3}\gamma \leqslant w(C),$$

and the connectedness of X implies the connectedness of the nerve of C. By 1.1, we obtain

$$w(C) \leqslant w(C_n) + 2\,\mathrm{mesh}(C_n) < w(C_n) + \tfrac{2}{3}\gamma,$$

whence $w(X) - \gamma < w(C_n)$. Combining the latter inequality with inequality (3), we see that

$$|w(C_n) - w(X)| < \gamma$$

provided $n \geqslant n_1$ and $n \geqslant n_2$. Thus 1.2 is proved.

Remark. It follows from 1.2 that, for tree-like continua, the width as defined in this paper is equal to that invented by Burgess [1]. On the other hand, the condition that the nerves of C_n are trees surely implies, by (2), that the continuum X from 1.2 is tree-like. It will be shown (see 3.7) that this condition cannot be omitted in 1.2.

2. The width of dendroids. Each arcwise connected tree-like continuum is said to be a *dendroid*. Clearly, all trees as well as all dendrites are dendroids. By a *simple triod* we mean a tree which is the union of three arcs having a common end-point and being pairwise disjoint except at that point. Since chains are related to arcs in a natural way, there should also exist a relationship between the concept of the width, as defined in Section 1, and another concept that expresses a similar idea by means of arcs instead of chains. To have such a relationship, however, one should assume the existence of sufficiently many arcs in the space, and the class of dendroids seems to be an appropriate range for this purpose. Before studying them, we prove two propositions of a more general character. For any metric space X, Arc(X) will denote the collection of all the arcs which are contained in X and of all the non-empty degenerate subsets of X. If $\{a\}$ is a degenerate set and B is a non-empty set, we denote the distance between them by $\varrho(a, B)$ rather than by $\varrho(\{a\}, B)$.

2.1. *If X is a metric space, $A \in \mathrm{Arc}(X)$ and $\gamma > 0$, then there exists a number $\varepsilon > 0$ such that each finite open cover C of X with* mesh$(C) < \varepsilon$ *has a chain $C' \in \mathrm{Chain}(C)$ satisfying the condition*

$$\mathrm{Sup}\{\varrho(a, |C'|):\ a \in A\} \leqslant \gamma.$$

Proof. If A is degenerate, 2.1 trivially holds since any degenerate chain C' composed of a set $C \in C$ which contains A has the distance $\varrho(A, C) = 0$. Let us then assume A is an arc and decompose it into $k \geqslant 3$ subarcs $A_1, \ldots, A_k$ such that $\delta(A_j) < \gamma$ and $A_i \cap A_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$ for $i, j = 1, \ldots, k$. We define

$$\varepsilon = \tfrac{1}{2}\mathrm{Min}\{\varrho(A_1 \cup \ldots \cup A_{i-1}, A_{i+1} \cup \ldots \cup A_k):\ i = 2, \ldots, k-1\}.$$

Let C be a finite open cover of X with mesh$(C) < \varepsilon$. The subcollection C^* of C consisting of all the elements of C which meet A has a connected nerve since A is connected. Two elements of C^* intersect A_1 and A_k, respectively, and thus they can be joined together by means of a chain $C' \subset C^*$. Thus $A_1 \cap |C'| \neq \emptyset$ and $A_k \cap |C'| \neq \emptyset$. We claim that $A_i \cap |C'| \neq \emptyset$, too, for $i = 2, \ldots, k-1$. If it were not so, the collection C' would split into two non-empty collections composed of sets which meet $A_1 \cup \ldots \cup A_{i-1}$ and $A_{i+1} \cup \ldots \cup A_k$, respectively. But, C' being a chain, there would exist non-disjoint elements C_1 and C_2 belonging to these collections, respectively, and we would get

$$\begin{aligned}\varrho(A_1 \cup \ldots \cup A_{i-1}, A_{i+1} \cup \ldots \cup A_k) &\leqslant \varrho[C_1 \cap (A_1 \cup \ldots \cup A_{i-1}),\ C_2 \cap (A_{i+1} \cup \ldots \cup A_k)] \\ &\leqslant \delta(C_1 \cup C_2) \leqslant \delta(C_1) + \delta(C_2) \leqslant 2\,\mathrm{mesh}(C) < 2\varepsilon,\end{aligned}$$

which contradicts the definition of ε. Hence $|C'|$ meets all the arcs $A_1, \ldots, A_k$. Given a point $a \in A$, there is an integer $j = 1, \ldots, k$ such that $a \in A_j$. The condition required in 2.1 now follows from the inequalities

$$\varrho(a, |C'|) \leqslant \varrho(a, A_j \cap |C'|) \leqslant \delta(A_j) < \gamma.$$

2.2. THEOREM. *If X is a non-empty compact metric space, then*

$$w(X) \leqslant \mathrm{Inf}\{\mathrm{Sup}\{\varrho(x, A):\ x \in X\}:\ A \in \mathrm{Arc}(X)\}.$$

Proof. Let r stand for the real number that is the right-hand side of the latter inequality and let $\gamma > 0$ be an arbitrary real number. There exists then an $A_0 \in \mathrm{Arc}(X)$ such that

$$\mathrm{Sup}\{\varrho(x, A_0):\ x \in X\} < r + \gamma \tag{4}$$

and let $\varepsilon_0 > 0$ be the real number whose existence is guaranteed by 2.1 for $A = A_0$. On the other hand, it follows from the definition of the width that there is a real number α_0 such that

$$w(X) - \gamma < \alpha_0$$

and, for each $\varepsilon > 0$, a finite open cover C of X exists with mesh$(C) < \varepsilon$ and $\alpha_0 \leqslant w(C)$, whence $w(X) < w(C) + \gamma$. In particular, for $\varepsilon = \varepsilon_0$, there exists such a finite open cover C_0 and, by 2.1, we have a chain $C_0' \in \mathrm{Chain}(C_0)$ with the following property: for each point $a \in A_0$, there is a point $y \in |C_0'|$ satisfying the condition

$$\mathrm{dist}(a, y) < \varrho(a, |C_0'|) + \gamma \leqslant 2\gamma. \tag{5}$$

According to the definition of $w(\mathcal{C}_0)$, an element $G_0 \in \mathcal{C}_0$ must exist such that

$$w(\mathcal{C}_0) \leqslant \varrho(G_0, |\mathcal{C}_0'|),$$

whence $w(X) < w(\mathcal{C}_0)+\gamma \leqslant \varrho(G_0, |\mathcal{C}_0'|)+\gamma$ and, taking a point $x_0 \in G_0$, we get

$$w(X) < \varrho(G_0, |\mathcal{C}_0'|)+\gamma \leqslant \varrho(x_0, |\mathcal{C}_0'|)+\gamma .$$

Since to each point $a \in A_0$ there corresponds a point $y \in |\mathcal{C}_0'|$ fulfilling (5), we obtain

$$\varrho(x_0, |\mathcal{C}_0'|) \leqslant \operatorname{dist}(x_0, y) \leqslant \operatorname{dist}(x_0, a)+\operatorname{dist}(a, y) < \operatorname{dist}(x_0, a)+2\gamma ,$$

which implies that $\varrho(x_0, |\mathcal{C}_0'|) \leqslant \varrho(x_0, A_0)+2\gamma$, whence

$$w(X) < \varrho(x_0, |\mathcal{C}_0'|)+\gamma \leqslant \varrho(x_0, A_0)+3\gamma .$$

By (4), the width of X thus satisfies the inequality $w(X) < r+4\gamma$, which completes the proof of 2.2.

2.3. THEOREM. *If X is a dendroid, then*

$$w(X) = \operatorname{Inf}\{\operatorname{Sup}\{\varrho(x, A)\colon x \in X\}\colon A \in \operatorname{Arc}(X)\}.$$

Proof. Keeping a notation of the preceding proof, let r be the number which is the right-hand side of this equality and let $\gamma > 0$ be an arbitrary number. If the inequality

(6) $$w(\mathcal{C}) < w(X)+\gamma$$

did not hold for some finite open covers $\mathcal{C}$ of X with $\operatorname{mesh}(\mathcal{C})$ as small as one wants, the number $\alpha = w(X)+\gamma$ would satisfy the condition from the definition of the width $w(X)$ and, consequently, we would have $\alpha \leqslant w(X)$ which is not the case as $\gamma > 0$. Therefore there exists a number $\varepsilon_0 > 0$ such that (6) holds if $\operatorname{mesh}(\mathcal{C}) < \varepsilon_0$.

The dendroid X being a tree-like continuum, there exists a finite open cover $\mathcal{C}_0$ of X such that

$$\operatorname{mesh}(\mathcal{C}_0) < \operatorname{Min}\{\varepsilon_0, \gamma\}$$

and the nerve of $\mathcal{C}_0$ is either degenerate or a tree. Then, by (6), we have $w(\mathcal{C}_0) < w(X)+\gamma$. It follows from the definition of $w(\mathcal{C}_0)$ that there exists a chain $\mathcal{C}_0' \in \operatorname{Chain}(\mathcal{C}_0)$ such that

(7) $$\operatorname{Max}\{\varrho(G, |\mathcal{C}_0'|)\colon G \in \mathcal{C}_0\} = w(\mathcal{C}_0) < w(X)+\gamma ,$$

and we claim that an $A_0 \in \operatorname{Arc}(X)$ can be picked up so that

(8) $$\operatorname{Sup}\{\varrho(x, A_0)\colon x \in |\mathcal{C}_0'|\} \leqslant \gamma .$$

Indeed, if $\mathcal{C}_0'$ is a degenerate chain $\{C\}$, it is enough to put $A_0 = \{a_0\}$, where $a_0 \in C$, and then $\varrho(x, A_0) = \operatorname{dist}(x, a_0) \leqslant \delta(C) \leqslant \operatorname{mesh}(\mathcal{C}_0) < \gamma$ for each $x \in |\mathcal{C}_0'| = C$. If $\mathcal{C}_0'$ is non-degenerate, we have $\mathcal{C}_0' = \{C_1, \ldots, C_k\}$, where $k \geqslant 2$ and $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$ for $i, j = 1, \ldots, k$. Let us select two distinct points $p \in C_1$ and $q \in C_k$. Since X is arcwise connected, there exists an arc $A_0 \subset X$ joining p and q.

Thus $C_1 \cap A_0 \neq \emptyset$ and $C_k \cap A_0 \neq \emptyset$. We show that $C_i \cap A_0 \neq \emptyset$ for $i = 1, \ldots, k$. If it were not so, $k \geqslant 3$ and there would be an integer $i_0 = 2, \ldots, k-1$ such that $A_0 \subset |\mathcal{C}_0 \setminus \{C_{i_0}\}|$. But the nerve of $\mathcal{C}_0$ being a tree, all the elements of $\mathcal{C}_0 \setminus \{C_{i_0}\}$ would group in two collections $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $C_1 \in \mathcal{C}_1$, $C_k \in \mathcal{C}_2$ and $|\mathcal{C}_1| \cap |\mathcal{C}_2| = \emptyset$. The arc A_0 would then be contained in the union of two disjoint open sets $|\mathcal{C}_1|$ and $|\mathcal{C}_2|$ each of them intersecting A_0, which contradicts the connectedness of A_0. Hence A_0 meets all the sets $C_1, \ldots, C_k$. Given a point $x \in |\mathcal{C}_0'|$, there is an integer $j = 1, \ldots, k$ such that $x \in C_j$, and (8) follows from the inequalities

$$\varrho(x, A_0) \leqslant \varrho(x, C_j \cap A_0) \leqslant \delta(C_j) \leqslant \operatorname{mesh}(\mathcal{C}_0) < \gamma .$$

By the definition of the number r, there exists a point $x_0 \in X$ such that $r-\gamma < \varrho(x_0, A_0)$. Let $G_0 \in \mathcal{C}_0$ be an element containing x_0 and let y be an arbitrary point of G_0. Clearly, to each point $x \in |\mathcal{C}_0'|$, there corresponds a point $a \in A_0$ with $\operatorname{dist}(x, a) < \varrho(x, A_0)+\gamma$, whence $\operatorname{dist}(x, a) < 2\gamma$, according to (8). As a result we obtain

$$\begin{aligned} r &< \varrho(x_0, A_0)+\gamma \leqslant \operatorname{dist}(x_0, a)+\gamma \\ &\leqslant \operatorname{dist}(x_0, y)+\operatorname{dist}(y, x)+\operatorname{dist}(x, a)+\gamma \\ &< \delta(G_0)+\operatorname{dist}(y, x)+3\gamma \\ &\leqslant \operatorname{mesh}(\mathcal{C}_0)+\operatorname{dist}(y, x)+3\gamma < \operatorname{dist}(y, x)+4\gamma , \end{aligned}$$

which implies that $r-4\gamma \leqslant \varrho(G_0, |\mathcal{C}_0'|) < w(X)+\gamma$, by (7). Thus $r-5\gamma < w(X)$ and, γ being an arbitrarily taken positive number, we have $r \leqslant w(X)$, whence $w(X) = r$, by 2.2. The proof of 2.3 is complete.

2.4. *If $X = A_0 \cup A_1 \cup A_2$ is a simple triod, where A_i are arcs having a common end-point v and $A_i \cap A_{i+1} = \{v\}$ for $i = 0, 1, 2$ and the subscripts of A_j taken* mod 3, *then*

$$w(X) = \operatorname{Min}\{\operatorname{Max}\{\varrho(x, A_{i+1} \cup A_{i+2})\colon x \in A_i\}\colon i = 0, 1, 2\}.$$

Proof. Let m stand for this Min Max. It follows from 2.2 that $w(X) \leqslant m$. If $A \in \operatorname{Arc}(X)$, there is a subscript $i = 0, 1, 2$ such that $A \subset A_{i+1} \cup A_{i+2}$, whence

$$\varrho(x, A_{i+1} \cup A_{i+2}) \leqslant \varrho(x, A)$$

for each point $x \in X$. Consequently, we get the inequalities

$$m \leqslant \operatorname{Max}\{\varrho(x, A_{i+1} \cup A_{i+2})\colon x \in A_i\} \leqslant \operatorname{Sup}\{\varrho(x, A)\colon x \in X\},$$

and 2.3 implies $m \leqslant w(X)$. Thus $w(X) = m$.

3. Some continuity properties. The span and the width, when treated as real-valued functions defined on collections of tree-like continua, seem to behave alike as far as their continuity is concerned. If X is a non-empty subset of a metric space Z and $\varepsilon > 0$, we call the set $\{z \in Z\colon \varrho(z, X) < \varepsilon\}$ the *ε-neighbourhood* of X in Z.

3.1. *Let X be a non-empty compact set contained in a metric space Z. If β is a real number and, for $n = 1, 2, \ldots$, there exists a subset Z_n of the* $(1/n)$*-neighbourhood of X in Z with $\beta \leqslant \sigma(Z_n)$, then $\beta \leqslant \sigma(X)$* (compare [7], p. 211).

Proof. Denote by $p_1, p_2\colon Z\times Z\to Z$ the standard projections of the product $Z\times Z$ onto Z. Let $\gamma>0$ be an arbitrary real number. Since $\beta-\gamma<\sigma(Z_n)$, it follows from the definition of the span that there exists a non-empty connected set $C_n\subset Z_n\times Z_n$ such that $p_1(C_n)=p_2(C_n)$ and $\beta-\gamma\leqslant\operatorname{dist}(x,y)$ for $(x,y)\in C_n$ $(n=1,2,\ldots)$. Let us select points $c_n\in C_n$ and observe that the set $Z_n\times Z_n$, hence also the set C_n, is contained in the $(\sqrt{2}/n)$-neighbourhood of $X\times X$ in $Z\times Z$. Consequently, there exist points $z_n\in X\times X$ such that $\operatorname{dist}(c_n,z_n)<\sqrt{2}/n$ for $n=1,2,\ldots$ By the compactness of $X\times X$, a subsequence $z_{n_1}, z_{n_2},\ldots$ (where $n_1<n_2<\ldots$) converges to a point $z\in X\times X$, whence also

$$\lim_{i\to\infty} c_{n_i}=z\,. \tag{9}$$

We define $D=\operatorname{Ls}_{i\to\infty}C_{n_i}$ (see [5], p. 337). Thus each point of D is the limit of a sequence of points of $C_{n_1}\cup C_{n_2}\cup\ldots$ It follows that $\beta-\gamma\leqslant\operatorname{dist}(x,y)$ for $(x,y)\in D$. Moreover, given a point $d\in D$, there exist points $u_j\in C_{n_{i_j}}$ (where $i_1<i_2<\ldots$) such that the sequence $u_1,u_2,\ldots$ converges to d. The point u_j belongs to the $(\sqrt{2}/n_{i_j})$-neighbourhood of $X\times X$, whence $d\in X\times X$, so that $D\subset X\times X$. On the other hand, since $p_1(C_{n_{i_j}})=p_2(C_{n_{i_j}})$, there are points $c'_j\in C_{n_{i_j}}$ such that $p_1(u_j)=p_2(c'_j)$. We conclude, as we did before for the points c_n, that a subsequence $c'_{j_1}, c'_{j_2},\ldots$ (where $j_1<j_2<\ldots$) converges to a point $z'\in X\times X$. By the definition of D, we have $z'\in D$, and

$$p_1(d)=\lim_{k\to\infty}p_1(u_{j_k})=\lim_{k\to\infty}p_2(c'_{j_k})=p_2(z')\,,$$

which implies that $p_1(D)\subset p_2(D)$. A symmetric argument shows that $p_2(D)\subset p_1(D)$, whence $p_1(D)=p_2(D)$.

Now, we claim that the set D is connected. Suppose it is not, and notice that $z\in D$, by (9). Then there exist two disjoint open subsets U and V of $Z\times Z$ such that $D\subset U\cup V$, $z\in U$ and $D\cap V\neq\emptyset$. Let $d'\in D\cap V$ be a point. As we have seen, there are points $v_m\in C_{n_{l_m}}$ (where $l_1<l_2<\ldots$) which converge to d', so that $v_m\in V$ for $m\geqslant m_1$, where m_1 is a positive integer. By (9), $c_{n_{l_m}}\in U$ for $m\geqslant m_2$. The connected set $C_{n_{l_m}}$ intersects both U and V provided $m\geqslant m_1$ and $m\geqslant m_2$. Therefore, almost all of the sets $C_{n_{l_m}}$ contain points $c''_m\notin U\cup V$ which, again, must have a subsequence converging to a point $z''\in D$. This implies that $z''\notin U\cup V$, contrary to the inclusion $D\subset U\cup V$. Hence D is a connected set.

Setting $\alpha=\beta-\gamma$ and $C_\alpha=D$ in the definition of $\sigma(X)$, we obtain $\beta-\gamma\leqslant\sigma(X)$. Since γ was an arbitrarily chosen positive number, the inequality $\beta\leqslant\sigma(X)$ follows and the proof of 3.1 is completed.

If X is a non-empty subset of a metric space Z and $\varepsilon>0$, we say that a continuous mapping $f\colon X\to Z$ is an *ε-translation* provided $\operatorname{dist}[x,f(x)]<\varepsilon$ for $x\in X$. Thus if $f\colon X\to Z$ is an ε-translation, then the set $f(X)$ is contained in the ε-neighbourhood of X in Z. It follows directly from the triangle inequality for distances that if $f\colon X\to Z$ is an ε-translation, then

$$|\operatorname{dist}(x,y)-\operatorname{dist}[f(x),f(y)]|<2\varepsilon\qquad(x,y\in X)\,. \tag{10}$$

3.2. *Let X be a non-empty set contained in a metric space Z. If β is a real number and, for $n=1,2,\ldots$, there exists a $(1/n)$-translation $f_n\colon X\to Z$ such that $\sigma[f_n(X)]\leqslant\beta$, then $\sigma(X)\leqslant\beta$.*

Proof. Assume α is a real number and $C\subset X\times X$ is a connected set such that $p_1(C)=p_2(C)$ and $\alpha\leqslant\operatorname{dist}(x,y)$ for $(x,y)\in C$, where p_1 and p_2 are the standard projections of $Z\times Z$ onto Z. The set

$$C_n=\{(f_n(x),f_n(y))\colon (x,y)\in C\}$$

is connected and $C_n\subset f_n(X)\times f_n(X)$. Moreover,

$$p_1(C_n)=f_n[p_1(C)]=f_n[p_2(C)]=p_2(C_n)$$

and it follows from (10) that, for any point $(f_n(x),f_n(y))$ of C_n, we have

$$\alpha-2/n\leqslant\operatorname{dist}(x,y)-2/n<\operatorname{dist}[f_n(x),f_n(y)]\,,$$

whence $\alpha-2/n\leqslant\sigma[f_n(X)]$. Consequently, the inequality $\alpha-2/n\leqslant\beta$ holds for $n=1,2,\ldots$ This means that $\alpha\leqslant\beta$ and, as a result, the least upper bound $\sigma(X)$ of such numbers α also satisfies the inequality $\sigma(X)\leqslant\beta$.

3.3. Lemma. *Let X be a non-empty subset of a bounded metric space Z and let $f\colon X\to Z$ be an ε-translation. If $\mathbf{C}$ is a finite open cover of $f(X)$ and $\mathbf{C}'=\{f^{-1}(G)\colon G\in\mathbf{C}\}$, then*

$$|\operatorname{mesh}(\mathbf{C})-\operatorname{mesh}(\mathbf{C}')|\leqslant2\varepsilon,\qquad |w(\mathbf{C})-w(\mathbf{C}')|\leqslant2\varepsilon\,.$$

Proof. By (10), for any pair of non-empty sets $A, B\subset f(X)$, we get the inequalities

$$|\delta(A)-\delta[f^{-1}(A)]|\leqslant2\varepsilon,\qquad |\varrho(A,B)-\varrho[f^{-1}(A),f^{-1}(B)]|\leqslant2\varepsilon\,,$$

which imply the two inequalities required in 3.3, respectively. It suffices to notice that some sets $G_1,\ldots,G_n$ of $\mathbf{C}$ form a chain if and only if the sets $f^{-1}(G_1),\ldots,f^{-1}(G_n)$ form a chain (in $\mathbf{C}'$), and

$$f^{-1}(G_1\cup\ldots\cup G_n)=f^{-1}(G_1)\cup\ldots\cup f^{-1}(G_n)\,.$$

3.4. *Let X be a non-empty compact set contained in a metric space Z. If β is a real number and, for $n=1,2,\ldots$, there exists a $(1/n)$-translation $f_n\colon X\to Z$ such that $\beta\leqslant w[f_n(X)]$, then $\beta\leqslant w(X)$.*

Proof. Since $\beta-1/n<w[f_n(X)]$ $(n=1,2,\ldots)$, it follows from the definition of the width that there exists a finite open cover $\mathbf{C}_n$ of $f_n(X)$ such that

$$\operatorname{mesh}(\mathbf{C}_n)<1/n,\qquad \beta-1/n\leqslant w(\mathbf{C}_n)\,.$$

We take the finite open cover $\mathbf{C}'_n=\{f_n^{-1}(G)\colon G\in\mathbf{C}_n\}$ of X $(n=1,2,\ldots)$ which, according to 3.3, fulfills the conditions

$$\operatorname{mesh}(\mathbf{C}'_n)\leqslant\operatorname{mesh}(\mathbf{C}_n)+2/n<3/n \tag{11}$$

and $\beta-3/n \leqslant w(C_n)-2/n \leqslant w(C_n')$. Setting $\alpha = \beta-3/n$ and $C = C_m'$ $(m \geqslant n)$ in the definition of $w(X)$, we obtain $\beta-3/n \leqslant w(X)$, by (11). The inequality $\beta \leqslant w(X)$ now follows.

3.5. *Let X be a tree-like continuum contained in a metric space Z. If β is a real number and, for $n = 1, 2, \ldots$, there exists a $(1/n)$-translation f_n: $X \to Z$ such that $Z_n = f_n(X)$ is a tree-like continuum with $w(Z_n) \leqslant \beta$, then $w(X) \leqslant \beta$.*

Proof. We can assume X is non-degenerate; otherwise $w(X) = 0$. Thus all but a finite number of the continua Z_n also are non-degenerate and, without loss of generality, we can as well assume that each Z_n is a non-degenerate tree-like continuum $(n = 1, 2, \ldots)$. According to 1.2, there exists a finite open cover C_n of Z_n $(n = 1, 2, \ldots)$ such that the nerve of C_n is a tree and

$$\operatorname{mesh}(C_n) < 1/n, \quad |w(C_n)-w(Z_n)| < 1/n .$$

The finite open cover C_n' of X from the proof of 3.4 fulfills condition (11), by 3.3. Furthermore, for $n = 1, 2, \ldots$, we have

$$w(C_n') \leqslant w(C_n)+2/n < w(Z_n)+3/n \leqslant \beta+3/n , \tag{12}$$

and the nerve of C_n' is obviously the same as the nerve of C_n; thus it is a tree. Since (11) implies (2) with C_n replaced by C_n', we can apply 1.2 again to conclude from (12) that

$$w(X) = \lim_{n\to\infty} w(C_n') \leqslant \lim_{n\to\infty} (\beta+3/n) = \beta .$$

Remark. Observe that the tree-likeness of Z_n in 3.5 implies, by 3.3, the tree-likeness of the continuum X itself. However, even if the continuum X is assumed to be tree-like, the conclusion of 3.5 is no longer true when the tree-likeness of the continua Z_n is dropped. We provide an example to explain this possibility (see 3.7).

3.6. Corollary. *If X is a tree-like continuum contained in a metric space and, for $n = 1, 2, \ldots$, a tree-like continuum X_n is the image of X under an ε_n-translation such that*

$$\lim_{n\to\infty} \varepsilon_n = 0 ,$$

then $\lim_{n\to\infty} \sigma(X_n) = \sigma(X)$ and $\lim_{n\to\infty} w(X_n) = w(X)$.

3.7. Example. *There exists a simple triod T on the plane R^2, $(1/n)$-translations f_n: $T \to R^2$ and finite open covers C_n of T $(n = 1, 2, \ldots)$ such that*

(i) $w(T) = 1$,

(ii) $w[f_n(T)] = 0$ *for* $n = 1, 2, \ldots$, *and*

(iii) $\lim_{n\to\infty} \operatorname{mesh}(C_n) = \lim_{n\to\infty} w(C_n) = 0$.

Proof. Let S_0, S_1 and S_2 be the straight-line segments joining the origin with the points $(0, 1)$, $(1, 0)$ and $(0, -1)$, respectively. Then $T = S_0 \cup S_1 \cup S_2$ is a simple triod and (i) follows from 2.4. Let f_n: $T \to R^2$ be a $(1/n)$-translation such that $f_n(T)$ is topologically a disk $(n = 1, 2, \ldots)$. We get (ii), by 2.2, which means that if D is a finite open cover of the disk $f_n(T)$ with mesh(D) sufficiently small (depending on n), then $w(D)$ is small, too. Thus there exists, for $n = 1, 2, \ldots$, a finite open cover D_n of $f_n(T)$ such that

$$\operatorname{mesh}(D_n) < 1/n, \quad w(D_n) < 1/n .$$

We define $C_n = \{f_n^{-1}(G)\colon G \in D_n\}$ $(n = 1, 2, \ldots)$. It follows from 3.3 that Cs satisfy condition (iii).

4. Simple triods and trees. The results of this section indicate that the problem of finding a relationship between the span and the width of tree-like continua reduces partially to another problem, rather combinatorial in its nature, namely that of connecting the width of a tree with the widths of simple triods contained in it.

4.1. *If T is a simple triod, then $w(T) \leqslant \sigma(T)$.*

Proof. By 2.4, this is a consequence of a lemma proved in [8].

Remark. The inequality in 4.1 cannot be replaced by the equality (see 4.5). Also, the assumption that T is a simple triod is essential in 4.1. To support the latter statement, we study, in 4.2 below, some properties of an example of a tree which has been constructed in [9].

4.2. Example. *There exists a tree T in the 3-space R^3 such that*

(i) $w(T) = 1$,

(ii) $\sigma(T) = \frac{1}{2}$, *and*

(iii) $T = A_0 \cup A_1 \cup A_2 \cup A_3$, *where A_i are arcs having a common end-point v and $A_i \cap A_j = \{v\}$ for $i \neq j$ $(i, j = 0, 1, 2, 3)$; thus T is a simple "4-od".*

Proof. The space R^3 with the ordinary Pythagorean distance will be used. Given two points $p, q \in R^3$, we denote by $\overline{pq}$ the straight-line segment having p and q as the end-points. For $i = 1, 2, 3$, we take the points

$$p_i = (\tfrac{1}{2}\cos\tfrac{2}{3}\pi i, \tfrac{1}{2}\sin\tfrac{2}{3}\pi i, 0), \quad q_i = (\cos\tfrac{2}{3}\pi(i+1), \sin\tfrac{2}{3}\pi(i+1), 0) ,$$

and let $q_0 = (0, 0, 1)$ and $v = (0, 0, 0)$. We define $A_0 = \overline{q_0 v}$ and $A_i = \overline{p_i q_i} \cup \overline{p_i v}$ $(i = 1, 2, 3)$. Then the union T of these arcs satisfies condition (iii). We also denote

$$B = A_1 \cup A_2 \cup A_3 ,$$

so that B is a simple triod and $T = A_0 \cup B$. Clearly, $\operatorname{dist}(x, v) \leqslant 1$ for $x \in B$. Since $v \in A_0$, we get

$$\operatorname{Sup}\{\varrho(x, A_0)\colon x \in T\} \leqslant 1 ,$$

which implies that $w(T) \leqslant 1$, by 2.2. Now, let $A \in \operatorname{Arc}(T)$. If $A \subset B$, we have

$$1 = \operatorname{dist}(q_0, v) = \varrho(q_0, B) \leqslant \varrho(q_0, A) ,$$

whence

$$1 \leqslant \operatorname{Sup}\{\varrho(x, A)\colon x \in T\} . \tag{13}$$

If $A \not\subset B$, there is a subscript $j = 1, 2, 3$ such that $A \subset A_n \cup A_j$. The set $A_0 \cup A_j$ is contained in the cylindrical section

$$S = \{(r\cos\theta, r\sin\theta, t):\ 0 \leqslant r \leqslant 1,\ \tfrac{2}{3}\pi j \leqslant \theta \leqslant \tfrac{2}{3}\pi(j+1),\ 0 \leqslant t \leqslant 1\}$$

and one of the points q_i, namely the point

$$q = (\cos\tfrac{2}{3}\pi(j+2), \sin\tfrac{2}{3}\pi(j+2), 0)$$

has the distance $\varrho(q, S) = 1$. Since $A \subset A_0 \cup A_j \subset S$, we obtain $\varrho(q, S) \leqslant \varrho(q, A)$, whence (13) holds again. By 2.3, this implies the inequality $1 \leqslant w(T)$, and so (i) is proved.

The set $T' = \overline{p_1 v} \cup \overline{p_2 v} \cup \overline{p_3 v}$ is a simple triod contained in T, and $w(T') = \frac{1}{2}$, by 2.4. It follows from 4.1 that $\frac{1}{2} \leqslant \sigma(T') \leqslant \sigma(T)$. Thus, to complete the proof of (ii), we have to show that $\sigma(T) \leqslant \frac{1}{2}$. Suppose, on the contrary, that $\sigma(T)$ exceeds $\frac{1}{2}$. Then there exists a number $\alpha_0 > \frac{1}{2}$, a continuum C and two continuous mappings f_1, f_2: $C \to T$ such that $f_1(C) = f_2(C)$ and $\alpha_0 \leqslant \operatorname{dist}[f_1(c), f_2(c)]$ for $c \in C$. The vertex v is a cut-point of T. If $v \notin f_1(C)$, the continuum $f_1(C)$ would be contained in one of the sets $A_i \setminus \{v\}$ $(i = 0, 1, 2, 3)$, which is impossible since the span of any arc is zero. Thus $v \in f_1(C)$. We denote

$$(14) \qquad V_i = f_i^{-1}(v), \quad X_i = f_i^{-1}(A_0), \quad Y_i = f_i^{-1}(B) \quad (i = 1, 2).$$

Let $\leqslant_0$ be the ordering of the arc A_0 from v to q_0, that is $(0, 0, t) \leqslant_0 (0, 0, t')$ if and only if $t \leqslant t'$. The sets

$$P_1 = \{c \in X_1 \cap X_2:\ f_1(c) \leqslant_0 f_2(c)\}, \quad P_2 = \{c \in X_1 \cap X_2:\ f_2(c) \leqslant_0 f_1(c)\}$$

are compact subsets of C whose union is $X_1 \cap X_2$. They are also disjoint since $f_1(c) \neq f_2(c)$ for $c \in C$. Hence

$$(15) \quad X_1 \cap X_2 = P_1 \cup P_2, \quad P_1 \cap P_2 = \emptyset, \quad V_i \cap X_1 \cap X_2 \subset P_i \quad (i = 1, 2).$$

We claim that a decomposition similar to (15) also exists for the set $Y_1 \cap Y_2$. More precisely, we are going to prove that there exist compact sets Q_1 and Q_2 satisfying the conditions

$$(16) \quad Y_1 \cap Y_2 = Q_1 \cup Q_2, \quad Q_1 \cap Q_2 = \emptyset, \quad V_i \cap Y_1 \cap Y_2 \subset Q_i \quad (i = 1, 2),$$

or, in other words, that the set $Y_1 \cap Y_2$ is not connected between $V_1 \cap Y_1 \cap Y_2$ and $V_2 \cap Y_1 \cap Y_2$. This will be achieved when we show that the set $Y_1 \cap Y_2$ is not connected between any two points belonging to these sets (see [6], p. 168). Let $d_i \in V_i \cap Y_1 \cap Y_2$ $(i = 1, 2)$ be points arbitrarily selected. By (14), we have $f_1(d_1) = v = f_2(d_2)$, whence $f_2(d_1) \neq v \neq f_1(d_2)$. Also by (14), the points $f_1(d_2)$ and $f_2(d_1)$ are in B, so that each of them is in one of the sets $A_i \setminus \{v\}$ $(i = 1, 2, 3)$. Thus there exists a subscript $k = 1, 2, 3$ such that the arc A_k contains neither $f_1(d_2)$ nor $f_2(d_1)$. Let l and m be the two remaining subscripts from $\{1, 2, 3\}$ arranged so that the arcs A_l and A_m are obtained from A_k by the counter-clockwise rotation through the angles $\frac{2}{3}\pi$ and $\frac{4}{3}\pi$, respectively. In other words, (k, l, m) is either $(1, 2, 3)$ or $(2, 3, 1)$ or $(3, 1, 2)$. We consider a retraction g: $B \to A_k \cup A_m$ which is defined in the following way. The mapping g maps A_l into A_m, $g(p_l) = v$, $g(q_l) = p_m$, and g is linear on both segments $\overline{p_l q_l}$ and $\overline{p_l v}$ whose union is A_l. Consequently, we have

$$(17) \qquad g^{-1}(x) = \begin{cases} \{x\} & \text{for} \quad x \in (A_k \cup A_m) \setminus \overline{p_m v}, \\ \overline{p_l v} & \text{for} \quad x = v, \\ \{x, y(x)\} & \text{for} \quad x \in \overline{p_m v} \setminus \{v\}, \end{cases}$$

where $y(x)$ is the point of the segment $\overline{p_l q_l}$ such that

$$(18) \qquad \frac{\operatorname{dist}[p_l, y(x)]}{\operatorname{dist}(p_l, q_l)} = \frac{\operatorname{dist}(v, x)}{\operatorname{dist}(v, p_m)}.$$

Next, we need to show that

$$(19) \qquad \delta[g^{-1}(x)] \leqslant \tfrac{1}{2} \quad (x \in A_k \cup A_m).$$

Since $\delta(\overline{p_l v}) = \operatorname{dist}(p_l, v) = \frac{1}{2}$, inequality (19) holds for $x \in (A_k \cup A_m) \setminus \overline{p_m v}$ or $x = v$, by (17). Assume then that x is a point of $\overline{p_m v}$ and $x \neq v$. It has to be shown that $\operatorname{dist}[x, y(x)] \leqslant \frac{1}{2}$ for such $x \in \overline{p_m v} \setminus \{v\}$. To this end, let us denote $\lambda = \operatorname{dist}(v, x)$, and observe that $\operatorname{dist}(p_l, q_l) = \frac{1}{2}\sqrt{7}$ and $\operatorname{dist}(v, p_m) = \frac{1}{2}$. Hence $0 \leqslant \lambda \leqslant \frac{1}{2}$ and

$$\operatorname{dist}[p_l, y(x)] = \lambda\sqrt{7},$$

by (18). Also, it follows from the definition of l and m that p_m is the mid-point of the segment $\overline{q_l v}$. Thus the point x belongs to $\overline{q_l v}$ and

$$\operatorname{dist}(q_l, x) = \operatorname{dist}(q_l, v) - \operatorname{dist}(v, x) = 1 - \lambda.$$

On the other hand, we get

$$\operatorname{dist}[q_l, y(x)] = \operatorname{dist}(q_l, p_l) - \operatorname{dist}[p_l, y(x)] = \sqrt{7}(\tfrac{1}{2} - \lambda),$$

and $\cos\theta = \frac{5}{14}\sqrt{7}$, where θ is the angle between the segments $\overline{q_l v}$ and $\overline{q_l p_l}$. As a result, we obtain

$$\begin{aligned} (\operatorname{dist}[x, y(x)])^2 &= (1-\lambda)^2 + 7(\tfrac{1}{2}-\lambda)^2 - 2\sqrt{7}(1-\lambda)(\tfrac{1}{2}-\lambda)\cos\theta \\ &= (1-\lambda)^2 + 7(\tfrac{1}{2}-\lambda)^2 - 5(1-\lambda)(\tfrac{1}{2}-\lambda) \\ &= \tfrac{1}{4} + 3\lambda(\lambda - \tfrac{1}{2}) \leqslant \tfrac{1}{4}, \end{aligned}$$

whence $\operatorname{dist}[x, y(x)] \leqslant \frac{1}{2}$ and, by (17), the proof of (19) is complete.

Let $\leqslant_k$ be the ordering of the arc $A_k \cup A_m$ from q_k to q_m, the end-points of this arc. The sets

$$Q_1' = \{c \in Y_1 \cap Y_2:\ gf_1(c) \leqslant_k gf_2(c)\},$$
$$Q_2' = \{c \in Y_1 \cap Y_2:\ gf_2(c) \leqslant_k gf_1(c)\}$$

are compact subsets of C whose union is $Y_1 \cap Y_2$, by (14). Since $\frac{1}{2} < \alpha_0 \leqslant \operatorname{dist}[f_1(c), f_2(c)]$ for $c \in C$, it follows from (19) that $gf_1(c) \neq gf_2(c)$ for

$c \in Y_1 \cap Y_2$. Therefore the sets Q_1' and Q_2' are disjoint. Moreover, the points $f_1(d_2)$ and $f_2(d_1)$ both belong to $A_l \cup A_m$, whence

$$gf_1(d_2),\, gf_2(d_1) \in g(A_l \cup A_m) = A_m,$$

and because $gf_i(d_i) = g(v) = v$ $(i = 1, 2)$, we conclude that

$$gf_1(d_1) = v \leqslant_k gf_2(d_1), \quad gf_2(d_2) = v \leqslant_k gf_1(d_2),$$

which means that $d_i \in Q_i'$ $(i = 1, 2)$. The decomposition of $Y_1 \cap Y_2$ into Q_1' and Q_2' then establishes the non-connectedness of $Y_1 \cap Y_2$ between d_1 and d_2. Consequently, we have also proved the existence of compact sets Q_1 and Q_2 which satisfy (16).

We now distinguish two cases to prove that

$$X_1 \cap Y_2 \neq \emptyset \neq X_2 \cap Y_1. \tag{20}$$

Let $u = (0, 0, \frac{1}{2})$.

Case 1. $u \in f_1(C)$. Since $f_1(C) = f_2(C)$, there exist points $c_i \in C$ such that $f_i(c_i) = u$ $(i = 1, 2)$. But $u \in A_0$, so that $c_i \in X_i$ $(i = 1, 2)$, by (14). Each point of T whose distance from u exceeds $\frac{1}{2}$ belongs to B. It follows from the inequalities

$$\tfrac{1}{2} < \alpha_0 \leqslant \operatorname{dist}[f_1(c_i), f_2(c_i)] \quad (i = 1, 2)$$

that $f_2(c_1), f_1(c_2) \in B$, whence $c_1 \in Y_2$ and $c_2 \in Y_1$, by (14). We get $c_1 \in X_1 \cap Y_2$ and $c_2 \in X_2 \cap Y_1$.

Case 2. $u \notin f_1(C)$. We know that $v \in f_1(C)$. Let w be the last point of the segment A_0 which belongs to $f_1(C)$, in the ordering $\leqslant_0$. The set $A_0 \cap f_1(C)$ is connected, $f_1(C)$ being a continuum. Thus, in this case, we have $w \in \overline{uv}$. Consequently, by the definition of w, each point of $f_1(C)$ whose distance from w exceeds $\frac{1}{2}$ must belong to B. Since $f_1(C) = f_2(C)$, there exist points $c_i' \in C$ such that $f_i(c_i') = w$ $(i = 1, 2)$. As in Case 1, we get $c_1' \in X_1 \cap Y_2$ and $c_2' \in X_2 \cap Y_1$, which completes the proof of (20).

The sets

$$M = P_1 \cup Q_2 \cup (X_2 \cap Y_1), \quad N = P_2 \cup Q_1 \cup (X_1 \cap Y_2)$$

are compact and non-empty, by (20). Also, we have

$$\begin{aligned} C = C \cap C &= f_1^{-1}(T) \cap f_2^{-1}(T) = f_1^{-1}(A_0 \cup B) \cap f_2^{-1}(A_0 \cup B) \\ &= (X_1 \cup Y_1) \cap (X_2 \cup Y_2) = (X_1 \cap X_2) \cup (X_1 \cap Y_2) \cup (X_2 \cap Y_1) \cup (Y_1 \cap Y_2) \\ &= P_1 \cup P_2 \cup (X_1 \cap Y_2) \cup (X_2 \cap Y_1) \cup Q_1 \cup Q_2 = M \cup N, \end{aligned}$$

by (14), (15) and (16). Since $A_0 \cap B = \{v\}$, it follows from (14) that $X_i \cap Y_i = V_i$ $(i = 1, 2)$. Hence

$$\begin{aligned} M \cap N &= (P_1 \cap Q_1) \cup (P_1 \cap X_1 \cap Y_2) \cup (Q_2 \cap P_2) \cup (Q_2 \cap X_1 \cap Y_2) \cup \\ &\quad \cup (P_2 \cap X_2 \cap Y_1) \cup (Q_1 \cap X_2 \cap Y_1) \cup (X_1 \cap X_2 \cap Y_1 \cap Y_2) \\ &\subset (P_1 \cap X_2 \cap Y_2) \cup (P_1 \cap X_2 \cap Y_2) \cup (Q_2 \cap Y_1 \cap X_1) \cup (Q_2 \cap Y_1 \cap X_1) \cup \\ &\quad \cup (P_2 \cap X_1 \cap Y_1) \cup (Q_1 \cap Y_2 \cap X_2) \cup (V_1 \cap V_2) \\ &= (P_1 \cap V_2) \cup (Q_2 \cap V_1) \cup (P_2 \cap V_1) \cup (Q_1 \cap V_2) \cup (V_1 \cap V_2) = \emptyset, \end{aligned}$$

by (15) and (16). This contradicts the assumption that C is a continuum. Condition (ii) is then proved, and so is 4.2.

4.3. *Let T be a tree which is not an arc. Let $v_1, \ldots, v_k$ be all the branch-points of T, and let n_i denote the ramification order of T at v_i, i.e. n_i is the number of components of $T \setminus \{v_i\}$ $(i = 1, \ldots, k)$. Denote*

$$m(T) = \Big[\Big(\sum_{i=1}^{k} n_i\Big) - (k+1)\Big]^{-1}.$$

Then there exists a simple triod $T' \subset T$ such that $m(T)w(T) \leqslant w(T')$ (see [9], p. 8).

Let us define separately $m(A) = 1$ for each arc A. Observe that if $T' \subset T$ are trees, then $m(T) \leqslant m(T')$. The next result follows from 4.1 and 4.3.

4.4. COROLLARY. *If T is a tree, then $m(T)w(T) \leqslant \sigma(T)$.*

Remarks. For some trees, the inequality in 4.4 provides a sharp estimation of the span. Indeed, the 4-od T from 4.2 has $m(T) = (4-2)^{-1} = \frac{1}{2}$ and $\sigma(T) = \frac{1}{2} = m(T)w(T)$. This estimation, however, is not the best one for all trees. We cite an interesting problem that seems to be important here. It is the following unsettled conjecture of Frances O. McDonald: is it true that each tree T which is not an arc contains a simple triod T' such that $w(T) \leqslant 2w(T')$? If the answer were "yes", we would get, by 4.1, the inequality $w(T) \leqslant 2\sigma(T)$ for all trees. Up to now, McDonald's conjecture has been proved for T being "n-ods" and $n = 4, 5, 6$ (see [9], p. 13). Thus, for all the 5-ods and 6-ods, it already provides a better estimation of the span than that given by 4.4.

4.5. EXAMPLE. *For each $\varepsilon > 0$, there exists a simple triod T on the plane R^2 such that $\sigma(T) = 1$ and $w(T') < \varepsilon$ for each simple triod $T' \subset T$.*

Proof. There exists an atriodic tree-like continuum $X \subset R^2$ such that $\sigma(X) > 0$ (see [3], pp. 100 and 106). Moreover, X is "triod-like" in the sense that there are finite open covers of X with the mesh arbitrarily small and with the nerve being a simple triod. These covers can be constructed by means of open disks on the plane, so that their nerves are embeddable in the unions of the disks (ibidem, see also [4], p. 76). It follows (compare 5.1 below) that there exists, for $n = 1, 2, \ldots$, a $(1/n)$-translation $f_n\colon X \to R^2$ such that $T_n = f_n(X)$ is a simple triod. Let $\gamma = \frac{1}{2}\varepsilon\sigma(X)$. We have

$$\lim_{n\to\infty} \sigma(T_n) = \sigma(X),$$

by 3.6, and therefore there is a positive integer n_0 such that $\frac{1}{2}\sigma(X) \leqslant \sigma(T_n)$ for $n \geqslant n_0$. We claim that there also exists an integer $n_1 \geqslant n_0$ such that $w(T') < \gamma$ for each simple triod $T'\subset T_{n_1}$. If it were not so, each T_n (where $n\geqslant n_0$) would contain a simple triod T_n' with $\gamma\leqslant w(T_n')$. Hence, by 2.4, there would exist arcs $A_{in}\subset T_n'\subset T_n$ $(i=0,1,2)$ such that $A_{0n}\cap A_{1n}\cap A_{2n}\neq\emptyset$ and none of the arcs A_{0n}, A_{1n}, A_{2n} is contained in the γ-neighbourhood of the union of the other two. All these arcs are contained in a bounded subset of the plane. Without loss of generality, we can assume that the three sequences of the arcs A_{in} $(i=0,1,2)$ converge to some three continua $C_i\subset X$ (when $n\to\infty$), respectively. Then also $C_0\cap C_1\cap C_2\neq\emptyset$ and none of the continua C_0, C_1, C_2 is contained in the union of the other two, which contradicts the fact that X is atriodic (see [10], p. 443). The existence of n_1 is thus proved.

Let $p\in R^2\setminus T_{n_1}$ be a point. We define an embedding $h\colon T_{n_1}\to R^2$ by taking $h(x)$ to be the point of the ray $\overrightarrow{px}$ $(x\in T_{n_1})$ such that

$$\operatorname{dist}[p,h(x)]=\operatorname{dist}(p,x)/\sigma(T_{n_1}),$$

whence $\operatorname{dist}[h(x),h(y)]=\operatorname{dist}(x,y)/\sigma(T_{n_1})$ for $x,y\in T_{n_1}$. The set $T=h(T_{n_1})$ is a simple triod with the span $\sigma(T)=\sigma(T_{n_1})/\sigma(T_{n_1})=1$. If $T'\subset T$ is a simple triod, then

$$w(T')=w[h^{-1}(T')]/\sigma(T_{n_1})<\gamma/\sigma(T_{n_1})\leqslant 2\gamma/\sigma(X)=\varepsilon.$$

5. The span of certain tree-like continua. The following well-known lemma establishes a relationship between finite open covers, their nerves and ε-translations.

5.1. Lemma. *Let X be a non-empty subset of the Hilbert space R^ω. If $\varepsilon>0$ and C is a finite open cover of X such that* $\operatorname{mesh}(C)<\varepsilon$, *then there exists a polyhedron $P\subset R^\omega$ contained in the ε-neighbourhood of X in R^ω and an ε-translation $\varkappa\colon X\to R^\omega$ such that P is topologically the nerve of C and $\varkappa(X)\subset P$. Moreover, if the nerve of C has dimension n, then R^ω can be replaced by the Euclidean $(2n+1)$-space* R^{2n+1} (see [5], pp. 319, 324 and 330).

Let Π be a collection of polyhedra. A compact metric space X is called *Π-like* provided, for each $\varepsilon>0$, there exists a finite open cover C of X such that $\operatorname{mesh}(C)<\varepsilon$ and the nerve of C, if non-degenerate, is topologically a polyhedron belonging to Π. Our next proposition involves the McDonald coefficient $m(T)$ as defined in 4.3.

5.2. *Let Π be a finite collection of trees and let X be a Π-like continuum. Denote*

$$m(\Pi)=\operatorname{Min}\{m(T)\colon T\in\Pi\}.$$

Then $m(\Pi)w(X)\leqslant\sigma(X)$.

Proof. We can assume X is non-degenerate; otherwise 5.2 states a trivial fact. Since then X is one-dimensional, it is embeddable in R^3. Let us also assume that $X\subset R^3$ and, moreover, that the space R^3 is remetrized so that the metric in R^3 is an extension of the given metric in X (see [2], p. 353). By 5.1, there exist trees $T_n\subset R^3$ and ε_n-translations $f_n\colon X\to R^3$ such that $\varepsilon_1,\varepsilon_2,\ldots$ converge to zero, T_n is topologically a member of Π and $f_n(X)\subset T_n$ $(n=1,2,\ldots)$. Since X is non-degenerate, all but a finite number of the sets $T_n'=f_n(X)$ are non-degenerate. Consequently, they are trees and the inclusions $T_n'\subset T_n$ imply the inequalities

$$m(\Pi)\leqslant m(T_n)\leqslant m(T_n').$$

It follows from 3.6 and 4.4 that

$$m(\Pi)w(X)=m(\Pi)\lim_{n\to\infty}w(T_n')=\lim_{n\to\infty}m(\Pi)w(T_n')$$
$$\leqslant\lim_{n\to\infty}m(T_n')w(T_n')\leqslant\lim_{n\to\infty}\sigma(T_n')=\sigma(X).$$

A continuum X is called *unicoherent* provided the common part of any two continua whose union is X is connected. We say that a continuum X is *P-unicoherent* provided there exists a collection Π of unicoherent connected polyhedra such that X is Π-like. If f is a mapping of a metric space X, we denote

$$\Delta_f=\{\operatorname{dist}(x,x')\colon f(x)=f(x'),\ x,x'\in X\}.$$

5.3. Theorem. *If $f\colon X\to Y$ is a continuous mapping of a P-unicoherent continuum X into an arc-like continuum Y, then*

$$[0,\sigma(X)]\subset\Delta_f.$$

Proof. Without loss of generality, let us assume that X is a subset of the Hilbert space R^ω and that the metric in R^ω is an extension of the given metric in X (see [2], p. 353). Also, since Y is one-dimensional (or degenerate), we can assume that $Y\subset R^3$. We are given a collection Π of unicoherent connected polyhedra such that X is Π-like. By 5.1, there exist polyhedra $P_m\subset R^\omega$ contained in the ε_m-neighbourhoods of X in R^ω, respectively, and ε_m-translations $f_m\colon X\to R^\omega$ such that $\varepsilon_1,\varepsilon_2,\ldots$ converge to zero, P_m is topologically a member of Π and $f_m(X)\subset P_m$ $(m=1,2,\ldots)$. Similarly, there exist arcs $A_n\subset R^3$ and η_n-translations $g_n\colon Y\to R^3$ such that $\eta_1,\eta_2,\ldots$ converge to zero and $g_n(Y)\subset A_n$ $(n=1,2,\ldots)$. Let $\Phi_n\colon R^\omega\to A_n$ $(n=1,2,\ldots)$ be a continuous extension of the mapping $g_nf\colon X\to A_n$ (see [6], p. 332).

Since P_m is a unicoherent locally connected continuum, the mapping $h=\Phi_n|P_m\colon P_m\to A_n$ fulfills the condition

$$[0,\sigma(P_m)]\subset\Delta_h$$

(see [8], p. 207). Given any number $\alpha\in[0,\sigma(X)]$, there exists a number $\alpha_m\in[0,\sigma(P_m)]$ such that

$$|\alpha-\alpha_m|\leqslant|\sigma(X)-\sigma(P_m)|,$$

and thus we also have $\alpha_m\in\Delta_h$. Consequently, there exist points $p_{mn}, p_{mn}'\in P_m$ with $\operatorname{dist}(p_{mn},p_{mn}')=\alpha_m$ and $h(p_{mn})=h(p_{mn}')$, whence

$$|\alpha-\operatorname{dist}(p_{mn},p_{mn}')|\leqslant|\sigma(X)-\sigma(P_m)|\quad(m,n=1,2,\ldots)\tag{21}$$

and

$$\Phi_n(p_{mn})=\Phi_n(p_{mn}')\quad(m,n=1,2,\ldots).\tag{22}$$

Let n be fixed for this part of the proof. Since the polyhedron P_m is contained in the ε_m-neighbourhood of X in R^ω, there exist points $x_{mn}, x_{mn}'\in X$ such that

$$\operatorname{dist}(p_{mn},x_{mn})<\varepsilon_m,\quad\operatorname{dist}(p_{mn}',x_{mn}')<\varepsilon_m\quad(m=1,2,\ldots),$$

whence

$$\lim_{m\to\infty} \operatorname{dist}(p_{mn}, x_{mn}) = \lim_{m\to\infty} \operatorname{dist}(p'_{mn}, x'_{mn}) = 0,$$

according to the assumption made about the sequence $\varepsilon_1, \varepsilon_2, \ldots$ By the compactness of X, there exists a sequence $m_1 < m_2 < \ldots$ of positive integers such that

$$\lim_{i\to\infty} x_{m_i n} = x_n, \quad \lim_{i\to\infty} x'_{m_i n} = x'_n,$$

where $x_n, x'_n \in X$. (Actually, the sequence $m_1, m_2, \ldots$ may depend on n, but this is irrelevant here, n being fixed.) Thus we also have

$$\lim_{i\to\infty} p_{m_i n} = x_n, \quad \lim_{i\to\infty} p'_{m_i n} = x'_n,$$

and therefore

$$g_n f(x_n) = \Phi_n(x_n) = \lim_{i\to\infty} \Phi_n(p_{m_i n}) = \lim_{i\to\infty} \Phi_n(p'_{m_i n}) = \Phi_n(x'_n) = g_n f(x'_n),$$

by (22). Moreover, it follows from 3.1 that

$$\limsup_{i\to\infty} \sigma(P_{m_i}) \leqslant \sigma(X),$$

and 3.2 implies

$$\sigma(X) \leqslant \liminf_{i\to\infty} \sigma[f_{m_i}(X)] \leqslant \liminf_{i\to\infty} \sigma(P_{m_i}),$$

since $f_{m_i}\colon X \to R^\infty$ is an ε_{m_i}-translation and $f_{m_i}(X) \subset P_{m_i}$ $(i = 1, 2, \ldots)$. As a result, we obtain

$$\lim_{i\to\infty} \sigma(P_{m_i}) = \sigma(X),$$

and condition (21) implies that

$$\operatorname{dist}(x_n, x'_n) = \lim_{i\to\infty} \operatorname{dist}(p_{m_i n}, p'_{m_i n}) = \alpha.$$

Again, X being compact, there exists a sequence $n_1 < n_2 < \ldots$ of positive integers such that

$$\lim_{j\to\infty} x_{n_j} = x, \quad \lim_{j\to\infty} x'_{n_j} = x',$$

where $x, x' \in X$. Consequently, we get

$$\operatorname{dist}(x, x') = \lim_{j\to\infty} \operatorname{dist}(x_{n_j}, x'_{n_j}) = \alpha$$

and $g_{n_j} f(x_{n_j}) = g_{n_j} f(x'_{n_j})$ for $j = 1, 2, \ldots$ But the mapping g_{n_j} is an η_{n_j}-translation, whence

$$\operatorname{dist}[f(x_{n_j}), f(x'_{n_j})] < 2\eta_{n_j} \quad (j = 1, 2, \ldots),$$

by (10). The latter inequality yields

$$f(x) = \lim_{j\to\infty} f(x_{n_j}) = \lim_{j\to\infty} f(x'_{n_j}) = f(x'),$$

so that $\alpha = \operatorname{dist}(x, x') \in \Delta_f$. Since α was an arbitrary number of the interval $[0, \sigma(X)]$, we have shown that this interval is a subset of Δ_f.

5.4. Corollary. *If Π is a finite collection of trees and $f\colon X \to Y$ is a continuous mapping of a Π-like continuum X into an arc-like continuum Y, then*

$$[0, m(\Pi) w(X)] \subset \Delta_f.$$

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