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Whitney properties

by

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Abstract. Some properties and structure of the "levels" $\mu^{-1}(t)$ are investigated, where X is a (metric) continuum and μ is a Whitney function for the space of all nonempty subcontinua of X.

1. Introduction. By a *continuum* we mean a nonempty compact connected metric space. The letter X will always denote a continuum. By the hyperspace of X we mean $C(X) = \{A: A \text{ is a (nonempty) subcontinuum of } X\}$ with the Hausdorff metric H [5]. In [17], in another context, Whitney defined a function $\mu: C(X)\to[0, \infty)$ satisfying

(1.1) μ is continuous on $C(X)$;

(1.2) if $A\subset B$ and $A \neq B$, then $\mu(A)<\mu(B)$;

(1.3) $\mu(\{x\}) = 0$ for each $x\in X$.

We will call any function from $C(X)$ to $[0,\infty)$ satisfying (1.1) through (1.3) a *Whitney map for* $C(X)$, and denote any such map by the symbol μ. Kelley [5] was the first person to introduce Whitney's function into the study of $C(X)$. The first explicit work done after Kelley on the nature of the sets $\mu^{-1}(t)$ was done in [3] where it was shown, among other results, that μ is both monotone and open. Next in [6] several results on the topological type of the sets $\mu^{-1}(t)$ were obtained. The next paper concerning the sets $\mu^{-1}(t)$ was [12]. Several papers on Whitney maps have recently been written (see our bibliography).

Let P be a topological property. We say that P is a *Whitney property* provided whenever X has property P, so does $\mu^{-1}(t)$ for any Whitney map μ for $C(X)$ and each $t<\mu(X)$. The purpose of this paper is to continue the work mentioned above. We give some general results about the levels $\mu^{-1}(t)$ (see, for example, 3.1 and 5.1) and some specific facts about $\mu^{-1}(t)$ for certain classes of continua (see, for example, 3.4, 3.5, and 4.4). Our results and examples show for many properties whether or not they are Whitney properties.

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We summarize results about whether or not a given property is a Whitney property in the following table. Results from this paper which appear in the table are listed according to how they are numbered in this manuscript. The column entitled "Primary Reference" gives the paper (resp., papers) where the result *first* (resp., simultaneously) appeared. The column entitled "Secondary Reference" lists papers where a simpler proof or a more general result is obtained. In all cases, when a theorem in this paper is listed as a secondary reference the result obtained here is more general than the result listed as the corresponding primary reference. The results in this paper were originally obtained without the authors' awareness of [15].

Topological properties of X	Primary reference Whitney property		Secondary reference
	Yes	No	
continuum	[3]		
arc	[6]		[12]
simple closed curve	[6]		[12]
pseudo-arc	[3]		
solenoid	Th. 4.5		
pseudo-solenoid	[15] [8]		
locally connected	[12]		
arcwise connected	[12]		
aposyndetic	[13]		
decomposable	Th. 3.4		
indecomposable		[15]	Th. 5.1, Ex. 5.4
hereditarily indecomposable	[5]		
unicoherent		[15]	Th. 5.1, Ex. 5.4
non-unicoherence		Ex. 5.5	
chainable	[6]		
indecomposable chainable	Th. 4.3		
circle-like		[15]	Th. 5.2
planar proper circle-like	[15] [8]		
non-planar circle-like	[15] [8]		
proper circle-like	[6]		
indecomposable circle-like		[15]	Th. 5.1, Ex. 5.4
hereditarily indecomposable tree-like	[7]		
fixed point property		Ex. 5.6	
cyclic		[15]	Ex. 5.5

In the last section we state several problems.

2. Terminology and notation. In this section we collect the notation and terminology used in this paper.

The continuum X is a point of $C(X)$ and it is called the *top* of $C(X)$. By the *base* of $C(X)$ we mean the collection of all one-point sets. It is denoted by $\hat{X}$ and is

isometric to X. If $A, B \in C(X)$ and $A \subset B$, then there exists a maximal collection of continua between A and B. Any such collection will be denoted by AB and called a *segment* between A and B. The existence of segments is proved in [2]. If $A \neq B$, then AB is an arc (any Whitney map on $C(X)$ defines a homeomorphism between AB and an arc). Hence, the segment AB can be parametrized in the sense that there exists a homeomorphism (or a constant map provided $A = B$) $\sigma\colon [0, 1] \to AB$ such that $\sigma(0) = A$. Any such map will also be called a *segment*. If in addition σ satisfies

$$\mu(\sigma(t)) = (1-t)\mu(A) + t\cdot\mu(B)\,,$$

then σ is called a *segment in the sense of Kelley from A to B* [5]. If $a \in X$, then

$$X(a) = \{A \in C(X)\colon a \in A\}\,.$$

More generally, if $A \in C(X)$, then

$$X(A) = \bigcup_{a \in A} X(a)\,.$$

Hence, $X(A)$ is the collection of elements of $C(X)$ each of which intersects A. If μ is a fixed Whitney map on $C(X)$, then we denote

$$X(A, \mu, t) = X(A) \cap \mu^{-1}(t)\,.$$

In case $A = \{a\}$ we simply write $X(a, \mu, t)$ instead of $X(\{a\}, \mu, t)$. We will also use the symbol

$$X[A] = \{B \in C(X)\colon A \subset B\}\,.$$

Notice that the sets $X(a)$ and $X[\{a\}]$ coincide.

We say that a continuum $A \in C(X)$ is *terminal in X* if for every two continua $B, C \in C(X)$ each of which contains A we have either $B \subset C$ or $C \subset B$ [4]. If A is a one-point set $\{a\}$, then we simply say that a is *terminal in X*. Observe that the continuum A is terminal in X if and only if $X[A]$ is a segment in $C(X)$. Hence, if a is a terminal point of X, $X(a, \mu, t)$ is a one-point set for each $0 \leqslant t \leqslant \mu(X)$.

Any (continuous) map $f\colon X \to Y$ into a continuum Y gives rise to a (continuous) map from $C(X)$ into $C(Y)$ which sends $A \in C(X)$ to the image of A under f. This map is denoted by $\hat{f}$ and called the *map induced by f*.

2.1. PROPOSITION. *Let $\hat{f}\colon C(X) \to C(Y)$ be the map induced by $f\colon X \to Y$. Assume*

$$\dim\{x \in X\colon \{x\} \neq f^{-1}f(x)\} \leqslant 0\,.$$

Then the following are true:

(i) *$\hat{f}$ embeds $C(X)\setminus\hat{X}$ into $C(Y)\setminus\hat{Y}$,*

(ii) *if $\mu\colon C(Y) \to [0, \infty)$ is a Whitney map, then $\mu\circ\hat{f}\colon C(X) \to [0, \infty)$ is also a Whitney map.*

Proof. If $A, B \in C(X)$, $A \notin \hat{X}$ and $A \setminus B \neq \emptyset$, then there exists a nondegenerate continuum $C \subset A \setminus B$. By the assumption about dimension, $f(C) \in C(Y)\setminus\hat{Y}$. Using this it is easy to complete the argument for 2.1.

We denote the unit closed interval [0, 1] by I and the unit circle in the complex plane by S^1, i.e., $S^1 = \{z \in C\colon |z| = 1\}$. A map $f\colon X \to Y$ is said to be an ε-mapping provided $\operatorname{diam} f^{-1}(y) < \varepsilon$ for each $y \in f(X)$. We say that a continuum X is *chainable* or *snake-like* (resp. *circle-like*) if for each $\varepsilon > 0$ there exists an ε-mapping from X onto I (resp., S^1). We say that a circle-like continuum X is *proper circle-like* [6] if there exists a map $f\colon X \to S^1$ which is not homotopic to a constant map. By $\bar{A}$ or $\operatorname{cl}(A)$ we mean the closure of A, and by $\operatorname{Fr}_Y(A)$ we mean $A \cap [\overline{Y-A}] \cap Y$.

3. Fundamental properties of some subsets of $C(X)$. In this section we prove a general result (3.1) about mapping simplexes into $C(X)$. One of the most important corollaries of this result is 3.2 which gives insight into the structure of the sets $X(A, \mu, t)$. Other results include the fact that decomposability is a Whitney property (3.4) and, furthermore, that $\mu^{-1}(t)$ is arcwise connected for t close to $\mu(X)$ when X is decomposable. First we give some notational conventions.

Let l_2 denote the Hilbert space of all square summable sequences $(x_1, x_2, \ldots, x_n, \ldots)$ of real numbers x_n. For each n we consider Euclidean n-space R^n as canonically embedded in l_2 so that $(x_1, x_2, \ldots, x_n) \in R^n$ is associated with $(x_1, x_2, \ldots, x_n, 0, 0, \ldots) \in l_2$. Thus, we frequently write $(x_1, x_2, \ldots, x_n)$ to mean $(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$. Let 0 denote $(0, 0, \ldots, 0, \ldots)$ and, for each $n = 1, 2, \ldots$, let e_n denote the point of l_2 whose ith-coordinate is zero if $i \neq n$ and one if $i = n$. If $S \subset l_2$, then let $\operatorname{conv}(S)$ denote the closed convex hull of S. For each $n = 1, 2, \ldots$, let $\Sigma^n = \operatorname{conv}\{0, e_1, \ldots, e_n\}$ and, for each $s \in [0, 1]$, let $\Sigma^n_s = \operatorname{conv}\{se_1, se_2, \ldots, se_n\}$. Note that $\Sigma^1 \subset \Sigma^2 \subset \ldots \subset \Sigma^n \subset \ldots$

The following theorem is a generalization of Lemma 1 of [12] and is related to Theorem 1 of [11].

3.1. Theorem. *Let* $t_0 \in [0, \mu(X)]$. *Let* $A_1, A_2, \ldots, A_n \in \mu^{-1}(t_0)$ *such that* $\bigcap_{i=1}^{n} A_i \neq \emptyset$, *and let K be a subcontinuum of* $\bigcap_{i=1}^{n} A_i$. *Then there is a continuous function* $f_n\colon \Sigma^n \to C(X)$ *such that*

(1) $$K \subset f_n(p) \subset \bigcup_{i=1}^{n} A_i \quad \text{for each } p \in \Sigma^n,$$

(2) $$f_n(e_i) = A_i \quad \text{for each } i = 1, 2, \ldots, n,$$

(3) $$f_n(0) = K,$$

(4) $$f_n[\Sigma^n_s] \subset \mu^{-1}[(1-s)\mu(K) + st_0] \quad \text{for each } s \in [0, 1],$$

(5) *if* $A_i \neq K = A_i \cap A_j$ *for all* $i, j = 1, 2, \ldots, n$ *with* $i \neq j$, *then* f_n *is a homeomorphism.*

Proof. The proof is by induction on n. Let $n = 1$. Let $\sigma\colon [0, 1] \to C(X)$ be a segment in the sense of Kelley from K to A_1. Let $f_1 = \sigma$. Clearly (1) through (5) hold for f_1. Now, assume inductively that we have defined $f_{n-1}\colon \Sigma^{n-1} \to C(X)$ such that (1) through (5) hold $(n-1 \geqslant 1)$. We define $f_n\colon \Sigma^n \to C(X)$ as follows. Let $\sigma\colon [0, 1] \to C(X)$ be a segment from K to A_n. Now, let $x = (x_1, x_2, \ldots, x_n) \in \Sigma^n$ be fixed and let

$x' = (x_1, x_2, \ldots, x_{n-1}) \in \Sigma^{n-1}$. Then, $x \in \Sigma^n_s$ for $s = \sum_{i=1}^{n} x_i$ and $x' \in \Sigma^{n-1}_{s'}$ for $s' = \sum_{i=1}^{n-1} x_i$. Now, by (4) of the inductive assumption,

$$\mu[f_{n-1}(x')] = (1-s')\mu(K) + s' t_0 .$$

Thus, since $s' \leqslant s$ and $t_0 \geqslant \mu(K)$,

$$\mu[f_{n-1}(x')] \leqslant (1-s)\mu(K) + st_0 .$$

Therefore, since $\mu(A_n) = t_0$ and by (1) of the inductive assumption $f_{n-1}(x') \supset K$, there exists $r_x \in [0, 1]$ (r_x not necessarily unique) such that

$$\mu[\sigma(r_x) \cup f_{n-1}(x')] = (1-s)\mu(K) + st_0 .$$

We define: $f_n(x) = \sigma(r_x) \cup f_{n-1}(x')$. By defining f_n at each point of Σ^n by this procedure, we obtain a function from Σ^n into $C(X)$. It can be shown, by a technique used in the proof of Lemma 1 of [12], that f_n is continuous on each Σ^n_s. It then follows easily that f_n is continuous on Σ^n. From the way we defined f_n, (4) obviously holds for f_n. Since (1) holds for f_{n-1} and since each value of f_n contains (as a subset) a value of f_{n-1}, (1) holds for f_n. Now observe that if $x = (x_1, x_2, \ldots, x_n) \in \Sigma^{n-1} \subset \Sigma^n$ (i.e., $x_n = 0$), then $\mu[f_{n-1}(x)] = \mu[f_n(x)]$ (because, in the notation above, $s = s'$). Therefore, since $f_n(x) \supset f_{n-1}(x)$, we have that $f_n(x) = f_{n-1}(x)$. This proves that f_n is an extension of f_{n-1}. Thus, since (3) holds for f_{n-1}, (3) holds for f_n; also, since (2) holds for f_{n-1}, (2) holds for f_n when $i \leqslant n-1$. To see that (2) holds when $i = n$, first note that $e_n \in \Sigma^n_1$. Thus, since (4) holds for f_n, $\mu[f_n(e_n)] = t_0$. Since (3) holds for f_{n-1},

$$f_n(e_n) = \sigma(r_{e_n}) \cup f_{n-1}(0) = \sigma(r_{e_n}) \cup K = \sigma(r_{e_n}) .$$

Hence, $\mu[\sigma(r_{e_n})] = \mu[f_n(e_n)] = t_0$ from which we conclude that $\sigma(r_{e_n}) = A_n$. Therefore, $f_n(e_n) = A_n$. This proves (2) holds for f_n (when $i = n$). To prove (5) holds for f_n, assume $A_i \neq K = A_i \cap A_j$ for all $i, j = 1, 2, \ldots, n$ with $i \neq j$. Let $x = (x_1, x_2, \ldots, x_n) \in \Sigma^n$ and let $y = (y_1, y_2, \ldots, y_n) \in \Sigma^n$ such that $x \neq y$. Let $x' = (x_1, x_2, \ldots, x_{n-1})$ and let $y' = (y_1, y_2, \ldots, y_{n-1})$. Assume $x' \neq y'$. Then, since f_{n-1} is a homeomorphism, $f_{n-1}(x') \neq f_{n-1}(y')$. Without loss of generality assume there is a point $w \in [f_{n-1}(x') - f_{n-1}(y')]$. Since (1) holds for f_{n-1}, $w \in [\bigcup_{i=1}^{n-1} A_i - K]$. Thus, since $A_n \cap [\bigcup_{i=1}^{n-1} A_i] = K$, $w \notin A_n$. Therefore, since $w \notin f_{n-1}(y')$ and $f_n(y) \subset A_n \cup f_{n-1}(y')$, we have that $w \notin f_n(y)$. Hence, $w \in [f_n(x) - f_n(y)]$ so $f_n(x) \neq f_n(y)$. Next, assume $x' = y'$. Then, since $x \neq y$, we have $x \in \Sigma^n_{s_1}$ and $y \in \Sigma^n_{s_2}$ with $s_1 \neq s_2$. Since (4) holds for f_n, $\mu[f_n(x)] = (1-s_1)\mu(K) + s_1 t_0$ and $\mu[f_n(y)] = (1-s_2)\mu(K) + s_2 t_0$. Now, K is a proper subcontinuum of each A_i and, thus, $\mu(K) \neq t_0$. It follows from this that if $\mu[f_n(x)] = \mu[f_n(y)]$, then $s_1 = s_2$. Therefore, $\mu[f_n(x)] \neq \mu[f_n(y)]$ and $f_n(x) \neq f_n(y)$. We have now shown that f is one-to-one, hence a homeomorphism.

3.2. Corollary. *If A is a subcontinuum of X, then $X(A, \mu, t)$ is a subcontinuum of $\mu^{-1}(t)$ whenever $0 \leqslant t \leqslant \mu(X)$. Furthermore*:

(1) *If $\mu(A) \leqslant t$, then $X(A, \mu, t)$ is arcwise connected (perhaps degenerate).*

(2) *If $\mu(A) > t$, then $\Lambda = \mu^{-1}(t) \cap C(A)$ is a subcontinuum of $X(A, \mu, t)$ and each member of $X(A, \mu, t) \setminus \Lambda$ can be "joined" to a member of Λ by an arc $\alpha \subset X(A, \mu, t)$.*

Proof. From the continuity of μ and properties of convergence in $C(X)$ it is easy to see that $X(A, \mu, t)$ is compact for any $t \in [0, \mu(X)]$. To prove (1), assume $\mu(A) \leqslant t_0$, t_0 fixed. By using a segment in $C(X)$ from A to X (see 2.3 of [5]) we produce $A_1 \in X(A, \mu, t_0)$ such that $A_1 \supset A$. Now, let $A_2 \in X(A, \mu, t_0)$. Then there is a point $a \in [A_2 \cap A]$. Letting $K = \{a\}$, we see that A_1, A_2, and K satisfy the hypotheses of 3.1. Hence, there is a continuous function $f_2\colon \Sigma^2 \to C(X)$ satisfying (1) through (4) of 3.1. By the first containment in (1), $f_2(p) \cap A \neq \emptyset$ for each $p \in \Sigma^2$. Hence, by (4) with $s = 1$, we have that $f_2[\Sigma_1^2] \subset X(A, \mu, t_0)$. Thus, since (by (2)) $f_2(e_1) = A_1$ and $f(e_2) = A_2$, it follows that there is an arc in $X(A, \mu, t_0)$ from A_2 to A_1. Therefore, since A_2 was an arbitrary member of $X(A, \mu, t_0)$, we have proved (1). Next, we prove (2). Assume $\mu(A) > t_0$, t_0 fixed. By 1.1 of [7], $\Lambda = \mu^{-1}(t_0) \cap C(A)$ is a subcontinuum of $X(A, \mu, t_0)$. Let $A_1 \in X(A, \mu, t_0) \setminus \Lambda$. Then, since $A_1 \cap A \neq \emptyset$ and $\bigcup \Lambda = A$, there exists $A_2 \in \Lambda$ such that $A_1 \cap A_2 \neq \emptyset$. Let $a \in [A_1 \cap A_2]$ and let $K = \{a\}$. The proof of (2) may now be completed by using 3.1 as we just did in the proof of (1).

3.3. Corollary. *If X contains an n-odd, then there exists $t_0 \in [0, \mu(X)]$ such that $\mu^{-1}(t_0)$ contains an $(n-1)$-cell.*

Proof. Let $M \subset X$ be an n-odd. Then, by definition [1], there is a subcontinuum K of M such that $M \setminus K$ is the union of n (nonempty) mutually separated sets $S_1, S_2, \ldots, S_n$. Let $B_i = S_i \cup K$ for each $i = 1, 2, \ldots, n$. Let

$$t_0 = \min\{\mu(B_i)\colon i = 1, 2, \ldots, n\}.$$

For each $i = 1, 2, \ldots, n$, let A_i be a subcontinuum of B_i such that $\mu(A_i) = t_0$ (such continua A_i exist by 2.3 of [5]). Let $f_n\colon \Sigma^n \to C(X)$ be as guaranteed by 3.1. Then, since (5) is satisfied by $K, A_1, A_2, \ldots, A_n$, f_n is a homeomorphism. The result now follows by using (4) which guarantees that $f_n(\Sigma_1^n) \subset \mu^{-1}(t_0)$.

3.4. Theorem. *Decomposability is a Whitney property.*

Proof. Let X be a decomposable continuum and let t_0 be fixed, $0 \leqslant t_0 < \mu(X)$. Let A and B be proper subcontinua of X such that $X = A \cup B$. Clearly, $\mu^{-1}(t_0) = X(A, \mu, t_0) \cup X(B, \mu, t_0)$ and, by 3.2, $X(A, \mu, t_0)$ and $X(B, \mu, t_0)$ are each subcontinua of $\mu^{-1}(t_0)$. Thus, if $X(A, \mu, t_0) \neq \mu^{-1}(t_0) \neq X(B, \mu, t_0)$, we are done. So, for the purpose of proof, assume that $X(A, \mu, t_0) = \mu^{-1}(t_0)$. Then, if $\mu(A) \leqslant t_0$, we have by (1) of 3.2 that $X(A, \mu, t_0)$ is arcwise connected, hence decomposable [9, p. 213]. Now, assume $\mu(A) > t_0$. Let $\Lambda = \mu^{-1}(t_0) \cap C(A)$. Since $A \neq X$, $\Lambda \neq \mu^{-1}(t_0)$. It now follows from (2) of 3.2 and from [9, p. 213] that $\mu^{-1}(t_0)$ is decomposable.

The following theorem provides as a corollary a partial solution to a question first posed in [12] and later in [15].

3.5. Theorem. *If X is a decomposable continuum, then there exists $t_a < \mu(X)$ such that $\mu^{-1}(t)$ is arcwise connected for each $t \geqslant t_a$.*

Proof. Let A and B be proper subcontinua of X such that $X = A \cup B$. Let $t_a = \max\{\mu(A), \mu(B)\}$. Let $t \geqslant t_a$ be fixed. Assume, for the purpose of proof, that $t_a = \mu(A)$. Then, since $\mu(B) \leqslant t$ and $A \cap B \neq \emptyset$, it follows that $X(A, \mu, t) = \mu^{-1}(t)$. Therefore, by 3.2, $\mu^{-1}(t)$ is arcwise connected.

The following corollary extends Theorem 5 of [12].

3.6. Corollary. *If X is a finite-dimensional decomposable continuum such that $\mu^{-1}(t)$ is homeomorphic to X for all $t < \mu(X)$, then X is an arc or a simple closed curve.*

Proof. From 3.5 above, we have that $\mu^{-1}(t)$ is arcwise connected for some t. Thus, since $\mu^{-1}(t)$ is homeomorphic to X, X is arcwise connected. Now, by Theorem 5 of [12], X must be an arc or a simple closed curve.

4. Whitney maps for snake-like and circle continua. In this section we give results about the hyperspaces of snake-like and circle-like continua. Theorem 4.3 answers a question posed in [15] and the next theorem, 4.4, gives more information than Theorem 5.1 of [15]. We then give some result for solenoids.

The following lemma is 2.3 of [6].

4.1. Lemma. *Let $t_0 \in [0, \mu(X)]$. Given $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon) > 0$ such that if $A, B \in \mu^{-1}(t_0)$ and $B \subset N(\eta, A)$, then $H(A, B) < \varepsilon$.*

4.2. Theorem. *Assume X is a chainable continuum and let $t_0 \in [0, \mu(X)]$. If Λ is a subcontinuum of $\mu^{-1}(t_0)$ such that $\bigcup \Lambda = X$, then $\Lambda = \mu^{-1}(t_0)$.*

Proof. Let Λ be a subcontinuum of $\mu^{-1}(t_0)$ such that $\bigcup \Lambda = X$. Let $A \in \mu^{-1}(t_0)$. We show $A \in \Lambda$. To do this let $\varepsilon > 0$. Choose $\eta = \eta(\varepsilon)$ so as to satisfy 4.1 above. Let $\{U_1, \ldots, U_n\}$ be an η-chain of open subsets of X covering X. Let

$$m = \min\{i\colon U_i \cap A \neq \emptyset\}$$

and let

$$s = \max\{i\colon U_i \cap A \neq \emptyset\}.$$

Now, let

$$\Lambda_1 = \{L \in \Lambda\colon L \cap U_i \neq \emptyset \text{ for some } i \leqslant m\}$$

and let

$$\Lambda_2 = \{L \in \Lambda\colon L \cap U_i \neq \emptyset \text{ for some } i \geqslant s\}.$$

Clearly, Λ_1 and Λ_2 are each open subsets of Λ. Since $\bigcup \Lambda = X$, $\Lambda_1 \neq \emptyset \neq \Lambda_2$.

First: Assume $\Lambda_1 \cup \Lambda_2 \neq \Lambda$. Then there exists $L_0 \in \Lambda$ such that $L_0 \subset \bigcup_{i=m+1}^{s-1} U_i$. Since X is chainable, $\{U_1, \ldots, U_n\}$ is a chain, and A is a subcontinuum of X such that

$A \cap U_m \neq \emptyset \neq A \cap U_s$, we have that $A \cap U_i \neq \emptyset$ for all $i = m+1, m+2, \ldots, s-1$. Hence, since $L_0 \subset \bigcup_{i=m+1}^{s-1} U_i$ and $\operatorname{diam}[U_i] < \eta$, $L_0 \subset N(\eta, A)$. Thus, by 4.1, $H(A, L_0) < \varepsilon$.

Second: Assume $A_1 \cup A_2 = A$. Then, since A is a continuum and each of A_1 and A_2 is a nonempty open subset of A, $A_1 \cap A_2 \neq \emptyset$. Let $K_0 \in [A_1 \cap A_2]$. Let $i_0 \leqslant m$ and $j_0 \geqslant s$ be such that $K_0 \cap U_{i_0} \neq \emptyset \neq K_0 \cap U_{j_0}$. Thus, by properties of chains used above, $K_0 \cap U_i \neq \emptyset$ for all $i = i_0, i_0+1, \ldots, j_0$. Now, $A \subset \bigcup_{i=m}^{s} U_i$ and therefore it follows easily that $A \subset N(\eta, K_0)$. Thus, by 4.1, $H(K_0, A) < \varepsilon$.

We have now shown that for any $\varepsilon > 0$, there is a member of $\mathcal{A}$ less than ε from A. Therefore, by compactness of $\mathcal{A}$, $A \in \mathcal{A}$.

4.3. Theorem. *The property of being an indecomposable chainable continuum is a Whitney property.*

Proof. Let $t_0 \in [0, \mu(X)]$. Let Γ_1 and Γ_2 be subcontinua of $\mu^{-1}(t_0)$ such that $\Gamma_1 \cup \Gamma_2 = \mu^{-1}(t_0)$. Let $X_1 = \bigcup \Gamma_1$ and let $X_2 = \bigcup \Gamma_2$. By 1.2 of [5], X_1 and X_2 are each subcontinua of X. Also,

$$X_1 \cup X_2 = [\bigcup \Gamma_1] \cup [\bigcup \Gamma_2] = \bigcup \mu^{-1}(t_0) = X.$$

Therefore, since X is indecomposable, $X_1 = X$ or $X_2 = X$. If $X_i = X$ then, by 4.2 $\Gamma_i = \mu^{-1}(t_0)$, $i = 1$ or 2. This proves $\mu^{-1}(t_0)$ is indecomposable. Its chainability is 6.2 (a) of [6].

4.4. Theorem. (a) *If X is a decomposable snake-like continuum, then there exists $t_0 < \mu(X)$ such that $\mu^{-1}(t)$ is an arc for each $t_0 \leqslant t < \mu(X)$.*

(b) *If X is a decomposable proper circle-like continuum, then there exists $t_0 < \mu(X)$ such that $\mu^{-1}(t)$ is a circle for each $t_0 \leqslant t < \mu(X)$.*

Proof. (a) By 3.5 there exists $t_0 < \mu(X)$ such that $\mu^{-1}(t)$ is arcwise-connected for each $t_0 \leqslant t$. Since $\mu^{-1}(t)$ is always snake-like [6] and each snake-like continuum is irreducible [10], $\mu^{-1}(t)$ is an arc.

(b) There exist two continua $A, B \in C(X)$ such that

$$X = A \cup B \quad \text{and} \quad A \neq X \neq B.$$

Since no proper subcontinuum of X separates X (this follows from [9, p. 435] and 3.2 of [6]) we can assume that

(1) $$A = \overline{X \setminus B} \quad \text{and} \quad B = \overline{X \setminus A} \quad \text{(see [9, p. 422])}.$$

Also, by [9, p. 435] and 3.2 of [6], $A \cap B$ is not connected. Hence

$$A \cap B = M \cup N$$

where M, N are nonempty closed disjoint sets. It follows that M and N are continua, for otherwise X would contain a triod, which is impossible [1]. Let

$$t_0 = \max\{\mu(A), \mu(B)\}$$

and let $t_0 \leqslant t < \mu(X)$. It follows that

(2) $$\mu^{-1}(t) = X(M, \mu, t) \cup X(N, \mu, t)$$

and each of these continua is arcwise connected by 3.2. We claim that

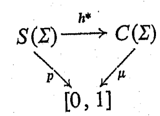
(3) $$X(M, \mu, t) \neq \mu^{-1}(t) \neq X(N, \mu, t).$$

Observe that A and B are irreducible between M and N, for otherwise, by (1), X would contain a proper non-unicoherent continuum. Let σ_A be a segment in $C(A)$ between M and A and let σ_B be a segment in $C(B)$ between M and B [5]. It follows that $\sigma\colon I \to C(X)$ given by $\sigma(s) = \sigma_A(s) \cup \sigma_B(s)$ is a segment in $C(X)$ between M and X. By the irreducibility, $\sigma(s) \cap N = \emptyset$ for each $s < 1$. Since $\mu(\sigma(0)) = \mu(M) < t_0 \leqslant t < \mu(X)$, there exists $s_0 < 1$ such that $\sigma(s_0) \in \mu^{-1}(t)$. Clearly, $\sigma(s_0) \in X(M, \mu, t) \setminus X(N, \mu, t)$, which proves that $X(N, \mu, t) \neq \mu^{-1}(t)$. The other inequality in (3) is proved analogously.

Since $\mu^{-1}(t)$ is a circle-like continuum [6], by (3) we see that $X(M, \mu, t)$ and $X(N, \mu, t)$ are snake-like arcwise connected continua. So, they are arcs. Hence by (2) $\mu^{-1}(t)$ is a union of two arcs and, therefore, is a circle (because it is a proper circle-like continuum [6]).

In [14], Rogers showed that the cone over a solenoid is homeomorphic to the hyperspace of the solenoid. In the following theorem we give a much better behaved homeomorphism which shows that "levels" can *all* be preserved with the same homeomorphism.

4.5. Theorem. *The property of being a solenoid is a Whitney property. More particularly, given any solenoid Σ and any Whitney map μ for $C(\Sigma)$ such that $\mu(\Sigma) = 1$, there is a homeomorphism h^* from the cone over Σ, $S(\Sigma)$, onto $C(\Sigma)$ such that for the standard projection $p\colon S(\Sigma) \to [0, 1]$ given by $p(x, t) = t$ the following diagram commutes*



Furthermore, for each $x \in \Sigma$ and each $t < t'$, $h^*((x, t)) \subset h^*(x, t')$.

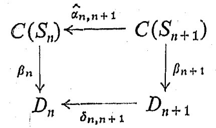
Proof. We prove the theorem only for the case of the dyadic solenoid. Let X be the dyadic solenoid, i.e., $X = \varprojlim \{S_n, \alpha_{nm}\}$ where $S_n = S^1 = \{z \in \mathbf{C}\colon |z| = 1\}$ and $\alpha_{n,n+1}(z) = z^2$ for each $z \in S_{n+1}$. Let $\sigma_n\colon X \to S_n$ denote projection for each n. Define $\alpha\colon C(X) \to \varprojlim \{C(S_n), \hat{\alpha}_{nm}\} = Y$ by letting $\alpha(A) = (\hat{\sigma}_1(A), \hat{\sigma}_2(A), \ldots, \hat{\sigma}_n(A), \ldots)$ for each $A \in C(X)$. It is easy to see that α is a homeomorphism of $C(X)$ onto Y; in fact, it is the inverse of Segal's homeomorphism [16]. Let $D_n = \{z \in \mathbf{C}\colon |z| \leqslant 1\}$ for each n. Let $p\colon R^1 \to S^1$ be given by $p(t) = e^{2\pi i t}$ for $t \in R^1$. Note that if $A \in [C(S_n) - \{S_n\}]$, then there are numbers $t_1 \leqslant t_2$ such that $p([t_1, t_2]) = A$; also if $p([t_1', t_2']) = A$, then $p(\frac{1}{2}(t_1 + t_2)) = p(\frac{1}{2}(t_1' + t_2'))$ and $|t_1 - t_2| = |t_1' - t_2'|$. For each n, let $\beta_n\colon C(S_n) \to D_n$ be given by

$$\beta_n(A) = [\max\{0, 1 - |t_1 - t_2|\}]\, p(\tfrac{1}{2}(t_1 + t_2)),$$

where t_1 and t_2 satisfy $p([t_1, t_2]) = A$ when $A \neq S_n$. It is easy to see that β_n is a homeomorphism of $C(S_n)$ onto D_n. For each n, let $\delta_{n,n+1}\colon D_{n+1}\to D_n$ be given by

$$\delta_{n,n+1}(z) = \begin{cases} 0 & \text{for } 0\leqslant|z|\leqslant\frac{1}{2}, \\ [2|z|-1][z/|z|]^2 & \text{for } \frac{1}{2}\leqslant|z|\leqslant 1. \end{cases}$$

If follows that the diagram



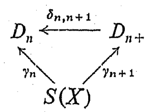
commutes for each n. Let $Z = \varprojlim\{D_n, \delta_{nm}\}$. Let $\beta\colon Y\to Z$ be the natural homeomorphism, i.e.,

$$\beta(A_1, A_2, \ldots, A_n, \ldots) = (\beta_1(A_1), \beta_2(A_2), \ldots, \beta_n(A_n), \ldots).$$

Next, let $I_n = [0, 1] = I$ for each n. Let $\varkappa_{n,n+1}\colon I_{n+1}\to I_n$ be given by $\varkappa_{n,n+1}(t) = \min\{2t, 1\}$ for each n and let $I^* = \varprojlim\{I_n, \varkappa_{nm}\}$. It is easy to see that I^* is an arc with end points $(0, 0, \ldots, 0, \ldots)$ and $(1, 1, \ldots, 1, \ldots)$; let $\lambda\colon I\to I^*$ be any homeomorphism of I onto I^* satisfying $\lambda(0) = (0, 0, \ldots, 0, \ldots)$ and $\lambda(1) = (1, 1, \ldots, 1, \ldots)$. For each n, let $\lambda_n = \varkappa_n\lambda\colon I\to I_n$. Let $S(X)$ denote the cone over X and, for each n, let $\gamma_n\colon S(X)\to D_n$ be given by

$$\gamma_n((x, t)) = [1-\lambda_n(t)]\alpha_n(x).$$

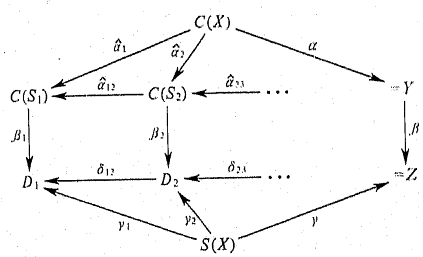
Straightforward computations show that the diagram



commutes. Let $\gamma\colon S(X)\to Z$ be the natural map, i.e.,

$$\gamma(x, t) = (\gamma_1((x, t)), \gamma_2((x, t)), \ldots, \gamma_n((x, t)), \ldots).$$

Since γ_n maps $S(X)$ onto D_n for each n, γ maps $S(X)$ onto Z. Computations show that γ is one-to-one. Thus, γ is a homeomorphism of $S(X)$ onto Z. Consider the following diagram:



Define $h\colon S(X)\to C(X)$ by $h = \alpha^{-1}\beta^{-1}\gamma$. Since each of α, β, and γ as a homeomorphism, h is a homeomorphism of $S(X)$ onto $C(X)$. Moreover,

(1) $$h((x, 0)) = \{x\},$$

(2) $$h((x, 1)) = X,$$

(3) if $(x, t), (x, t')\in S(X)$ with $t<t'$, then $h((x, t))\subset h((x, t'))$.

Now let $\varphi\colon S(X)\to S(X)$ be the homeomorphism given by $\varphi((x, t)) = (x, \mu h((x, t)))$ for each (x, t) Define $h^*\colon S(X)\to C(X)$ by $h^* = h\circ\varphi^{-1}$. Since (1) through (3) above hold for h, they hold for h^*. Also, the diagram in the statement of the theorem commutes.

5. The hyperspaces of continua obtained by some identifications. In this section we give some general results which are useful for recognizing certain subsets of hyperspaces.

5.1. THEOREM. *Let X be a continuum which is irreducible between points a and b and assume a and b are each terminal in X. Let $Y = X/_{\{a,b\}}$ be the continuum obtained from X by identifying a and b and let p be the corresponding identification point in Y, i.e., $p = \{a, b\}\in Y$. Let $\nu\colon X\to Y$ be the quotient map and let μ be a Whitney map on $C(Y)$. Then:*

(1) *$\mu_1 = \mu\cdot\hat{\nu}$ is a Whitney map on $C(X)$,*

(2) *$\hat{\nu}$ is an embedding of $C(X)\setminus\{\{a\}, \{b\}\}$ into $C(Y)$,*

(3) *$S = \hat{\nu}[X(a)]\cup\hat{\nu}[X(b)]$ is a simple closed curve,*

(4) *for each $0<t<\mu(Y)$, $\hat{\nu}$ maps $\mu_1^{-1}(t)$ homeomorphically onto $\mathrm{cl}[\mu^{-1}(t)\setminus Y(p, \mu, t)]$,*

(5) *$Y(p)$ is a 2-cell and S is its manifold boundary,*

(6) *for each $0<t<\mu(Y)$, $Y(p, \mu, t)$ is an arc with endpoints $\hat{\nu}(A_t)$ and $\hat{\nu}(B_t)$, where $\{A_t\} = X(a, \mu_1, t)$ and $\{B_t\} = X(b, \mu_1, t)$; moreover,*

$$\{\hat{\nu}(A_t), \hat{\nu}(B_t)\} = \mathrm{Fr}_{\mu^{-1}(t)}[Y(p, \mu, t)],$$

(7) $$C(Y) = \hat{\nu}[C(X)]\cup Y(p) \quad \textit{and} \quad \hat{\nu}[C(X)]\cap Y(p) = S.$$

Proof. Statements (1) and (2) follow from 2.1. It is easy to prove (3). We now prove (4) (see the diagram following the proof). By (2) it suffices to show

$$\hat{\nu}[\mu_1^{-1}(t)] = \mathrm{cl}[\mu^{-1}(t)\setminus Y(p, \mu, t)].$$

To do this, let $A\in\mu_1^{-1}(t)$. By definition of μ_1 in (1), $\hat{\nu}(A)\in\mu^{-1}(t)$. Assume $\hat{\nu}(A)\in Y(p, \mu, t)$. Then, $a\in A$ or $b\in A$ and we assume without loss of generality that $a\in A$. Since $\mu_1^{-1}(t)$ is a nondegenerate continuum [3, p. 1032] and since $X(a, \mu_1, t)$ and $X(b, \mu_1, t)$ are each one-point sets, we have

$$A\in\mathrm{cl}[\mu_1^{-1}(t)\setminus(X(a, \mu_1, t)\cup X(b, \mu_1, t))].$$

This implies $\hat{\nu}(A) \in \mathrm{cl}[\mu^{-1}(t) \setminus Y(p, \mu, t)]$. Now, let $B \in \mathrm{cl}[\mu^{-1}(t) \setminus Y(p, \mu, t)]$. For each $n = 1, 2, \ldots$, let $B_n \in [\mu^{-1}(t) \setminus Y(p, \mu, t)]$ such that the sequence $\{B_n\}$ converges to B. Since $p \notin B_n$ for any n, $\nu^{-1}(B_n)$ is a continuum for each $n = 1, 2, \ldots$ Let K be the limit of a convergent subsequence of $\{\nu^{-1}(B_n)\}$. It follows easily that K is a continuum, $K \in \mu_1^{-1}(t)$, and $\hat{\nu}(K) = B$. This completes the proof of (4). Next we prove (5). First, define functions

$$\alpha\colon Y(p) \to X(a) \quad \text{and} \quad \beta\colon Y(p) \to X(b)$$

by the equalities

$\alpha(Z) =$ the component of $\nu^{-1}(Z)$ which contains a;
$\beta(Z) =$ the component of $\nu^{-1}(Z)$ which contains b.

It is easy to see that if $Z \in Y(p)$ then $\nu^{-1}(Z)$ has at most two components (one containing a and one containing b). It follows from this that

(a) $\nu[\alpha(Z) \cup \beta(Z)] = Z$ for each $Z \in Y(p)$,

(b) α is continuous on $Y(p) \setminus \{Y\}$.

To see this let $Z \in [Y(p) \setminus \{Y\}]$ and let $\{Z_n\}$ be a sequence of members of $Y(p) \setminus \{Y\}$ converging to Z. There is a subsequence $\{Z_{n_i}\}$ of $\{Z_n\}$ such that both sequences $\{\alpha(Z_{n_i})\}$ and $\{\beta(Z_{n_i})\}$ converge; let their limits be denoted, respectively, by A and B. Using (a) and the continuity of ν it follows that $\nu(A \cup B) = Z$. Now, since $a \in \alpha(Z_{n_i})$ and $b \in \beta(Z_{n_i})$ for each i, $a \in A$ and $b \in B$. Thus, since A and B are continua and $\nu(A \cup B) = Z \neq Y$, $A \cap B = \emptyset$. Also, since $a \in A$ and $b \in B$, $\nu(A \cup B) = Z$ implies $\nu^{-1}(Z) = A \cup B$. It now follows that $\alpha(Z) = A$, i.e., $\{\alpha(Z_{n_i})\}$ converges to $\alpha(Z)$. Therefore, α is continuous at Z and we have proved (b). Now, for each $Z \in Y(p)$, let

$$f(Z) = (\mu(Z), \mu_1[\alpha(Z)]).$$

Noting that $\mu_1[\alpha(Z)] = \mu \circ \hat{\nu}[\alpha(Z)] \leqslant \mu(Z)$, we see that f is a function from $Y(p)$ into $\{(s, t) \in R^2\colon 0 \leqslant t \leqslant s \leqslant \mu(Y)\} = T$. Let D be the 2-cell obtained from T by shrinking the convex segment from $(\mu(Y), 0)$ to $(\mu(Y), \mu(Y))$ to a point and let $\lambda\colon T \to D$ be the natural map. We will show

$$h = \lambda \circ f\colon Y(p) \to D$$

is a homeomorphism onto D. First we prove

(c) $$h[Y(p)] = D.$$

To prove (c) first note that $f(Y) = (\mu(Y), \mu(Y))$. Thus, it suffices to show that if $(s, t) \in T$ such that $s < \mu(Y)$, then there is a $Z \in Y(p)$ such that $f(Z) = (s, t)$. Let $(s, t) \in T$ with $s < \mu(Y)$. Choose $E \in X(a)$ such that $\mu_1(E) = t$. Now, there is $A \in X(a)$ and $B \in X(b)$ such that $\mu_1(A) = s = \mu_1(B)$. Clearly, $\hat{\nu}(A) \in Y(p, \mu, s)$ and $\hat{\nu}(B) \in Y(p, \mu, s)$. Also, $\alpha[\hat{\nu}(A)] = A$ and $\alpha[\hat{\nu}(B)] = \{a\}$ (the second equality follows from the fact that $s < \mu(X)$) and hence $a \notin B$. Since $t \leqslant s$, we have $a \in E \subset A$.

Therefore, since $Y(p, \mu, s)$ is a continuum (by 3.2) and since, by (b), α is a continuous function on $Y(p, \mu, s)$ into the arc $X(a)$, it now follows that there is a $Z \in Y(p, \mu, s)$ such that $\alpha(Z) = E$. Clearly, $Z \in Y(p)$ and $f(Z) = (s, t)$. This completes the proof of (c).

(d) h is one-to-one.

From the formula for f it is easy to see that (d) follows from the following:

(e) For each $0 < s < \mu(Y)$, α is one-to-one on $Y(p, \mu, s)$.

To prove (e) let $K, L \in Y(p, \mu, s)$ such that $\alpha(K) = \alpha(L)$. Suppose $\beta(K) \neq \beta(L)$. Since b is terminal in X, $\beta(K) \subset \beta(L)$ or $\beta(L) \subset \beta(K)$. Without loss of generality we assume $\beta(K) \subset \beta(L)$. Suppose

(*) $$\alpha(K) \cup \beta(K) = \alpha(L) \cup \beta(L).$$

Then, since $\beta(K)$ is a proper subset of $\beta(L)$, $\beta(L) \cap \alpha(K) \neq \emptyset$. Thus, $\alpha(K) \cup \beta(L) = X$ (by irreducibility). It now follows easily that

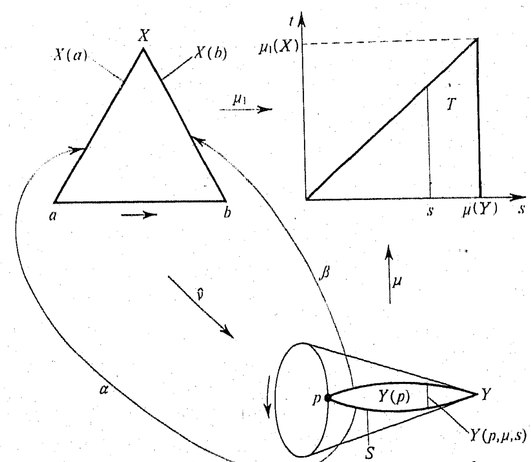
$$\alpha(K) \cup \beta(K) = X = \alpha(L) \cup \beta(L)$$

which, by (a), implies $K = Y = L$. This contradicts the assumption $s < \mu(Y)$. Therefore, (*) is false and we have that $\alpha(K) \cup \beta(K)$ is a proper subset of $\alpha(L) \cup \beta(L)$. Hence, by (a), K is a proper subcontinuum of L. This contradicts the fact that $\mu(K) = s = \mu(L)$. We have now proved that $\beta(K) = \beta(L)$ and thus, by (a), $K = L$. This proves (e).

To complete the proof that h is a homeomorphism we need to show

(f) h is continuous.

However, (f) follows easily from (b) and the continuity of μ at Y.



We have now proved (5). To prove (6) note that, by (b) and (e), α is a homeomorphism of $Y(p, \mu, t)$ into $X(a)$. It is easy to see from part of the proof of (c) that $\alpha[Y(p, \mu, t)]$ has endpoints $\{a\}$ and A_t where $\{A_t\} = X(a, \mu_1, t)$. It follows that $Y(p, \mu, t)$ has endpoints $\hat{v}(A_t)$ and $\hat{v}(B_t)$, A_t and B_t as in (6). The fact that

$$\{\hat{v}(A_t), \hat{v}(B_t)\} = \mathrm{Fr}_{\mu^{-1}(t)}[Y(p, \mu, t)]$$

can be proved with an argument similar to the one used to prove (4).

Finally, (7) can easily be verified using (3).

The proof of the next theorem is similar to that of the above theorem and is omitted.

5.2. Theorem. *Let p_i be a terminal point of a continuum X_i, $i = 0, 1$, and let X be the one-point union of X_0 and X_1 obtained by identification of p_0 and p_1. Consider X_i as the subset of X and let $p_0 = p_1 = p$. If μ is a Whitney map on $C(X)$, then we have*:

(1) *$X(p)$ is a 2-cell with its manifold boundary*

$$\partial X(p) = \{p\}X_0 \cup X_0X \cup \{p\}X_1 \cup X_1X.$$

The interior of $X(p)$ relative to $C(X)$ is equal

$$\mathrm{Int}X(p) = X(p)\setminus(\{p\}X_0 \cup \{p\}X_1).$$

(Observe that the segments are unique.)

(2) *For $0<t<\mu(X)$ the set $X(p, \mu, t)$ is an arc with endpoints A_t, B_t such that*
(a) *$A_t \in \{p\}X_0$ for $t \leqslant \mu(X_0)$; $A_t \in X_0X$ for $t \geqslant \mu(X_0)$*,
(b) *$B_t \in \{p\}X_1$ for $t \leqslant \mu(X_1)$; $B_t \in X_1X$ for $t \geqslant \mu(X_1)$.*

5.3. Theorem. *Let X be a continuum such that*

$$X = X_0 \cup X_1 \quad \text{and} \quad X_0 \cap X_1 = Y,$$

where X_i, $i = 0, 1$, is a proper subcontinuum of X, and Y is a continuum terminal in both X_0 and X_1 such that each subcontinuum of X intersecting both $X_0\setminus Y$ and $X_1\setminus Y$ contains Y. Let $t_0 = \max\{\mu(X_0), \mu(X_1)\}$. Then for each $t_0<t<\mu(X)$, the continuum $\mu^{-1}(t)$ is an arc.

Proof. By definition of t_0 we have $\mu^{-1}(t) = X(Y, \mu, t)$, and the result follows easily from the assumption that Y is terminal in X_0 and in X_1.

All but three of the examples we intended including in this section to illustrate our results appear in [15]. All these examples can be constructed from the arc, $\sin(1/x)$-continuum, and the two snake-like examples in [9, p. 205] by some simple identifications. The results in this section can then be used to verify the pertinent properties. The reader is referred to [15] for the examples and the statements of the properties they have. We include the following three examples.

5.4. Example. Let X be the pseudoarc and let a and b be two points from distinct composants of X. Then a and b are terminal in X and X is irreducible between them. Let Y be the indecomposable circle-like continuum obtained from X by identification of a and b, and let μ be a Whitney map on $C(Y)$. Now, applying 5.1 we can easily determine the topological type of the level $\mu^{-1}(t)$ for $0<t<\mu(Y)$. In fact, $\mu_1^{-1}(t)$ is a pseudoarc [3] and hence, by (4) and (6), $\mu^{-1}(t)$ is the union of a pseudoarc P and an arc A such that the intersection $A \cap P$ consists of two points which are the endpoints of A and which lie in distinct composants of P. Note that $\mu^{-1}(t)$ is decomposable for each $0<t<\mu(Y)$.

5.5. Example. There exists a non-unicoherent 1-dimensional continuum X containing a simple closed curve such that, for some $t_0<\mu(X)$, $\mu^{-1}(t)$ is an arc for each $t_0<t<\mu(X)$. Let X be the continuum obtained by identifying $(0, -1)$ with $(0, -1)$ and $(0, 1)$ with $(0, 1)$ in two copies of the $\sin(1/x)$-continuum. Let S be the simple closed curve in X. Let X_i, $i = 0, 1$, be continua such that

$$X = X_0 \cup X_1 \quad \text{and} \quad X_0 \cap X_1 = S.$$

The properties of X can be easily checked by using 5.3.

5.6. Example. Let X be the $\sin(1/x)$-circle, i.e., X is the continuum obtained by identifying two terminal points of the $\sin(1/x)$-continuum about which it is irreducible. Then, X has the fixed point property but, by (b) of 4.4, $\mu^{-1}(t)$ is a simple closed curve for some t. Hence, the fixed point property is not a Whitney property.

6. Problems. The table in Section 1 lists topological properties for which it is now known whether or not they are Whitney properties. The canonical problem is of course:

1. For a given topological property determine whether it is a Whitney property. In relation to this problem, we are especially interested in, and do not know the answer for, the following properties of X: acyclic, ANR, AR, contractibility, Hilbert cube, homogeneity, λ-connected, $\mathrm{Sh}(X)\leqslant \mathrm{Sh}(Y)$, and weakly chainable.

In 4.2 we showed that for chainable continua X, no proper subcontinuum of $\mu^{-1}(t)$ covers X for any t.

2. What class of continua does this condition characterize? It is easy to see from 8.3 of [5] that every hereditarily indecomposable continuum satisfies this condition. Also, it is not difficult to prove that the condition implies unicoherence (of X).

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