Continuous images of continua and 1-movability

by

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Abstract. It is proved that pointed 1-movability, which is an invariant of pointed shape, is also an invariant for continuous mappings of continua. Several characterizations of pointed 1-movability are given. It seems to the author that the characterization involving the Mittag-Leffler property for inverse sequences of fundamental groups is the most useful. We prove that every continuum which is not pointed 1-movable contains a subcontinuum which is indecomposable and not pointed 1-movable. In the final section we show that all continua embeddable into 2-manifolds are pointed 1-movable.

0. Introduction. In a paper by K. Borsuk [7] the notion of n-movability is introduced and studied. We find that the case of 1-movability is especially interesting in the category of pointed metric continua. Our main result states that it is an invariant of morphisms in that category, that is: continuous images of pointed 1-movable continua are pointed 1-movable. Since continua of trivial shape are pointed movable and this property implies n-movability for each $n \geqslant 1$, the above result extends a recent theorem of the author which says that 1-dimensional continuous images of continua of trivial shape are pointed movable (see [11] for the precise statement of this theorem). We follow K. Borsuk [6] in writing $\mathrm{Sh}(X, x_0)$ for the shape of a pointed compactum (X, x_0). It is easy to see that if (X, x_0) is n-movable (movable) and fundamentally dominates (Y, y_0), $\mathrm{Sh}(X, x_0) \geqslant \mathrm{Sh}(Y, y_0)$, then so is (Y, y_0). Hence the 1-movability of pointed continua is an invariant of continuous mappings and an invariant of shape. We prove also that hereditarily decomposable continua and continua embeddable in surfaces are pointed movable. Hence their continuous images are pointed 1-movable.

These results generalize a theorem of Borsuk [5] on the movability of plane compacta. Bing [1] has proved that no non-planar circle-like continuum can be embedded into an orientable surface. The above results combined with a theorem in [10] (see also [18]) generalize Bing's theorem to arbitrary surfaces. This fact was first observed by M. C. McCord [17], who proved it using an algebraic argument.

We assume that the reader is familiar with the equivalence of the Mardešić–Segal approach to shape theory to that of Borsuk (in both the absolute and the pointed case), which can be found in [15].

1. Some simple consequences of 1-movability. Let X be a continuum contained in $M \in \mathrm{ANR}(\mathfrak{M})$ and let x_0 be a point of X. We say that X is *movable rel.* M if for every neighbourhood U of X in M there exists a neighbourhood $U_0 \subset U$ of X which can be shrunk inside U into any neighbourhood of X in M (see [5]). If in addition the homotopies can be choosen in such a way that they keep x_0 fixed, then we say that (X, x_0) is *movable rel.* M (see [6]).

The pointed continuum (X, x_0) is said to be 1-*movable rel.* M if for every neighbourhood U of X in M there exists a neighbourhood $U_0 \subset U$ of X such that each loop $\omega\colon (I, \dot{I}) \to (U_0, x_0)$ can be deformed inside (U, x_0) into any neighbourhood of X in M; i.e., for each neighbourhood V of X there exists a homotopy $\varphi\colon (I, \dot{I}) \times I \to (U, x_0)$ connecting ω with some loop in (V, x_0). Here I is the unit interval $[0, 1]$ of reals and $\dot{I} = \{0, 1\}$.

This definition of 1-movability (in the pointed category) is equivalent to that of Borsuk [7].

1.1. THEOREM. *The pointed continuum (X, x_0) is 1-movable rel. M iff for every neighbourhood U of X in M there exists a neighbourhood $U_0 \subset U$ of X such that each path $\omega\colon (I, \dot{I}) \to (U_0, X)$ can be deformed inside (U, X) into any neighbourhood of X; i.e., for each neighbourhood V of X there is a homotopy $\varphi\colon (I, \dot{I}) \times I \to (U, X)$ connecting ω with a path in V.*

Proof. $\Rightarrow$. Let U be a neighbourhood of X in M. Then there is a neighbourhood U_0 of X such that

(1) any loop in (U_0, x_0) can be deformed inside (U, x_0) into any neighbourhood of X.

Let V be a neighbourhood of X and let ω_0 be a path in U_0 with endpoints in X. By the connectedness of X there exist two paths $\omega_1\colon I \to V \cap U_0$ and $\omega_2\colon I \to V \cap U_0$ satisfying the conditions:

$$\omega_1(0) = \omega_2(0) = x_0, \quad \omega_1(1) = \omega_0(0), \quad \omega_2(1) = \omega_0(1).$$

It follows that

$$\omega_0 \simeq \omega_1^{-1} * \omega_1 * \omega_0 * \omega_2^{-1} * \omega_2 \quad \text{in } U_0 \text{ rel. } \dot{I}. \tag{2}$$

By condition (1) there exists a loop ω_3 in (V, x_0) such that

$$\omega_3 \simeq \omega_1 * \omega_0 * \omega_2^{-1} \quad \text{in } U \text{ rel. } \dot{I}. \tag{3}$$

Conditions (2) and (3) imply

$$\omega_0 \simeq \omega_1^{-1} * \omega_3 * \omega_2 \quad \text{in } U \text{ rel. } \dot{I}.$$

The path on the right-hand side is in V. So we have proved that any path in U_0 with endpoints in X can be shrunk rel. $\dot{I}$ inside U to a path in V.

$\Leftarrow$. Let U be a neighbourhood of X. By our assumption there exists a neighbourhood $U_0 \subset U$ of X such that every path ω in U_0 with endpoints in X can be shrunk inside (U, X) into any neighbourhood of X. It is easy to observe that if there is a homotopy inside (U, X) joining ω with some path in $V \supset X$, then by some simple modification of that homotopy we can obtain a homotopy in (U, X) joining ω with some path in V and keeping the endpoints of ω fixed. Hence every loop in (U_0, x_0) can be deformed inside (U, x_0) into any neighbourhood of X. This completes the proof.

1.2. THEOREM. *Let X and X' be homeomorphic continua and let $x_0 \in X \subset M \in \mathrm{ANR}(\mathfrak{M})$, $x_0' \in X' \subset M' \in \mathrm{ANR}(\mathfrak{M})$. If (X, x_0) is 1-movable rel. M, then (X', x_0') is 1-movable rel. M'.*

Proof. Let U' be a neighbourhood of X' in M'. To complete the proof it remains by 1.1 to construct a neighbourhood $U_0' \subset U'$ of X' in M' such that each path in U_0' with endpoints in X' can be deformed inside (U', X') into any neighbourhood of X' in M'. Let h be a homeomorphism from X onto X'. There are neighbourhoods G of X in M and G' of X' in M' and two mappings $f\colon G \to M'$, $g\colon G' \to M$ such that

$$f(x) = h(x) \text{ for } x \in X \quad \text{and} \quad g(x') = h^{-1}(x') \text{ for } x' \in X'.$$

Using the Borsuk homotopy extension theorem it is easy to construct a neighbourhood $H \subset U' \cap g^{-1}(f^{-1}(U'))$ of X' in M' and a homotopy $\psi\colon H \times I \to U'$ such that

(1) $\psi(x', 0) = x'$, $\psi(x', 1) = f(g(x'))$ for $x' \in H$ and $\psi(x', t) = x'$ for $(x', t) \in X' \times I$.

Since $U = f^{-1}(U')$ is a neighbourhood of X in M, there is a neighbourhood $U_0 \subset U$ of X in M such that every path in (U_0, X) can be shrunk inside (U, X) into any neighbourhood of X. We claim that

$$U_0' = g^{-1}(U_0) \cap H \quad (\subset U')$$

is the desired neighbourhood of X' in M'.

So let ω be a path in U_0' with endpoints in X' and let V be a neighbourhood of X' in M'. Then $g \circ \omega$ is a path in U_0 with endpoints in X and $f^{-1}(V)$ is a neighbourhood of X in M. Hence there is a homotopy $\varphi\colon I \times I \to U$ such that

(2) $\varphi(t, 0) = g \circ \omega(t)$, $\varphi(t, 1) \in f^{-1}(V)$, $\varphi(0, s) = g(\omega(0))$ and $\varphi(1, s) = g(\omega(1))$ for each $t, s \in I$.

Now for each $(t, s) \in I \times I$ let

$$\varphi'(t, s) = \begin{cases} \psi(\omega(t), 2s) & \text{for } 0 \leqslant s \leqslant \frac{1}{2}, \\ f \circ \varphi(t, 2s-1) & \text{for } \frac{1}{2} \leqslant s \leqslant 1. \end{cases}$$

By (1) and (2) one easily checks that φ' is well-defined continuous and $\varphi'(t, s) \in U'$. Hence $\varphi'\colon I \times I \to U'$ is a homotopy joining ω with some path in V inside U'. This completes the proof.

It is known that in the definition of movability and pointed movability of continua the choice of a particular point and a particular $\mathrm{ANR}(\mathfrak{M})$-space containing a topological copy of a given continuum is inessential [6]. Theorems 1.1 and 1.2 say that an analogous fact holds for the notion of the 1-movability rel. M of a pointed continuum. Hence we may speak about the *movability, pointed movability* and *pointed* 1-*movability* of a continuum (or the 1-*movability* of a pointed continuum) without referring to a particular point of that continuum and to a particular ANR($\mathfrak{M}$)-space containing it (every metric continuum embeds into the Hilbert cube $Q \in$ ANR).

1.3. PROPOSITION. *For a continuum X we have the following implications:*

$$(X \textit{ movable}) \Leftarrow (X \textit{ pointed movable}) \Rightarrow (X \textit{ pointed } 1\textit{-movable}).$$

The next theorem shows the importance of 1-movability.

1.4. THEOREM. *If a continuum is movable and pointed* 1-*movable, then it is pointed movable.*

Proof. Without loss of generality we may assume that the given continuum X is a subset of the Hilbert cube Q. Let U be a neighbourhood of X in Q. By our assumption there exists some neighbourhood $U_1 \subset U$ such that for every neighbourhood V of X and for every loop ω in (U_1, x_0) there exists a loop ω_V in V such that ω is homotopic to ω_V in (U, x_0).

Since X is movable, there exists a neighbourhood $U_0 \subset U_1$ of X such that for each neighbourhood V of X there exists a contraction of U_0 inside U_1 into V. Let V be any neighbourhood of X. To prove the theorem it suffices to show that U_0 can be contracted inside U into V in such a way that x_0 is fixed during this contraction. There exists a homotopy $\varphi\colon U_0 \times I \to U_1$ such that

$$\varphi(x, 0) = x \quad \text{and} \quad \varphi(x, 1) \in V \quad \text{for every } x \in U_0. \tag{1}$$

We may assume that each of U, U_1, U_0, V is a connected ANR-set. By the Borsuk homotopy extension theorem φ can be modified so that

$$\varphi(x_0, 1) = x_0. \tag{2}$$

Let $\omega(t) = \varphi(x_0, t)$ for $t \in I$. Hence ω is a path in (U_1, x_0). So there is a homotopy α from $(I, \dot{I}) \times I$ into (U, x_0) such that

$$\alpha(t, 0) = \omega(t) \quad \text{and} \quad \alpha(t, 1) \in V. \tag{3}$$

Let $\omega_V(t) = \alpha(t, 1)$, $t \in I$. Set $A = U_0 \times (1)$ and let $\beta\colon A \to V$ be defined by $\beta(x) = \varphi(x, 1)$. Again using Borsuk's homotopy extension theorem we obtain a homotopy $\gamma\colon A \times I \to V$ such that

$$\gamma(a, 0) = \beta(a), \quad \gamma((x_0, 1), t) = \omega_V(1-t) \quad \text{for} \quad a \in A,\ t \in I. \tag{4}$$

Let $B = U_0 \times I$ and consider a subset C of $B \times I$ given by

$$C = (x_0) \times I \times (0) \cup (x_0) \times (1) \times I \cup (x_0) \times I \times (1) \cup (x_0) \times (0) \times I.$$

Let $\delta\colon C \to U$ be defined by

$$\delta(x_0, u, v) = \begin{cases} \varphi(x_0, u) & \text{for } v = 0, \\ \gamma(x_0, 1, v) & \text{for } u = 1, \\ x_0 & \text{otherwise}. \end{cases}$$

Let $\tau\colon (I, \dot{I}) \to (C, (x_0, 0, 0))$ be defined by

$$\tau(t) = \begin{cases} (x_0, 4t, 0) & \text{for } 0 \leqslant t \leqslant \frac{1}{4}, \\ (x_0, 1, 4t-1) & \text{for } \frac{1}{4} \leqslant t \leqslant \frac{1}{2}, \\ (x_0, -4t+3, 1) & \text{for } \frac{1}{2} \leqslant t \leqslant \frac{3}{4}, \\ (x_0, 0, -4t+4) & \text{for } \frac{3}{4} \leqslant t \leqslant 1. \end{cases}$$

Since C is homeomorphic to the circle, τ is a representative of a generator of the fundamental group of C. By (3) and (4) one easily sees that composition $\delta \circ \tau\colon (I, \dot{I}) \to (U, x_0)$ is homotopic to a trivial loop. Hence δ can be extended onto $D = (x_0) \times I \times I$. Let $\eta\colon D \to U$ be such an extension. Let

$$E = U_0 \times (0) \cup (x_0) \times I \cup A \subset B$$

and let $\chi\colon E \times I \cup B \to U$ be defined as follows:

$$\chi(y, v) = \begin{cases} y & \text{for } y \in U_0 \times (0), \\ \gamma(y, v) & \text{for } y \in A, \\ \eta(y, v) & \text{for } (y, v) \in D, \\ \varphi(y) & \text{for } v = 0. \end{cases}$$

By the homotopy extension property there exists an extension $\lambda\colon B \times I \to U$ of the map χ, $U \in$ ANR. Setting $\psi(x, t) = \lambda(x, t, 1)$ one obtains a map from $U_0 \times I$ into U such that: $\psi(x, 0) = x$, $\psi(x, 1) \in V$ and $\psi(x_0, t) = x_0$ for each $x \in U_0$ and $t \in I$. This completes the proof.

We will need the following modification of 1-movability. A subset A of a space Y is said to be 1-*movable in* Y provided there exists a neighbourhood U_0 of A in Y such that for every neighbourhood V of A in Y and for every path $\omega\colon (I, \dot{I}) \to (U_0, A)$ there exists a homotopy $\varphi\colon (I, \dot{I}) \times I \to (Y, A)$ joining ω with some path in V. The set U_0 is called an *admissible* neighbourhood of A. Clearly, we may always modify φ to a homotopy keeping the endpoints of ω fixed.

This notion is related to 1-movability in the following manner.

1.5. PROPOSITION. *A continuum $X \subset M \in$ ANR($\mathfrak{M}$) is pointed* 1-*movable if and only if X is* 1-*movable in U for every neighbourhood U of X in M.*

To state our next results we need the following

1.6. LEMMA. *Let A_0 and A_1 be subcontinua of $M \in$ ANR($\mathfrak{M}$). Assume that A_i is* 1-*movable in G_i, where G_i is a neighbourhood of A_i in M, $i = 0, 1$. Then $A_0 \cup A_1$ is* 1-*movable in $G_0 \cup G_1$.*

Proof. In the case where A's are disjoint it suffices to take disjoint admissible neighbourhoods of A's in G's. Suppose now that A's intersect. Let U_i be an admissible neighbourhood of A_i in G_i and let $U_2 \subset U_0 \cap U_1$ be an open neighbourhood (open in M) of $A = A_0 \cap A_1$ satisfying the condition:

(1) for each $x \in U_2$ there exist $y \in A$ and an arc $\widehat{xy} \subset U_0 \cap U_1$.

The existence of U_2 follows from the local contractibility of M. The compact sets $B_i = A_i \setminus U_2$ are disjoint. Let V_i be an open neighbourhood of B_i in M, $i = 0, 1$, such that $V_0 \cap V_1 = \emptyset$ and

(2) for each $x \in V_i$ there exist a point $y \in B_i$ and an arc $\widehat{xy} \subset U_i$, $i = 0, 1$.

Clearly, $U = V_0 \cup V_1 \cup U_2$ is a neighbourhood of $A_0 \cup A_1$ in $G_0 \cup G_1$. We shall prove that it is admissible. Let V be a neighbourhood of $A_0 \cup A_1$ in $G_0 \cup G_1$ and let $\omega\colon (I, \dot{I}) \to (U, A_0 \cup A_1)$. The family $\mathcal{U}$ consisting of the sets V_0, V_1, U_2 is an open covering of the compact set $\omega(I)$. Hence there is a subdivision

$$t_0 = 0 < t_1 < \ldots < t_{n+1} = 1$$

of I such that $\omega([t_j, t_{j+1}])$ is a subset of an element of $\mathcal{U}$ for each $j \leqslant n$. Let $p_j = \omega(t_j)$ for $j \leqslant n+1$. According to (1) and (2), for each $j \leqslant n+1$, there exists a point q_j satisfying the following conditions:

(3) if $p_j \in U_2$, then $q_j \in A$ and there exists a path τ_j in $U_0 \cap U_1$ from p_j to q_j,

(4) if $p_j \in V_i \setminus U_2$, then $q_j \in A_i$ and there exists a path τ_j in U_i from p_j to q_j.

Let ω_j be a path defined by $\omega_j(t) = \omega(t_j + t(t_{j+1} - t_j))$, $j \leqslant n$. Conditions (3) and (4) imply that

$$\omega \simeq \tau_0 * (\tau_0^{-1} * \omega_0 * \tau_1) * \ldots * (\tau_n^{-1} * \omega_n * \tau_{n+1}) * \tau_{n+1}^{-1} \quad \text{in } U_0 \cup U_1 \text{ rel. } \dot{I}.$$

Hence to finish the proof we need only to show that each of the paths $\sigma_j = \tau_j^{-1} * \omega_j * \tau_{j+1}$, $j \leqslant n$, is either in (U_0, A_0) or in (U_1, A_1). Observe that $\omega_j(I)$ is contained either in U_2 or in V_i for some $i = 0, 1$. If $\omega_j(I) \subset U_2$, then $\sigma_j(I) \subset U_0 \cap U_1$ and $\sigma_j(\dot{I}) \subset A$, by (3). Hence in this case σ_j is a path in both (U_0, A_0) and (U_1, A_1). Suppose now that $\omega_j(I) \subset V_i$ for some i. If p_j and p_{j+1} are in U_2, the assertion follows from (3). If $p_j \notin U_2$, then $p_{j+1} \in U_2 \cup V_i$ and by (3) and (4) we obtain $\sigma_j\colon (I, \dot{I}) \to (U_i, A_i)$. The remaining case, $p_{j+1} \notin U_2$, is symmetric to the above one. This completes the proof.

1.7. Corollary. *Let A_0 and A_1 be subcontinua of $M \in \mathrm{ANR}(\mathfrak{M})$. If both of them are 1-movable in M, so is their union $A_0 \cup A_1$.*

Combining this corollary with 1.5, we obtain

1.8. Theorem. *If some continuum can be represented as a union of two pointed 1-movable subcontinua, then it is pointed 1-movable.*

Our last result in this section is based on the next two lemmas.

1.9. Lemma. *Let $A_1, A_2, \ldots$ be a decreasing sequence of continua in $M \in \mathrm{ANR}(\mathfrak{M})$ such that A_n is not 1-movable in M for each $n = 1, 2, \ldots$ Then $\bigcap_n A_n$ is not 1-movable in M.*

Proof. Let A denote the intersection and suppose that A is 1-movable in M. Let U be an open neighbourhood of A in M which is admissible for A. Then $A_n \subset U$ for some integer n. Since U is not admissible for A_n, there exist a neighbourhood V of A_n and a path ω in U with endpoints in A_n which cannot be shrunk into V rel. $\dot{I}$. Let ω_i be some path in $U \cap V$ from $\omega(i)$ to some point of A. Such paths exist by the connectedness of A_n. Then

$$\tau = \omega_0^{-1} * \omega * \omega_1\colon (I, \dot{I}) \to (U, A) \quad \text{and} \quad \omega \simeq \omega_0 * \tau * \omega_1^{-1} \text{ rel. } \dot{I}.$$

Since U is admissible for A, there is a path τ' in V such that $\tau \simeq \tau'$ rel. $\dot{I}$. It follows that $\omega_0 * \tau' * \omega_1^{-1}$ is a path in V homotopic to ω rel. $\dot{I}$, contrary to the construction of V. Hence our supposition about A is false.

1.10. Lemma. *Let X be a subcontinuum of $M \in \mathrm{ANR}(\mathfrak{M})$. Assume that X is not 1-movable in M. Then there exists a continuum $X_0 \subset X$ such that X_0 is not 1-movable in M but every proper subcontinuum of X_0 is 1-movable in M. Moreover, X_0 is indecomposable.*

Proof. The existence of X_0 follows from 1.9 and the Brouwer reduction theorem. Suppose that X_0 is a union of two proper subcontinua A_0 and A_1. Hence both A_0 and A_1 are 1-movable in M. By 1.7 we conclude that X_0 is 1-movable in M, a contradiction.

As a corollary from the above fact we have

1.11. Theorem. *Every continuum which is not pointed 1-movable contains some indecomposable subcontinuum with the same property.*

In the next section we establish several results from group theory which will be used to obtain some additional information on pointed 1-movability.

2. Some properties of groups. A group G is the *free product* of its subgroups A and B, notation: $G = A * B$, if each element g of G can be uniquely written in the form

$$g = c_1 \cdot c_2 \cdot \ldots \cdot c_n,$$

where $c_i \in A \cup B$, $c_i \neq 1$, and no two successive elements c_i, c_{i+1} belong together to A or B. The following result can be found in [8].

2.1. *If G and H are finitely generated free nonabelian groups and $f\colon G \to H$ is an epimorphism, then G can be represented as a free product of its subgroups A and B such that $f|A\colon A \to H$ is an isomorphism and $B \subset \ker f$.*

2.2. *Let A and B be finitely generated groups and let $f\colon A \to B$ be an epimorphism. Let B' be a finite set generating B such that $1 \in B$. Then there exists a finite set A' generating A such that $1 \in A'$ and $f(A') = B'$.*

Proof. Choose a finite set M generating A such that

(1) $$B' \subset f(M).$$

Denote by A^* the free nonabelian group generated by M, $A^* = \langle M \rangle$, and let $\varphi\colon A^* \to A$ be a homomorphism such that

(2) $$\varphi(x) = x \quad \text{for each } x \in M.$$

Likewise, let $B^* = \langle B' \rangle$ and let $\psi\colon B^* \to B$ be a homomorphism such that

(3) $$\psi(y) = y \quad \text{for each } y \in B'.$$

Suppose $B' = \{b_1, b_2, \ldots, b_n\}$. For each index i, $1 \leqslant i \leqslant n$, there exists, by (1) an element $e_i \in M$ such that

(4) $$f(e_i) = b_i.$$

For each $x \in M \setminus \{e_1, e_2, \ldots, e_n\}$ choose an element $b_x \in B^*$ such that

(5) $$\psi(b_x) = f(x).$$

The function $\gamma\colon M \to B^*$ defined by the formula

$$\gamma(x) = \begin{cases} b_i & \text{if} \quad x = e_i \quad \text{for some } i \leqslant n, \\ b_x & \text{if} \quad x \neq e_i \quad \text{for each } i = 1, 2, \ldots, n \end{cases}$$

can be extended to a homomorphism $g\colon A^* \to B^*$, because M is a free base for A^*. Conditions (2), (3), (4) and (5) imply

(6) $$f \circ \varphi = \psi \circ g,$$

because M is a set of generators for A^*. Observe that $B' \subset g(M)$, hence g is an epimorphism. By 2.1 there exist subgroups C, D of A^* such that $A^* = C * D$, $D \subset \ker g$, and $g|C\colon C \to B^*$ is an isomorphism. For each $i = 1, 2, \ldots, n$ choose an element $c_i \in C$ such that $g(c_i) = b_i$. Since each subgroup of a free group is free itself and $\operatorname{rank} C + \operatorname{rank} D = \operatorname{rank} A^*$, the group D is a finitely generated subgroup of A^*. Let D' be a finite set generating D. Then $E = D' \cup \{c_1, \ldots, c_n\}$ generates A^* and

(7) $$g(E) = B'.$$

Put $A' = \varphi(E) \cup \{1\}$. Since φ is an epimorphism (see (2)), the set A' is finite and generates A. Conditions (3), (6) and (7) imply

$$f(A') = \psi \circ g(E) \cup \{1\} = \psi(B') \cup \{1\} = B',$$

which completes the proof.

Let $\underline{H} = \{H_n, f_n\}$ be an inverse sequence of groups and homomorphisms. As usual, we write f_{nm}, $n < m$, instead of $f_n \circ \ldots \circ f_{m-1}$ and f_{nn} instead of 1_{X_n}. If $\underline{H}' = \{H_{n_i}, g_i\}$, where $n_i < n_{i+1}$ and $g_i = f_{n_i} \circ \ldots \circ f_{n_{i+1}-1}$, then we say that $\underline{H}'$ is a *subsequence* of $\underline{H}$. The sequence $\underline{H}$ is said to be *finitely generated* (*free*) if each group H_n is finitely generated (free nonabelian, respectively). $\underline{H}$ is an *epi-sequence* if each bonding homomorphism is an epimorphism. We say that $\underline{H}$ has a *regular system of generators* if for each index $n \geqslant 1$ there exists a finite set H_n^* generating H_n and satisfying the condition

$$f_n(H_{n+1}^*) \subset H_n^*.$$

2.3. *Let $\underline{H} = \{H_n, f_n\}$ be an inverse sequence of groups such that for each $n \geqslant 1$ there exist A_n, G_n, H_n^* satisfying the conditions:*

1. *H_n^* is a set of generators for H_n,*
2. *$f_n(H_{n+1}^*) \subset H_n^*$,*
3. *G_n is a finitely generated subgroup of H_n,*
4. *$f_n(G_{n+1}) = G_n$,*
5. *A_n is a finite set,*
6. *$A_n \cup G_n \subset H_n^*$,*
7. *each element $h \in H_n^*$ can be written in the form $h = g_1 a g_2$ for some $g_1, g_2 \in G_n$ and $a \in A_n$, i.e., $H_n^* \subset G_n A_n G_n$.*

Then $\underline{H}$ has a regular system of generators.

Proof. First we show that there exists a sequence of functions $\{\alpha_n\colon A_n \to H_n\}$ such that

(1) for each $a \in A_n$ we have $\alpha_n(a) = g_1 a g_2$ for some $g_1, g_2 \in G_n$,

(2) $f_n \alpha_{n+1}(a) \in \alpha_n(A_n)$ for each $a \in A_{n+1}$.

Let α_1 be the inclusion and suppose we have constructed α_i, $i \leqslant n$, satisfying the above conditions. Now we define α_{n+1}. Take an $a \in A_{n+1}$. By conditions 2, 6 and 7 there exist elements $e_1, e_2 \in G_n$ and $a_1 \in A_n$ such that

(3) $$f_n(a) = e_1 a_1 e_2.$$

By 4 there exist elements $e_1', e_2' \in G_{n+1}$ such that

(4) $$f_n(e_i') = e_i^{-1} \quad \text{for} \quad i = 1, 2.$$

From condition (1) we infer that there exist $g_1, g_2 \in G_n$ such that

(5) $$\alpha_n(a_1) = g_1 a_1 g_2.$$

By condition 4 there exist elements $g_1', g_2' \in G_{n+1}$ such that

(6) $$f_n(g_i') = g_i \quad \text{for} \quad i = 1, 2.$$

Now we define $\alpha_{n+1}(a)$ in the following way:

$$\alpha_{n+1}(a) = (g_1' e_1') a (e_2' g_2').$$

Since $g_1' e_1', e_2' g_2' \in G_{n+1}$, condition (1) is fulfilled for α_{n+1}. By conditions (3), (4), (5) and (6) we have: $f_n \alpha_{n+1}(a) = f_n(g_1') f_n(e_1') f_n(a) f_n(e_2') f_n(g_2') = g_1 e_1^{-1} e_1 a_1 e_2 e_2^{-1} g_2 = g_1 a_1 g_2 = \alpha_n(a_1)$, which implies condition (2).

Using 3, 4 and 2.2, we can construct a sequence of finite sets $\{B_n\}$ such that B_n generates G_n and

(7) $$f_n(B_{n+1}) = B_n.$$

According to 1, 5 and 7 we see that $A_n \cup B_n$ is a finite set generating H_n. Setting $H'_n = B_n \cup \alpha_n(A_n)$, one obtains a finite set generating H_n (by (1)). Condition (2) implies $f_n(\alpha_{n+1}(A_{n+1})) \subset \alpha_n(A_n)$. Combining this result with (7), we infer that $f_n(H'_{n+1}) \subset H'_n$. It follows that $\{H'_n\}$ is a regular system of generators for $\underline{H}$. This completes the proof.

An inverse sequence of groups $\underline{H} = \{H_n, f_n\}$ is said to be a *Mittag-Leffler sequence*, briefly: an ML-*sequence*, if for each index n there exists an $n_0 \geqslant n$ such that $f_{nn_0}(H_{n_0}) = f_{nm}(H_m)$ for each $m \geqslant n_0$.

2.4. *If an inverse sequence has a regular system of generators, then it is an* ML-*sequence. Every finitely generated* ML-*sequence contains some subsequence with a regular system of generators. If an inverse sequence contains some subsequence which is an* ML-*sequence, the original sequence is an* ML-*sequence.*

The proof is easy and is left to the reader (Hint: use 2.2).

The sequence $\underline{H}$ is called *movable* if for each n there exists an $n_0 \geqslant n$ such that for each $m \geqslant n$ there is a homomorphism h from H_{n_0} to H_m such that $f_{nm} \circ h = f_{nn_0}$.

It is easily seen (comp. [11]) that

2.5. *If a free sequence has a regular system of generators, then it is movable.*

The sequence $\underline{H}$ is called *simply movable* if for each index n there is a homomorphism $h_n\colon H_{n+1} \to H_{n+2}$ such that $f_n = f_{nn+2} \circ h_n$. Let us observe that setting $H'_n = \operatorname{im} f_n$ one obtains $H'_n = f_n(H'_{n+1})$. This shows that each simply movable sequence is an ML-sequence. It is evident that each movable sequence contains a simply movable subsequence. Combining these facts with 2.4 and 2.5, we obtain

2.6. *For every finitely generated sequence of groups $\underline{H}$ we have the following implications*:

($\underline{H}$ *is free and an* ML-*sequence*)

$\Downarrow$

($\underline{H}$ *is movable*) $\Rightarrow$ ($\underline{H}$ *is an* ML-*sequence*) $\Leftarrow$ ($\underline{H}$ *has a regular system of generators*)

($\underline{H}$ *is an* ML-*sequence*) $\Leftarrow$ ($\underline{H}$ *contains a subsequence with a regular system of generators*),

3. Modifications of ANR-sequences. If $(\underline{X}, x_0) = \{(X_n, x_n), f_n\}$ is a pointed ANR-sequence, then we denote

$$\pi(\underline{X}, x_0) = \{\pi(X_n, x_n), (f_n)_\#\},$$

where $\pi(X_n, x_n)$ denotes the fundamental group of X_n with respect to the point $x_n \in X_n$, and $(f_n)_\#\colon \pi(X_{n+1}, x_{n+1}) \to \pi(X_n, x_n)$ is the homomorphism induced by the map $f_n\colon (X_{n+1}, x_{n+1}) \to (X_n, x_n)$. The symbol $\operatorname{inv\,lim}(\underline{X}, x_0)$ denotes the inverse limit of the inverse sequence $(\underline{X}, x_0)$. The sequence $(\underline{X}, x_0)$ is said to be *associated* with every pointed space homeomorphic to $\operatorname{inv\,lim}(\underline{X}, x_0)$.

Theorem below will be used in our subsequent considerations.

3.1. THEOREM. *Let $(\underline{X}, x_0)$ be a pointed* ANR-*sequence such that $\pi(\underline{X}, x_0)$ is an* ML-*sequence. Then there exists a pointed* ANR-*sequence $(\underline{Y}, y_0)$ such that $\pi(\underline{Y}, y_0)$ is an epi-sequence and*

$$\operatorname{inv\,lim}(\underline{Y}, y_0) = \operatorname{inv\,lim}(\underline{X}, x_0).$$

Proof. Let $(\underline{X}, x_0) = \{(X_n, x_n), f_n\}$ and put $G_n = \pi(X_n, x_n)$, $h_n = (f_n)_\#$. By 2.6 there is a subsequence of $\underline{G} = \{G_n, h_n\}$ which has a regular system of generators. Without loss of generality we may assume that $\underline{G}$ has a regular system of generators, say $\{G^*_n\}$ (otherwise we consider the subsequence with this property). For each n there is a set $F_n \subset G^*_n$ such that $F_n = h_{nm}(G^*_m)$ for almost all $m \geqslant n$. Without loss of generality we may assume that $F_n = h_{nm}(G^*_m)$ for each $m > n$. It follows that $F_n = h_n(F_{n+1})$ for each $n \geqslant 1$. The group $[F_n]$ generated by F_n in G_n is equal to $\operatorname{im} h_{nm}$ for each $m > n$. Now we shall construct some pointed ANR-sequence $(\underline{Y}, y_0) = \{(Y_n, y_n), g_n\}$ with the required properties. Let $\hat{X}_1$ denote the cone over X_1 and regard X_1 as the base of $\hat{X}_1$. Put $(Y_1, y_1) = (\hat{X}_1, x_1)$. Let $n \geqslant 2$ and suppose we have constructed ANR-sets $(Y_1, y_1), \ldots, (Y_{n-1}, y_{n-1})$ and the mappings $g_1, \ldots, g_{n-2}$ satisfying the following conditions:

(1) $$(X_j, x_j) \subset (Y_j, y_j),$$

(2) $$g_{j-1}(Y_j) \subset X_{j-1},$$

(3) $i_{j-1} \circ f_{j-1} = g_{j-1} \circ i_j$, where $i_k\colon (X_k, x_k) \to (Y_k, y_k)$ denotes the inclusion map,

(4) $(i_j \circ f_j)_\#$ is an epimorphism,

for every $j = 1, 2, \ldots, n-1$. Now we construct (Y_n, y_n) and g_{n-1} such that conditions (1)-(4) are fulfilled for $j = n$. The space Y_n will be obtained from X_n by attaching to X_n a finite number of 2-cells corresponding to the elements of the finite set $A = G^*_n \setminus F_n$. If $A = \emptyset$, then put $(Y_n, y_n) = (X_n, x_n)$ and note that this set satisfies the conditions, with $g_{n-1} = i_{n-1} \circ f_{n-1}$. Suppose $A \neq \emptyset$ and pick a point a from this set. Let $\omega\colon (I, \dot{I}) \to (X_n, x_n)$ be a loop representing a. Since $h_{n-1}(a) \in F_{n-1}$, there existis an element $[\omega_1] \in F_n$ such that

(5) $$f_{n-1} \circ \omega \simeq f_{n-1} \circ \omega_1 \quad \text{rel. } \dot{I}.$$

Let $\dot{I}^2$ denote the boundary of the square $I^2 = I \times I$ and let $\varphi_a\colon \dot{I}^2 \to X_n$ be defined by the formula:

$$\varphi_a(t, s) = \begin{cases} \omega(t) & \text{for} \quad s = 0, \\ x_n & \text{for} \quad t = 0 \text{ or } t = 1, \\ \omega_1(t) & \text{for} \quad s = 1. \end{cases}$$

Attach the 2-cell to X_n according to the map φ_a and observe that by (5) the projection f_{n-1} can be extended onto the attached cell. Let $y_n = x_n$ and let Y_n be the space obtained from X_n by the process of attaching 2-cells by means of the maps φ_a, $a \in A$. By the above remark the mapping f_{n-1} can be extended to a map

$$g\colon (Y_n, y_n) \to (X_{n-1}, x_{n-1}).$$

Put $g_n = i_{n-1} \circ g$. The space X_n is naturally embedded in Y_n. Moreover, the space Y_n is an ANR-set by a result of Borsuk and Whitehead ([2] p. 116). Since G_n^* generates G_n, $(i_n)_\#(G_n^*) = (i_n)_\#(F_n)$ and $F_n \subset \operatorname{im} h_n$, condition (4) is fulfilled for $j = n$, (comp. [19], p. 146). All the remaining conditions follow from the above remarks. This completes the construction.

Hence we may assume that conditions (1)-(4) are satisfied for each $j \geq 1$. Conditions (1) (2) and (3) imply that $\operatorname{invlim}(\underline{Y}, y_0) = \operatorname{invlim}(\underline{X}, x_0)$. Conditions (3) and (4) imply that $\{\pi(Y_n, y_n), (g_n)_\#\}$ is an epi-sequence, which completes the proof.

4. Other characterizations of pointed 1-movability and its invariance under continuous mappings. This and the next section are devoted to the proof of the following two theorems.

4.1. THEOREM. *Pointed 1-movability is an invariant of continuous mappings. Precisely, if f is a continuous map from a pointed 1-movable continuum X onto Y, then Y is pointed 1-movable.*

4.2. THEOREM. *If X is a continuum and x_0 a point of X, then the following are equivalent:*

(A) *X is pointed 1-movable,*

(B) *for every ANR-sequence $(\underline{X}, x_0)$ associated with (X, x_0) the corresponding sequence of fundamental groups $\pi(\underline{X}, x_0)$ is an ML-sequence,*

(C) *there exists an ANR-sequence $(\underline{X}, x_0)$ such that $\pi(\underline{X}, x_0)$ is an epi-sequence,*

(D) *there exists a decreasing sequence of ANR-sets $X_1, X_2, \ldots$ such that $X = \bigcap_n X_n$ and for each $n \geq 1$ the inclusion map*

$$(X_{n+1}, x_0) \hookrightarrow (X_n, x_0)$$

induces an epimorphism of fundamental groups,

(E) *there exists a decreasing sequence of locally connected continua $X_1, X_2, \ldots$ such that $X = \bigcap_n X_n$ and for each n the inclusion*

$$(X_{n+1}, x_0) \hookrightarrow (X_n, x_0)$$

induces an epimorphism of fundamental groups,

(F) *there exists an ANR-sequence $(\underline{X}, x_0)$ associated with (X, x_0) such that $\pi(\underline{X}, x_0)$ is an ML-sequence.*

Theorems 4.1 and 4.2 will be proved simultaneously in the following manner. First we prove the implications:

(α) (F) $\Rightarrow$ (A) $\Rightarrow$ (B) $\Rightarrow$ (C) $\Rightarrow$ (D) $\Rightarrow$ (E).

Next we shall prove that

(β) *if $f\colon (X, x_0) \to (Y, y_0)$ is a continuous surjective map and (X, x_0) is a pointed continuum satisfying condition (E), then (Y, y_0) satisfies condition (F), where (X, x_0), $(\underline{X}, x_0)$ are replaced by (Y, y_0), $(\underline{Y}, y_0)$, respectively.*

Clearly, propositions (α) and (β) together imply both 4.1 and 4.2.

Proof of (α). (F)$\Rightarrow$(A)$\Rightarrow$(B). Let $(\underline{X}, x_0) = \{(X_n, x_n), f_n\}$ be an ANR-sequences associated with a pointed continuum (X, x_0). Using similar considerations to those in [13] (or the construction presented in [12]), one proves that the 1-movability of (X, x_0) is equivalent to the following property of $(\underline{X}, x_0)$: for each $n \geq 1$ there is an $n_0 \geq n$ such that for each $m \geq n_0$ and each loop $\omega\colon (I, \dot{I}) \to (X_{n_0}, x_{n_0})$ there is a loop $\omega'\colon (I, \dot{I}) \to (X_m, x_m)$ such that $f_{nn_0} \circ \omega \simeq f_{nm} \circ \omega'$. However, this property says that $\pi(\underline{X}, x_0)$ is an ML-sequence. Hence the pointed 1-movability of X is equivalent to the fact that $\pi(\underline{X}, x_0)$ is an ML-sequence for any ANR-sequence $(\underline{X}, x_0)$ associated with (X, x_0). On the other hand, by a classical result of Freudenthal, every compactum possesses an ANR-sequence associated with it. This proves the above implications.

(B)$\Rightarrow$(C). This follows from 3.1.

(C)$\Rightarrow$(D). This is proved in [12].

The last implication, (D)$\Rightarrow$(E), is trivial. This completes the proof of proposition (α).

Proposition (β) is much more complicated and its proof will be given in the next section.

5. Proof of (β). The proof is given at the end of this section after several auxiliary lemmas. For the definitions of the undefined terms used in this section the reader is referred to [19].

A map $f\colon X \to Y$ is said to be *nondegenerate* provided each fibre $f^{-1}(y)$ is totally disconnected. If K is a simplicial complex, then by $|K|$ we denote the underlying space of K.

5.1. LEMMA. *Let K and L be locally finite complexes and let $f\colon |K| \to |L|$ be a nondegenerate map simplicial with respect to K and L. If A is a continuum in $|K|$ and V is a neighbourhood of A in $|K|$, then there exist subdivisions K' of K and L' of L, and a finite connected subcomplex N of K' such that $|N|$ is a neighbourhood of A contained in V and f is simplicial with respect to K' and L'. If x is a point of A, the subdivision K' can be chosen is such a way that x is a vertex of K'.*

5.2. LEMMA. *Let (X, x_0) be a pointed polyhedron and let G be a subgroup of $\pi(X, x_0)$. Then there is a covering projection $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ such that $\operatorname{im} p_\# = G$* (see [19], p. 82).

5.3. LEMMA. *Let X be an ANR-set and let $p\colon \tilde{X} \to X$ be a covering projection. Assume that A is a compact subset of $\tilde{X}$ and U is a neighbourhood of A in $\tilde{X}$. Then there is a number $\varepsilon > 0$ satisfying the following condition: if $f\colon C \to X$ is a mapping of a connected space C such that $\operatorname{diam} f(C) < \varepsilon$ and $\tilde{f}\colon C \to \tilde{X}$ is a lifting of f, i.e., $p \circ \tilde{f} = f$, such that $\tilde{f}(C) \cap A \neq \emptyset$, then $\tilde{f}(C) \subset U$.*

Proof. Since $\tilde{X}$ is metrizable, A is a compactum. Suppose that the conclusion of the lemma fails. So for each $n \geq 1$ there is a connected space C_n and two mappings $f_n\colon C_n \to X$, $\tilde{f}_n\colon C_n \to \tilde{X}$ such that

(1) $$f_n = p \circ \tilde{f}_n,$$

(2) $$\operatorname{diam} f_n(C_n) < 1/n,$$

(3) $$\tilde{f}_n(C_n) \cap A \neq \emptyset,$$

(4) $$\tilde{f}_n(C_n) \not\subset U.$$

By (3) there is a point $a_n \in \tilde{f}_n(C_n) \cap A$. The sequence $\{a_n\}$ contains some subsequence converging to a point $a \in A$. Without loss of generality we may assume that $\{a_n\}$ converges to a. Now take an evenly covered open neighbourhood V of $p(a)$ in X. Then $p^{-1}(V)$ is the union of open disjoint sets V_s, $s \in S$, such that $p|V_s\colon V_s \to V$ is a homeomorphism. Assume $a \in V_{s'}$. Then $U' = U \cap V_{s'}$ is a neighbourhood of a in $\tilde{X}$, hence there is an index n_1 such that

(5) $$a_n \in U' \quad \text{for} \quad n \geqslant n_1.$$

The set $p(U')$ is a neighbourhood of $p(a)$ in X. Observe that by (1) the points $p(a_n)$ converge to $p(a)$ and $p(a_n) \in f_n(C_n)$. Applying (2), we infer that there is an index n_2 such that

(6) $$f_n(C_n) \subset p(U') \quad \text{for} \quad n \geqslant n_2.$$

Take $n \geqslant n_1, n_2$. Since $p(U') \subset V$, by (6) we have $f_n(C_n) \subset V$. Thus $\tilde{f}_n(C_n) \subset \bigcup V_s$. Since $\tilde{f}_n(C_n)$ is connected, and (5) implies $a_n \in U' \cap \tilde{f}_n(C_n) \subset V_{s'} \cap \tilde{f}_n(C_n)$, we have $\tilde{f}_n(C_n) \subset V_{s'}$. Since $g = p|V_{s'}$ maps $V_{s'}$ homeomorphicly onto V, $U' \subset V_{s'}$ and $g(\tilde{f}_n(C_n)) \subset g(U')$, we have $\tilde{f}_n(C_n) \subset U' \subset U$, contrary to (4). This completes the proof.

5.4. LEMMA. *Let X be a compactum contained in a metric space M. Let f be a map from X into an* ANR-*set Y and let $f_i\colon V_i \to Y$, $i = 1, 2$, be two extensions of f to neighbourhoods V_i of X in M. Then for each $\varepsilon > 0$ there is a neighbourhood V of X in M such that $V \subset V_1 \cap V_2$ and $f_1|V \underset{\varepsilon}{\simeq} f_2|V$ rel. X, i.e., there is a homotopy $h\colon V \times I \to Y$ joining these maps, keeping each point of X fixed and such that* diam $h(\{z\} \times I) < \varepsilon$ *for each $z \in V$.*

The proof is easy and is left to the reader.

5.5. LEMMA. *Let $(\underline{Y}, y_0) = \{(Y_n, y_n), g_n\}$ be an inverse sequence of pointed connected polyhedra with surjective bonding maps $g_n\colon (Y_{n+1}, y_{n+1}) \to (Y_n, y_n)$, and let $(Y, y_0) = \operatorname{invlim}(\underline{Y}, y_0)$. Assume that (Y, y_0) can be represented as a continuous image of a pointed continuum (X, x_0) satisfying condition* (E) *of* 4.2. *Then for each $n \geqslant 1$ there exist: a compact connected polyhedron $(U_n, \tilde{y}_n)$, triangulations (L_n, y_n), $(N_n, \tilde{y}_n)$ of respectively (Y_n, y_n), $(U_n, \tilde{y}_n)$, and mappings $h_n\colon (U_n, y_n) \to (Y_n, y_n)$, $\hat{g}_n\colon (U_{n+1}, \tilde{y}_{n+1}) \to (U_n, \tilde{y}_n)$ satisfying the conditions:*

(a) *h_n is nondegenerate and simplicial with respect to N_n and L_n, $\tilde{y}_n$ is a vertex of N_n,*

(b) $$h_n(U_n) = Y_n,$$

(c) $$h_n \circ \hat{g}_n = g_n \circ h_{n+1},$$

(d) $$(g_n)_{\#}(\operatorname{im}(h_{n+1})_{\#}) = \operatorname{im}(h_n)_{\#}.$$

Proof. Let $f\colon (X, x_0) \to (Y, y_0)$ be a surjective map. Let $f_n\colon (X, x_0) \to (Y_n, y_n)$ denote the composition $\pi_n \circ f$, where π_n is the natural projection from (Y, y_0) into (Y_n, y_n). Observe that each f_n is surjective and

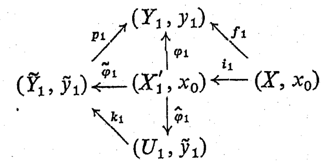
(1) $$f_n = g_n \circ f_{n+1}.$$

Put $M = X_1$ (see 4.2 (E)). Since $Y_1 \in$ ANR, there is a neighbourhood W of X in M and some extension φ_1' of f_1 onto (W, x_0). Since X is the intersection of the decreasing sequence $X_1, X_2, \ldots$, there is an index m_1 such that $X_1' = X_{m_1} \subset W$. Let φ_1 denote the restriction of φ_1' to (X_1', x_0). According to 5.2 there is a covering projection $p_1\colon (\tilde{Y}_1, \tilde{y}_1) \to (Y_1, y_1)$ such that

(2) $$\operatorname{im}(p_1)_{\#} = \operatorname{im}(\varphi_1)_{\#}.$$

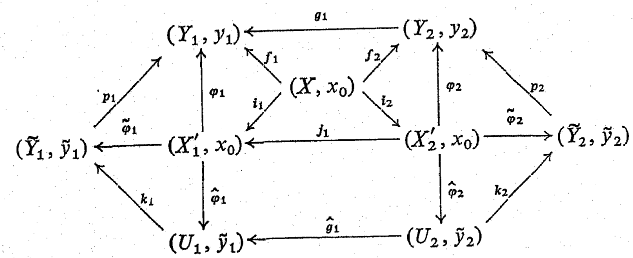
By the lifting theorem there exists a lifting $\tilde{\varphi}_1\colon (X_1', x_0) \to (\tilde{Y}_1, \tilde{y}_1)$ of φ_1. Let $A_1 = \tilde{\varphi}_1(X_1')$. Since each covering projection into a polyhedron may be regarded as a simplicial nondegenerate map (see [19], p. 144), by 5.1 there is a triangulation (L_1, y_1) of (Y_1, y_1) and a finite connected complex $(N_1, \tilde{y}_1)$, $\tilde{y}_1$ being a vertex of N_1, such that $U_1 = |N_1|$ is a neighbourhood of A_1 in $\tilde{Y}_1$ and p_1 is nondegenerate and simplicial with respect to N_1 and L_1. Let $i_1\colon (X, x_0) \to (X_1', x_0)$ and $k_1\colon (U_1, \tilde{y}_1) \to (\tilde{Y}_1, \tilde{y}_1)$ denote the inclusions and let $\hat{\varphi}_1\colon (X_1', x_0) \to (U_1, \tilde{y}_1)$ be defined by $\tilde{\varphi}_1$. Hence the following diagram commutes:

(D$_1$)



If we set $h_1 = p_1 \circ k_1$, conditions (a) and (b) are satisfied for $n = 1$. As we shall show, the following sublemma completes the proof of 5.5.

SUBLEMMA. *In the notation of the diagram: there is a neighbourhood V of X in M such that for each locally connected continuum $X_2' \subset V \cap X_1'$ which contains X the diagram* (D$_1$) *can be completed to a (not necessarily commutative) diagram*



with the following properties: p_2 *is a covering projection*, i_2, j_1 *and* k_2 *are inclusion maps*, U_2 *is a compact connected neighbourhood of* $A_2 = \tilde{\varphi}_2(X_2')$ *in* $\tilde{Y}_2$, $\operatorname{im}(p_2)_\# = \operatorname{im}(\varphi_2)_\#$, *there exist finite complexes* L_2 *and* N_2 *such that* $U_2 = |N_2|$, $Y_2 = |L_2|$ *and*

(3) $h_2 = p_2 \circ k_2$ *is nondegenerate and simplicial with respect to* N_2 *and* L_2, $\tilde{y}_2$ *is a vertex of* N_2,

(4) $$h_2(U_2) = Y_2,$$

(5) $$g_1 \circ h_2 = h_1 \circ \tilde{g}_1,$$

(6) *if* $(j_1)_\#$ *is an epimorphism, then* $(g_1)_\#(\operatorname{im}(h_2)_\#) = \operatorname{im}(h_1)_\#$,

(7) *the diagram* (D_2) *obtained from* (D_1) *by replacing all subscripts* 1 *by* 2 *commutes*.

Observe that by (3)-(5) properties (a) and (b) are fulfilled for $n = 2$, and property (c) for $n = 1$. Property (d) follows from (6) because by our assumption about X we can find some locally connected continuum $X_2' \subset V \cap X_1'$ containing X and such that $(j_1)_\#$ is an epimorphism. Moreover, the assumption about X, condition (7), and the remaining properties stated in the sublemma enable us to prove a similar sublemma for (D_2) (with X_2' suitably chosen). In the same manner we can define recursively diagrams (D_3), (D_4), ... and prove a similar sublemma for each of them (choosing suitable locally connected continua $X_3', X_4', \ldots$). This, by induction, completes the proof of 5.5. Thus it remains to prove the sublemma.

Proof of sublemma. Since $\operatorname{Int} U_1$ is a neighbourhood of A_1 in $\tilde{Y}_1$ and $Y_1 \in \mathrm{ANR}$, we infer from 5.3 that there exists a number $\varepsilon>0$ such that

(8) if ω is a mapping of a connected space C into Y_1 such that $\operatorname{diam}\omega(C)<\varepsilon$ and $\tilde{\omega}\colon C\to \tilde{Y}_1$ is a lifting of ω, then the condition $\tilde{\omega}(C) \cap A_1 \neq \emptyset$ implies $\tilde{\omega}(C)\subset \operatorname{Int} U_1$.

Let $\psi_1\colon (V_1, x_0)\to(Y_1, y_1)$ be an extension of φ_1 over some neighbourhood V_1 of X_1' in M, and let $\psi_2\colon (V_2, x_0)\to(Y_2, y_2)$ be an extension of f_2 over a neighbourhood V_2 of X in M.

For every $x \in X$ we have by (1): $\psi_1(x) = \varphi_1(x) = \varphi_1 \circ i_1(x) = f_1(x) = g_1 \circ f_2(x) = g_1 \circ \psi_2(x)$. So ψ_1 and $g_1 \circ \psi_2$ are both extensions of the same map from X into Y_1. Since V_1 is a neighbourhood of X in M, by 5.4 there is a neighbourhood V of X in M such that $V\subset V_1 \cap V_2$ and

(9) $$\psi_1|V \simeq g_1 \circ \psi_2|V \quad \text{rel. } X.$$

We shall prove that V is a desired neighbourhood of X. Let $X_2' \subset V \cap X_1'$ be a locally connected continuum containing X. Setting $\varphi_2 = \psi_2|X_2'$ one obtains the equality

(10) $$f_2 = \varphi_2 \circ i_2,$$

and using condition (9), we infer that there is a map $h\colon X_2' \times I\to Y_1$ such that the following conditions hold:

(11) $h(x, 0) = \varphi_1 \circ j_1(x)$, $h(x, 1) = g_1 \circ \varphi_2(x)$, $h(x_0, t) = y_1$ and $\operatorname{diam} h(\{x\} \times I) < \varepsilon$ for every $x \in X_2'$ and $t \in I$.

According to 5.2 there exists a covering projection $p_2\colon (\tilde{Y}_2, \tilde{y}_2)\to(Y_2, y_2)$ such that $\operatorname{im}(p_2)_\# = \operatorname{im}(\varphi_2)_\#$. Since Y_2 is a polyhedron with a triangulation, say L, we may assume that $\tilde{Y}_2$ is an infinite polyhedron with a triangulation K such that p_2 is simplicial with respect to K and L (see [19], p. 144). Observe also that p_2 is nondegenerate. Being a locally connected continuum, X_2' is arcwise connected and locally arcwise connected. Hence the inclusion $\operatorname{im}(\varphi_2)_\# \subset \operatorname{im}(p_2)_\#$ implies the existence of a lifting $\tilde{\varphi}_2$ of φ_2. Thus we obtain

(12) $$\varphi_2 = p_2 \circ \tilde{\varphi}_2.$$

Using (11) we have $\varphi_1 \circ j_1 \simeq g_1 \circ \varphi_2$ rel. x_0. It follows that

(13) $$(\varphi_1 \circ j_1)_\# = (g_1 \circ \varphi_2)_\#.$$

From this equality we derive in succesion: $\operatorname{im}(g_1 \circ p_2)_\# = \operatorname{im}(g_1)_\# \circ (p_2)_\# = (g_1)_\#(\operatorname{im}(p_2)_\#) = (g_1)_\#(\operatorname{im}(\varphi_2)_\#) = \operatorname{im}(g_1 \circ \varphi_2)_\# = \operatorname{im}(\varphi_1 \circ j_1)_\# \subset \operatorname{im}(\varphi_1)_\# = \operatorname{im}(p_1)_\#$. Since $\tilde{Y}_2$ is arcwise connected and locally arcwise connected, the inclusion $\operatorname{im}(g_1 \circ p_2)_\# \subset \operatorname{im}(p_1)_\#$ implies the existence of a map $\tilde{g}_1\colon (\tilde{Y}_2, \tilde{y}_2)\to(\tilde{Y}_1, \tilde{y}_1)$ satisfying the condition

(14) $$g_1 \circ p_2 = p_1 \circ \tilde{g}_1$$

(see the lifting theorem in [19], p. 76). Since each covering projection has the homotopy lifting property (see [19], p. 67) and by (11) we have $p_1 \circ (\tilde{\varphi}_1 \circ j_1)(x) = h(x, 0)$, there is a map $\tilde{h}\colon X_2' \times I\to \tilde{Y}_1$ such that

(15) $$\tilde{h}(x, 0) = \tilde{\varphi}_1 \circ j_1(x) \quad \text{and} \quad p_1 \circ \tilde{h} = h.$$

Now we shall show that

(16) $$\tilde{h}(X_2' \times I)\subset \operatorname{Int} U_1 \quad \text{and} \quad \tilde{h}(\{x_0\} \times I) = \{\tilde{y}_1\}.$$

According to (11) and (15) we have $\tilde{h}(\{x_0\} \times I)\subset p_1^{-1}(y_1)$ and the latter space is discrete. Therefore the equality $\tilde{h}(x_0, 0) = \tilde{\varphi}_1(x_0) = \tilde{y}_1$ implies the second part of (16). To prove the first part, consider an arbitrary point $x \in X_2'$ and the map $\omega\colon I\to Y_1$ given by $\omega(t) = h(x, t)$. By (11) we have $\operatorname{diam}\omega(I)<\varepsilon$. It follows from (15) that the map $\tilde{\omega}\colon I\to \tilde{Y}_1$ defined by $\tilde{\omega}(t) = \tilde{h}(x, t)$ is a lifting of ω. Moreover, $\tilde{\omega}(0) = \tilde{h}(x, 0) = \tilde{\varphi}_1 \circ j_1(x) \in \tilde{\varphi}_1(X_1') = A_1$, hence $\tilde{\omega}(I) \cap A_1 \neq \emptyset$. Thus condition (8) implies $\tilde{\omega}(I)\subset \operatorname{Int} U_1$. Since this is true for every $x \in X_2'$, condition (16) is proved.

Note that conditions (11) and (15) together imply $p_1 \circ \tilde{h}(x, 1) = g_1 \circ \varphi_2(x)$ for every $x \in X_2'$. Hence, setting $\psi(x) = \tilde{h}(x, 1)$, we obtain by (16) a map $\psi\colon (X_2', x_0)\to(\tilde{Y}_1, \tilde{y}_1)$ satisfying the conditions

(17) $$p_1 \circ \psi = g_1 \circ \varphi_2 \quad \text{and} \quad \psi(X_2')\subset \operatorname{Int} U_1.$$

Using (14), (12) and (17) we obtain $p_1 \circ \tilde{g}_1 \circ \tilde{\varphi}_2 = g_1 \circ \varphi_2 = p_1 \circ \psi$. Since $\tilde{g}_1 \circ \tilde{\varphi}_2(x_0) = \tilde{y}_1 = \psi(x_0)$ and X_2' is connected, we have $\psi = \tilde{g}_1 \circ \tilde{\varphi}_2$ (see [19], p. 67). Setting $A_2 = \tilde{\varphi}_2(X_2')$ we obtain from the last equality and (17) the following inclusion:

(18) $$\tilde{g}_1(A_2)\subset \operatorname{Int} U_1.$$

Let $G = \tilde{g}_1^{-1}(\operatorname{Int} U_1)$. Hence G is an open subset of $\tilde{Y}_2$ and $A_2 \subset G$. So by 5.1 there exist subdivisions K_2, L_2 of K, L, respectively, and a finite complex $N_2 \subset K_2$ such that $\tilde{y}_2$ is a vertex of N_2 and

(19) $U_2 = |N_2| \subset G$ is a compact connected neighbourhood of A_2 in $\tilde{Y}_2$, and $h_2 = p_2 \circ k_2$ is simplicial with respect to N_2 and L_2, where k_2 denotes the inclusion map (so we have (3)).

It follows that $\tilde{g}_1(U_2) \subset U_1$. Since $\tilde{g}_1(\tilde{y}_2) = \tilde{y}_1$, by setting $\hat{g}_1(x) = \tilde{g}_1(x)$ for every $x \in U_2$ one obtains a well-defined map $\hat{g}_1: (U_2, \tilde{y}_2) \to (U_1, \tilde{y}_1)$. In particular, we have

(20) $$\tilde{g}_1 \circ k_2 = k_1 \circ \hat{g}_1 .$$

By (14) and (20) we have $g_1 \circ h_2 = g_1 \circ p_2 \circ k_2 = p_1 \circ \tilde{g}_1 \circ k_2 = p_1 \circ k_1 \circ \hat{g}_1 = h_1 \circ \hat{g}_1$, which is (5). Since f_2 is onto, by (10) and (12) we see that $p_2(A_2) = Y_2$. But $h_2(x) = p_2(x)$ for $x \in A_2$. Since $A_2 \subset U_2$, condition (4) follows. Let $\hat{\varphi}_2: (X_2', x_0) \to (U_2, \tilde{y}_2)$ be defined by $\hat{\varphi}_2(x) = \tilde{\varphi}_2(x)$. Let us observe that in this notation condition (7) is also fulfilled. It remains to show (6). Thus suppose that $(j_1)_\#: \pi(X_2', x_0) \to \pi(X_1', x_0)$ is an epimorphism. Let us note that $\varphi_i = p_i \circ k_i \circ \hat{\varphi}_i$ and $\operatorname{im}(\varphi_i)_\# = \operatorname{im}(p_i)_\#$; hence by (12) we obtain $\operatorname{im}(\varphi_i)_\# = \operatorname{im}(h_i)_\#$, for $i = 1, 2$. Hence condition (13) implies $(g_1)_\#(\operatorname{im}(h_2)_\#) = (g_1)_\#(\operatorname{im}(\varphi_2)_\#) = \operatorname{im}(g_1 \circ \varphi_2)_\# = \operatorname{im}(\varphi_1 \circ j_1)_\# = (\varphi_1)_\#(\operatorname{im}(j_1)_\#) = \operatorname{im}(\varphi_1)_\# = \operatorname{im}(h_1)_\#$, which is exactly (6). This completes the proof of the sublemma, and therefore the proof of 5.5.

Let X be the underlying space of a simplicial complex K, and let ω be a path in X. We say that ω is an *s-path* provided that ω is simplicial with respect to some subdivision of I and K. If, in addition, ω is an embedding or $\omega(I)$ is degenerate, then ω is said to be an *sh-path*. Recall that by a *torn loop in X based at $x_0 \in X$*, briefly: a *t-loop in (X, x_0)*, we mean a pair of paths (ω^0, ω^1) in X such that $\omega^0(0) = \omega^1(1) = x_0$ (see [11]). In such a case we write $(\omega^0, \omega^1): I \to (X, x_0)$. A t-loop (ω^0, ω^1) is called an *st-loop* (an *sht-loop*) provided both ω^0 and ω^1 are s-paths (sh-paths, resp.).

5.6. Lemma. *If $X = |K|$, x_0 is a vertex of K and (ω^0, ω^1) is a t-loop in (X, x_0) such that $\omega^0(1)$ and $\omega^1(0)$ are vertices of K, then there exist paths $\varphi, \varphi_1, \psi, \psi_1$ satisfying the conditions*:

(1) *φ_1, ψ_1 are loops in (X, x_0),*

(2) *φ is an sh-path from x_0 to $\omega^0(1)$,*

(3) *ψ is an sh-path from $\omega^1(0)$ to x_0,*

(4) $\omega^0 \simeq \varphi_1 * \varphi$ *rel.* $\dot{I}$,

(5) $\omega^1 \simeq \psi * \psi_1$ *rel.* $\dot{I}$.

Proof. Point x_0 and the endpoints of the paths are vertices of K belonging to the same component of K; hence there exist paths φ and ψ satisfying (2) and (3). The loops $\varphi_1 = \omega^0 * \varphi^{-1}$ and $\psi_1 = \psi^{-1} * \omega^1$ have the desired properties. This completes the proof.

Let $f: (X, x_0) \to (Y, y_0)$ be a mapping, ω a loop in (Y, y_0) and (ω^0, ω^1) a t-loop in (X, x_0). Then we say that (ω^0, ω^1) is a *t_f-lifting* of ω, and write $\omega = f(\omega^0, \omega^1)$, provided the following conditions are fulfilled:

(i) $$f(\omega^0(1)) = f(\omega^1(0)),$$

(ii) $$\omega(t) = \begin{cases} f \circ \omega^0(2t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2}, \\ f \circ \omega^1(2t-1) & \text{for } \frac{1}{2} \leqslant t \leqslant 1 . \end{cases}$$

If (ω_1^0, ω_1^1) is another t-loop in (X, x_0) such that $\omega^0 \simeq \omega_1^0$ and $\omega^1 \simeq \omega_1^1$, then $f(\omega^0, \omega^1) \simeq f(\omega_1^0, \omega_1^1)$ (all homotopies rel. $\dot{I}$). In other words, the loops $f(\omega^0, \omega^1)$ and $f(\omega_1^0, \omega_1^1)$ generate the same element of the fundamental group of (Y, y_0), i.e., $[f(\omega^0, \omega^1)] = [f(\omega_1^0, \omega_1^1)]$. The following homotopies are easily checked:

$$f(\omega^0, \omega^1) \simeq f \circ \omega^0 * f \circ \omega^1 ,$$

$$f \circ \sigma * \tau \simeq f \circ \sigma * f \circ \tau ,$$

where σ, τ are paths in X such that $\sigma(1) = \tau(0)$.

5.7. Lemma. *Let $X = |K|$ and $Y = |L|$ be compact connected polyhedra and let x_0 be a vertex of K. Assume that $f: (X, x_0) \to (Y, y_0)$ is a surjective map nondegenerate and simplicial with respect to K and L. Then for $H = \pi(Y, y_0)$, $G = \operatorname{im} f_\#$,*

$$H^* = \{h \in H:\ h = [f(\omega^0, \omega^1)] \text{ for some } t\text{-loop } (\omega^0, \omega^1) \text{ in } (X, x_0)\}$$

and

$$A = \{h \in H:\ h = [f(\omega^0, \omega^1)] \text{ for some } sht\text{-loop } (\omega^0, \omega^1) \text{ in } (X, x_0)\}$$

we have:

(1) *H^* is a set of generators for H,*

(2) *G is a finitely generated subgroup of H,*

(3) $G \cup A \subset H^*$,

(4) *A is a finite set,*

(5) *for each $h \in H^*$ there exist $g_1, g_2 \in G$ and $a \in A$ such that $h = g_1 a g_2$.*

Proof. Condition (1) follows from 6.4 of [11]. The group G is finitely generated because the fundamental group of every compact polyhedron is finitely generated. So we have (2). Condition (3) is obvious. By a slight modification of the proof of 6.3 in [11] we obtain condition (4). To prove (5) consider an arbitrary $h \in H^*$. Hence $h = [f(\omega^0, \omega^1)]$ for some t-loop (ω^0, ω^1) in (X, x_0). Using an argument almost identical with that used in the proof of 6.2 in [11] (Case II of the proof), we may assume that $\omega(\frac{1}{2})$ is a vertex of L, where $\omega = f(\omega^0, \omega^1)$. Since f is simplicial and nondegenerate, the points $\omega^0(1)$ and $\omega^1(0)$ are vertices of K. Hence by 5.6 there exist paths φ, φ_1, ψ, ψ_1 satisfying the conclusion of 5.6. So we have

$$h = [f(\omega^0, \omega^1)] = [f(\varphi_1 * \varphi, \psi * \psi_1)] = [f \circ \varphi_1 * f \circ \varphi * f \circ \psi * f \circ \psi_1]$$
$$= [f \circ \varphi_1 * f(\varphi, \psi) * f \circ \psi_1].$$

The paths φ_1, ψ_1 are loops in (X, x_0). Hence $g_1' = [\varphi_1]$ and $g_2' = [\psi_1]$ are well-defined elements of $\pi(X, x_0)$. Moreover, $[f(\varphi, \psi)] \in A$. Thus

$$h = f_{\#}(g_1')\cdot[f(\varphi, \psi)]\cdot f_{\#}(g_2'),$$

which proves (5).

Now we are ready to prove proposition (β). Since for every pointed continuum there exists some pointed inverse sequence of compact connected polyhedra with surjective bonding maps associated with that continuum [9], the following theorem implies (β).

5.8. THEOREM. *Let $(\underline{Y}, y_0)$ be a pointed inverse sequence of compact connected polyhedra with surjective bonding maps. If $(Y, y_0) = \operatorname{invlim}(\underline{Y}, y_0)$ can be represented as a continuous image of a pointed continuum (X, x_0) satisfying condition* (E) *of* 4.2, *then $\pi(\underline{Y}, y_0)$ is an* ML-*sequence.*

Proof. Let $(\underline{Y}, y_0) = \{(Y_n, y_n), g_n\}$ and let $f\colon (X, x_0)\to(Y, y_0)$ be a surjective map. According to 5.5 for each $n\geqslant 1$ there exist: a compact connected polyhedron $(U_n, \tilde{y}_n)$, triangulations (L_n, y_n), $(N_n, \tilde{y}_n)$ of (Y_n, y_n), $(U_n, \tilde{y}_n)$, resp., mappings $h_n\colon (U_n, \tilde{y}_n)\to(Y_n, y_n)$, $\hat{g}_n\colon (U_{n+1}, \tilde{y}_{n+1})\to(U_n, \tilde{y}_n)$ such that the following conditions are satisfied:

(1) h_n is nondegenerate and simplicial with respect to N_n and L_n,

(2) $$h_n(U_n) = Y_n,$$

(3) $$h_n\circ\hat{g}_n = g_n\circ h_{n+1},$$

(4) $$(g_n)_{\#}(\operatorname{im}(h_{n+1})_{\#}) = \operatorname{im}(h_n)_{\#}.$$

Let us denote:

$$H_n = \pi(Y_n, y_n), \quad G_n = \operatorname{im}(h_n)_{\#},$$

$$H_n^* = \{h\in H_n\colon h = [h_n(\omega^0, \omega^1)] \text{ for some } t\text{-loop } (\omega^0, \omega^1) \text{ in } (U_n, \tilde{y}_n)\},$$

$$A_n = \{h\in H_n\colon h = [h_n(\omega^0, \omega^1)] \text{ for some } sht\text{-loop } (\omega^0, \omega^1) \text{ in } (U_n, \tilde{y}_n)\}.$$

Observe that by (3) we have

(5) $$(g_n)_{\#}(H_{n+1}^*)\subset H_n^*.$$

According to (1) and (2) we can apply 5.7 to the map h_n. Thus we obtain:

(6) H_n^* generates H_n,

(7) G_n is a finitely generated subgroup of H_n,

(8) $G_n\cup A_n\subset H_n^*$,

(9) A_n is a finite set,

(10) each $h\in H_n^*$ can be written in the form $h = g_1 a g_2$ for some $g_1, g_2\in G_n$ and $a\in A_n$.

By conditions (4)-(10) the hypotheses of 2.3 are fulfilled for the sequence $\pi(\underline{Y}, y_0)$. Hence, applying 2.3, we see that $\pi(\underline{Y}, y_0)$ has a regular system of generators. By 2.4 we conclude that this sequence is an ML-sequence, which completes the proof.

6. Movable curves. Recall that by a *curve* we mean any 1-dimensional continuum. We have the following

6.1. THEOREM (A. Trybulec). *If X is a curve, then the following are equivalent:*

(1) *X is movable,*

(2) *X is pointed movable,*

(3) *X is pointed 1-movable.*

Proof. The equivalence (1)⇔(2) was proved by A. Trybulec in [20]. Hence, by 1.3, it remains to prove (3)⇒(2). To prove this let $x_0\in X$ be some point and let $(\underline{X}, x_0)$ be some inverse sequence of graphs associated with (X, x_0) (see [9]) (by a *graph* we mean a compact connected 1-dimensional polyhedron). Hence if (X, x_0) is 1-movable, then by 4.2 we infer that $\pi(\underline{X}, x_0)$ is an ML-sequence. Since the fundamental group of a graph is finitely generated and free (see [19]), $\pi(\underline{X}, x_0)$ is a finitely generated free sequence. By 2.6 it is movable. Using 5.10 of [11] and the fundamental result of [13], we get the pointed movability of X. This completes the proof.

The following results follow from the preceding theorems:

6.2. THEOREM. *Curves being continuous images of movable curves are movable.*

6.3. THEOREM. *Every hereditarily decomposable continuum is pointed movable. Continuous images of hereditarily decomposable continua are pointed 1-movable.*

This theorem follows from the observation that hereditarily decomposable continua are at most 1-dimensional [16]. The first part of the above theorem is a solution of Problem 3 from [11].

6.4. THEOREM. *Every nonmovable curve contains some indecomposable subcurve which is also nonmovable.*

In [11] the present author has proved that 1-dimensional continuous images of tree-like continua are pointed movable. Since tree-like continua are of trivial shape ([11], 2.1), and every continuum with a trivial shape is pointed movable, Theorems 4.1 and 6.1 generalize this result.

7. Movability of compacta embeddable in surfaces. Recall some commonly used terminology (see e.g. [3]). By a *bounded surface* we understand a nonempty continuum such that each of its points has a neighbourhood homeomorphic to the unit square. Such homeomorphs are called *disks*. The points of a bounded surface M which have a neighbourhood homeomorphic to the Euclidean plane E^2 constitute a set $\mathring{M}$ called the *interior of M*. The complement $M\setminus\mathring{M}$ is denoted by $\dot{M}$ and is called the *boundary of M*. If $\dot{M}$ is empty, then M is called a *surface*. If $\dot{M}$ is not empty, it is the union of a finite number of mutually disjoint simple closed curves. By a *perforated disk* we mean the set which remains after removing from the sphere S^2 the interiors of a finite number of mutually disjoint disks. If M has a nonempty boundary and any simple closed curve in $\mathring{M}$ separates M, then M is a perforated disk. The maximal number of mutually disjoint simple closed curves in $\mathring{M}$ which together do not separate M is called the *genus of* M and is denoted by $\gamma(M)$. Hence M is a perforated disk iff the boundary of M is nonempty and $\gamma(M) = 0$. The genus of M is not changed by the operation of removing the interior of a disk lying in $\mathring{M}$. Observe that if N is a bounded surface in M, then $\gamma(N) \leqslant \gamma(M)$. Hence if $\{M_n\}$ is a strictly decreasing sequence of bounded surfaces in M, that is, $M_{n+1} \subset \mathring{M}_n$ for each $n \geqslant 1$, then there is an integer $k \geqslant 0$ such that $\gamma(M_n) = k$ for almost all n. If X is a continuum in $\mathring{M}$ and $X \neq M$, then there is a strictly decreasing sequence of bounded surfaces in M such that X is the intersection of that sequence. If S is a simple closed curve in $\dot{M}$, D is a disk and h is a homeomorphism of $\dot{D}$ onto S, then the *matching* $D \overset{h}{\cup} M$ is another bounded surface containing M with the same genus as that of M. Hence for any bounded surface M there is a surface $M' \supset M$ such that $\gamma(M') = \gamma(M)$. The following evident fact will be useful in the sequel:

7.1. Lemma. *Let N and N' be bounded surfaces in a surface M such that $M = N \cup N'$ and $N \cap N' = \dot{N}$. Then*

$$\gamma(M) \geqslant \gamma(N) + \gamma(N') + r - 1,$$

where r is the number of simple closed curves in $\dot{N}$.

7.2. Theorem. *Let X be a subcontinuum of a surface M and let $x_0 \in X$. Then (X, x_0) is movable.*

Proof. We may assume that $X \neq M$. Let $\{M_n\}$ be a strictly decreasing sequence of bounded surfaces in M such that $\dot{M}_1$ is not empty and $X = \bigcap_n M_n$. We may also assume that $\gamma(M_n) = c$ for each $n \geqslant 1$. Let M^* be a surface containing M_1 such that $\gamma(M^*) = \gamma(M_1)$. Let $G_1, G_2, \ldots$ be the components of $M^* \setminus X$. We assert that

(1) each G_i is homeomorphic to E^2.

To prove (1) we need only to show that any compact set A in G_i is contained in the interior of a disk $D \subset G_i$. We may assume that A is connected. Since M's converge to X, there is an integer n such that M_n is disjoint with A. Let D be the component of $M^* \setminus \mathring{M}_n$ which contains A. Then D and $D' = M^* \setminus \mathring{D}$ are bounded surfaces with nonempty boundaries such that $M = D \cup D'$ and $D \cap D' = \dot{D}$. It follows from 7.1 that

$$\gamma(M^*) \geqslant \gamma(D) + \gamma(D') + r - 1,$$

where $r \geqslant 1$ is the number of simple closed curves in $\dot{D}$. Since $M_n \subset D' \subset M^*$, we see that $c = \gamma(M_n) \leqslant \gamma(D') \leqslant \gamma(M^*) = c$; hence the above inequality states that

$$\gamma(D) = 1 - r = 0.$$

Therefore D is a perforated disk with a connected boundary. It follows that D is a disk in G_i containing A in its interior, which proves (1).

Let $D_1^i, D_2^i, \ldots$ be a strictly increasing sequence of disks such that

$$G_i = \bigcup_{k \geqslant 1} D_k^i, \quad i \geqslant 1,$$

and for each $m \geqslant 1$ put

$$U_m = M^* \setminus \bigcup_{i+k \leqslant m+1} \mathring{D}_k^i.$$

It follows that $\{U_m\}$ is a strictly decreasing sequence of bounded surfaces converging to X such that the inclusion $j_m\colon (U_{m+1}, x_0) \to (U_m, x_0)$ induces an epimorphism of the corresponding fundamental groups. (Let us note that almost all U's lie in the original surface M.) Since U_m is a bounded surface, there is a subset $Y_m \subset U_m$ containing x_0 which is homeomorphic to the wedge of a finite number of circles (which may reduce to the point x_0) such that x_0 corresponds to the centre of the wedge and Y_m is a strong deformation retract of U_m (because $\dot{U}_1$ is not empty). Thus there is a homotopy $\varphi_m\colon U_m \times I \to U_m$ such that $\varphi_m(x, 0) = x$, $\varphi_m(x, 1) \in Y_m$ for $x \in U_m$ and $\varphi_m(x, t) = x$ for $(x, t) \in Y_m \times I$. Consider the following diagram, $m \geqslant 1$,

$$\begin{array}{ccc} (U_m, x_0) & \xleftarrow{j_m} & (U_{m+1}, x_0) \\ {\scriptstyle k_m}\downarrow\uparrow{\scriptstyle i_m} & & {\scriptstyle k_{m+1}}\downarrow\uparrow{\scriptstyle i_{m+1}} \\ (Y_m, x_0) & \xleftarrow{f_m} & (Y_{m+1}, x_0) \end{array}$$

where i_m, j_m are inclusion maps, $k_m(x) = \varphi_m(x, 1)$ and $f_m(x) = k_m(x)$. Since $(j_m)_\#$ is an epimorphism and $(k_m \circ i_m)_\#$, $(i_m \circ k_m)_\#$ are the identities on the corresponding fundamental groups, $(f_m)_\#$ is an epimorphism for each $m \geqslant 1$. If follows that $(Y, y_0) = \operatorname{invlim}\{(Y_m, x_0), f_m\}$ is a movable continuum because $\dim Y \leqslant 1$ and the diagrams commute (see 4.2 and 6.1). Since $(X, x_0) = \operatorname{invlim}\{(U_m, x_0), j_m\}$ and the diagrams commute up to homotopy, we have $\operatorname{Sh}(X, x_0) = \operatorname{Sh}(Y, y_0)$ (see [15]). Movability is preserved even by fundamental domination; hence (X, x_0) is movable, which completes the proof.

If each component of a compactum is movable, the compactum is also movable [5]. Hence the following theorem results from 7.2 and answers a question raised by Professor K. Borsuk in a conversation with the author.

7.3. Corollary. *Every compactum embeddable in a surface is movable.*

We close this paper with the following problem:

Problem. *Let X be a movable continuum and let $x_0 \in X$. Does there exist a locally connected continuum Y such that $\operatorname{Sh}(X, x_0) = \operatorname{Sh}(Y, y_0)$, where y_0 is a point of Y?*

I acknowledge my gratitude to Professor K. Borsuk for his help in the preparation of this paper. I am grateful to A. Trybulec and J. Olędzki for an interesting discussion on the subject of this paper. I am also grateful to the reviewer for several interesting suggestions.

Addendum. In his recent paper entitled: *One-dimensional shape properties and three-manifolds* (Studies in Topology, U.N.C.C. Proceedings), D. R. Mc Millan has independently obtained several results from this paper. In particular, he has also proved the invariance of pointed 1-movability under continuous mappings and the movability of pointed compacta lying in surfaces.

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Reçu par la Rédaction le 24. 9. 1974
Accepté par la Rédaction le 15. 9. 1975

Whitney properties

by

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Abstract. Some properties and structure of the "levels" $\mu^{-1}(t)$ are investigated, where X is a (metric) continuum and μ is a Whitney function for the space of all nonempty subcontinua of X.

1. Introduction. By a *continuum* we mean a nonempty compact connected metric space. The letter X will always denote a continuum. By the hyperspace of X we mean $C(X) = \{A: A \text{ is a (nonempty) subcontinuum of } X\}$ with the Hausdorff metric H [5]. In [17], in another context, Whitney defined a function $\mu: C(X)\to[0, \infty)$ satisfying

(1.1) μ is continuous on $C(X)$;

(1.2) if $A\subset B$ and $A \neq B$, then $\mu(A)<\mu(B)$;

(1.3) $\mu(\{x\}) = 0$ for each $x \in X$.

We will call any function from $C(X)$ to $[0,\infty)$ satisfying (1.1) through (1.3) a *Whitney map for* $C(X)$, and denote any such map by the symbol μ. Kelley [5] was the first person to introduce Whitney's function into the study of $C(X)$. The first explicit work done after Kelley on the nature of the sets $\mu^{-1}(t)$ was done in [3] where it was shown, among other results, that μ is both monotone and open. Next in [6] several results on the topological type of the sets $\mu^{-1}(t)$ were obtained. The next paper concerning the sets $\mu^{-1}(t)$ was [12]. Several papers on Whitney maps have recently been written (see our bibliography).

Let P be a topological property. We say that P is a *Whitney property* provided whenever X has property P, so does $\mu^{-1}(t)$ for any Whitney map μ for $C(X)$ and each $t<\mu(X)$. The purpose of this paper is to continue the work mentioned above. We give some general results about the levels $\mu^{-1}(t)$ (see, for example, 3.1 and 5.1) and some specific facts about $\mu^{-1}(t)$ for certain classes of continua (see, for example, 3.4, 3.5, and 4.4). Our results and examples show for many properties whether or not they are Whitney properties.

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* This paper was begun when the second author was a visitor at The Polish Academy of Sciences, Warsaw, and was completed when the first author was a visitor at the University of Georgia, Athens, Georgia 30602.