Finite T_0-spaces and universal mappings

by

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Abstract. A universal map $f\colon X\to Y$ is a continuous function such that for every continuous function $g\colon X\to Y$ there exists $x\in X$ such that $f(x) = g(x)$. Universal maps arise naturally as a generalization of the fixed point property. In this paper, we study universal mappings of finite T_0-spaces. Analogous to defining polyhedral trees we define T_0-trees using finite T_0-spaces to play the role of the usual intervals. An examination of universal mappings between finite T_0-trees is made.

Introduction. A universal map $f\colon X\to Y$ is a continuous function such that for every continuous function $g\colon X\to Y$ there exists $x\in X$ such that $f(x) = g(x)$. Universal maps arise naturally as a generalization of the fixed point property (universal mappings appear also in other contexts; see [2] and references given there).

In this paper, we study universal mappings (and the fixed point property) of finite T_0-spaces. Analogous to defining polyhedral trees we define T_0-trees. Other topics introduced are Artinian spaces (see Section 1) and P-spaces (see Section 2). Section 3 contains a summary of the results from [3], the first general results related to universal mappings into finite spaces, with finite T_0-spaces playing the role of the usual intervals. In Section 4, the concept of height of a finite T_0-space is introduced in a manner similar to dimension with respect to polyhedra. In Sections 5, 6, and 7 we discuss trees and universal mappings into trees with relationships between P-spaces and trees studied in detail in Section 5 and general properties of trees studied in Section 6. Section 6 also contains examples which illustrate the differences between finite T_0-trees and polyhedral trees. The primary result of this paper is Theorem 7.2. A more general but also more complicated result is possible, but fairly complete theorems on universal mappings of trees are difficult at the present stage of information (compare A. Wallace [6] and H. Schirmer [5]).

Conventions and notations

1. Everywhere in this paper, with the exception of Section 3, all topological spaces are assumed to be T_0-spaces.
2. $A\subset B$ means that the set A is contained in B but $A\neq B$.
3. $A\subseteq B$ means that $A\subset B$ or $A = B$.

4. A point x is called open (closed) if the set $\{x\}$ is open (closed).
5. $\bar{A}$ stands for the topological closure of A.
6. $\bar{x}$ denotes the closure $\{\bar{x}\}$ of the singleton $\{x\}$.
7. A discrete family of closed sets is a family such that the union of any of its subfamilies is closed.
8. U_x denotes the smallest open set containing the point x, provided such exists.
9. Examples of spaces will be given by using "∘" for an open point and "•" for a closed point. A thin line will indicate the paths between points, and dark arrows will indicate mappings.

Section 1. Artinian spaces. A topological space is called *Noetherian* if every decreasing sequence of closed subsets is finite. By analogy a space is *Artinian* if every decreasing sequence of open subsets is finite; i.e., every family of open subsets has a minimal member.

PROPOSITION 1.1. *If X is an Artinian space and $x \in X$, then there exists an open subset U_x such that U_x is the smallest open subset containing the point x. That is if $x \in V$, V open in X, then $U_x \subseteq V$.*

COROLLARY 1.2. *A function $f\colon X \to Y$ into an Artinian space is continuous if and only if $f^{-1}(U_y)$ is open for each $y \in Y$.*

COROLLARY 1.3. *A function $f\colon X \to Y$ between two Artinian spaces is continuous if and only if $f(U_x) \subseteq U_{f(x)}$ for every $x \in X$. In particular if $\{f(x)\}$ is open, then $f(U_x) = \{f(x)\}$.*

PROPOSITION 1.4. *If $f\colon X \to Y$ is a continuous map of an Artinian space X onto an arbitrary space Y, then Y is Artinian.*

PROPOSITION 1.5. *Every subspace of an Artinian space is Artinian.*

PROPOSITION 1.6. *Every finite space is Artinian. If every proper open subset of a space X is finite, then X is Artinian.*

EXAMPLE 1.7. Let N denote the natural numbers and let

$$T = \{\varphi, \{1\}, \{1, 2\}, \ldots\} \cup \{N\}.$$

Then (N, T) is an Artinian space.

EXAMPLE 1.8. Let X be an initial interval of ordinal numbers. Let

$$T = \{\{x\colon x < y\}\colon y \in X\} \cup \{X\}.$$

Then (X, T) is an Artinian space. If X contains an infinite ordinal p, then X contains an infinite proper open subset.

QUESTION 1.9. Is the cartesian product of two Artinian spaces Artinian?

Section 2. P-spaces. In this section we study a class of rather simple spaces. They serve as examples for later sections and as test spaces. Any result about P-spaces can be considered as the first step toward a more general result.

A topological space X is called a *P-space* if it has exactly one point $p \in X$ such that $\{p\}$ is an open subset of X.

EXAMPLE 2.1. The Alexandroff 2-point space $X = \{p, q\}$ with topology $\{\varphi, \{p\}, \{p, q\}\}$ is a P-space.

EXAMPLE 2.2. The 3-point space $X = \{p, q, r\}$ with the topology

$$\{\varphi, \{p\}, \{p, q\}, \{p, r\}, X\}$$

is a P-space.

EXAMPLE 2.3. Let X be an arbitrary non-empty set. Let $p \in X$. The family $B_p = \{\{p, q\}\colon q \in X\}$ is a base for a topology T_p on X. The space (X, T_p) is a P-space. For $G \subseteq X$, $G \in T_p$ if and only if $G = \emptyset$ or $p \in G$.

The spaces furnished by the above examples give us our most important examples of P-spaces. In fact any Artinian P-space with topology τ has the property that the topology τ can be refined to a T_p topology as given in Example 2.3.

THEOREM 2.4. *Let (X, T) be an Artinian space. The following are equivalent:*
(i) *(X, T) is a P-space,*
(ii) *(X, T) is a one-to-one continuous image of the space (X, T_p) as described in Example 2.3,*
(iii) *there exists $p \in X$ such that $\{p\}$ is dense in (X, T).*

Proof. In an Artinian space (X, T) every non-empty open set contains a minimal non-empty open set. Since every Artinian space (X, T) is a T_0-space such a minimal open set must be a singleton $\{p\}$. In a P-space (X, T) with an open point p there is exactly one such minimal open set, namely $\{p\}$. This shows that $T \subseteq T_p$ (see Example 2.3). Thus we have proved (i)⇒(ii). The other implications (ii)⇒(iii)⇒(i) are easy.

COROLLARY 2.5. *If $\{p\}$ is the unique open singleton of an Artinian P-space X and if $f\colon X \to Y$ is a continuous map onto a space Y, then Y is an Artinian P-space and $\{f(p)\}$ is its unique open singleton.*

COROLLARY 2.6. *Every Artinian P-space is connected.*

Proof. The proof follows from property (iii) of the theorem.

EXAMPLE 2.7. The compact subspace $\{0\} \cup [1, 2]$ of the real line is a P-space, but does not satisfy property (iii) of Theorem 2.4. Thus the assumption that X is Artinian is essential.

Remark 2.8. Properties (ii) and (iii) of Theorem 2.4 are equivalent for arbitrary spaces.

P-spaces give simple examples of spaces which have the fixed point property — as is illustrated by the following theorems.

THEOREM 2.9. *Every finite P-space has the fixed point property.*

Proof. Let X be a finite P-space, and $p \in X$ be the unique open point of X. Let $f\colon X \to X$ be an arbitrary continuous map. If $p \in f(X)$, then $f^{-1}(p)$ is a non-empty open subset of X. Hence, by (ii) of Theorem 2.4, $p \in f^{-1}(p)$, i.e., p is a fixed point of f. If $p \notin f(X)$, then consider $f_1 = f \mid f(X)$. Since $f(X)$ is also a P-space (see Corollary 1.5) and it has less points than X, the theorem is proved by induction on the number of points in X. It is clearly true for one point spaces.

Remark 2.10. If (X, T_p) is a P-space as given by Example 2.3, then (X, T_p) has the fixed point property.

Proof. $f(p) = p$ or otherwise $f(X) = f(\bar{p}) \subseteq \overline{f(p)} = f(p)$ and $f(f(p)) = f(p)$.

EXAMPLE 2.11. The infinite Artinian P-space given in Example 1.7 fails to have the fixed point property.

Remark 2.12. Consider the category FinTop of finite topological spaces. As was remarked in [3] the fixed point property (and the property of being a universal map) is invariant under the functor FinTop → FinTop which is the identity on mappings and which replaces open sets by closed sets and replaces closed sets by open sets in every finite topological space. Thus it follows that finite p'-spaces, defined as T_0-spaces which have exactly one closed point, also have the fixed point property. When we consider only finite spaces we can use the functor FinTop → FinTop as defined above to obtain dual theorems for all the above results. It is easy to see that the proof of Theorem 2.4 can be dualized word by word; thus, we have a dual version of Theorem 2.4 for Noetherian P'-spaces.

Section 3. Universal images. A continuous map $f\colon X \to Y$ is called *universal* if for every continuous map $g\colon X \to Y$ there exists $x \in X$ such that $g(x) = f(x)$. $f\colon X \to Y$ is called *bi-onto* (see [4]) if for every clopen set $U \subseteq X$, $f(U) = Y$ or $f(X \setminus U) = Y$. A connected space E is called a *universal image* if for every space X every continuous bi-onto map $f\colon X \to E$ is universal.

Hausdorff universal images were characterized by W. Holsztyński and S. Kwapień [4] as connected spaces E which admit a linear order $<$ with respect to which there is a first and a last element and such that the sets $\{x \in E\colon x < a\}$ and $\{x \in E\colon a < x\}$ are open in E. Eilenberg's theorem [1] characterizing what he called ordered connected Hausdorff spaces was crucial in this context. The characterization of universal images and Eilenberg's theorem about ordering were generalized to arbitrary topological spaces by W. Holsztyński. These results are stated in the following Theorems, 3.1–3.3.

THEOREM 3.1. *Let X be an arbitrary connected space. Then the deleted square $X^{(2)} = X \times X \setminus \Delta$ is connected or splits into the union of exactly two components U and V such that $V = \{(x, y)\colon (y, x) \in U\}$. The second case holds if and only if X is at least a 2-point space (except for the special case of a 2-point space with a non-T_0 topology) such that*

(*) *there is a linear order $<$ on X such that for every $a \in X$, $\{x \in X\colon x < a\}$ and $\{x \in X\colon a < x\}$ are open subsets in the relative topology of $X \setminus \{a\}$.*

Note. If the space has at least three points, then condition (*) implies that the topology is T_0.

THEOREM 3.2. *Let X be an arbitrary connected space with at least two points. The following are equivalent:*

(a) *X is a universal image,*

(b) *X is a universal image for connected spaces,*

(c) *X is a T_0-space with two distinct non-curting points and admits a linear order $<$ such that condition (*) is satisfied.*

Condition (b) given in Theorem 3.2 simply means that for every connected space Z, every continuous map of Z onto X is a universal map.

THEOREM 3.3. *Let T_n (respectively T_n') be a topology for $I_n = \{1, 2, \ldots, n\}$ such that the smallest neighborhood of k is $\{k\}$ if k is odd (respectively even), and it is $\{k-1, k, k+1\} \cap I_n$ if k is even (respectively odd). Then every finite n-point universal image is homeomorphic to either (I_n, T_n) or (I_n, T_n'). Thus, up to a homeomorphism, there exists only one n-point universal image when n is even or equal to 1. There are exactly two non-homeomorphic n-point universal images when n is odd and greater than 1.*

Remark 3.4. Two examples of finite universal images were known prior to the above theorem. Alster [4] had observed the two point Alexandroff space is a universal image and F. Pedersen did the same for a three point space which turned out to be homeomorphic to (I_3, T_3').

If $f\colon X \to Y$ is a universal map, then Y has the fixed point property. In particular, every universal image has the fixed point property. Using theorems from this section and examples from Section 2 we produce examples of finite spaces which have the fixed point property and which are not universal images.

LEMMA 3.5. *Let X be a connected space and let $<$ be a linear order in X such that (*) holds. Then*

$$\{\bar{x}\} \leqslant \{y \in X\colon \forall z < x,\ z \leqslant y \text{ and } \forall z > x,\ z \geqslant y\}.$$

Proof. Let $a < b < c$. Since $\{p \in X\colon b < p\}$ is open in $X \setminus \{b\}$ the closure of $\{p \in X\colon p < b\}$ in X is contained in $\{p \in X\colon p < b\} \cup \{b\}$. Thus $c \notin \{\bar{a}\}$. By a similar argument $a \notin \{\bar{c}\}$. This proves the lemma.

COROLLARY 3.6. *If a P-space X has at least 4 points, then X is not a universal image. In particular, if X is an Artinian P-space with at least 4 points, then X has the fixed point property without being a universal image.*

Remark 3.7. (X, T) where $X = \{a, b, c\}$ and $T = \{\Phi, \{a\}, \{a, b\}, X\}$ is the only 3 point P-space which is not a universal image.

Remark 3.8. Let X be an n-point space which is not a universal image. The method developed in [4] can be followed to construct a $n(n-1)$-point connected space and a non-universal continuous map of this space onto X. The resulting map will be the first coordinate projection from the deleted square $X^{(2)}$ onto X.

Section 4. Height of a topological space. All spaces are assumed to be non-empty.

The following notion is analogous to the notion of dimension of a cell-complex.

The *height* of a space X is the least integer n (or ∞) such that if $a_0, a_1, \ldots, a_m$ is a sequence of distinct points with $a_{i-1} \in \bar{a}_i$ for $i = 1, \ldots, m$, then $m \leqslant n$. The following remarks point out some properties of the notion of height.

Remark 4.1. The height of a space X is equal to the supremum of the height of the connectivity components of X.

Remark 4.2. If X is a subspace of Y, then height $(X) \leqslant$ height (Y).

Remark 4.3. If $f\colon X \to Y$ is a bijective continuous map, then height $(X) \leqslant$ height (Y).

Remark 4.4. Height $(X \times Y) =$ height (X) + height (Y).

THEOREM 4.5. *The height of an Artinian space X is the least integer n such that for every sequence $a_0, a_1, \ldots, a_m \in X$, where $U_{a_k} \subset U_{a_{k-1}}$ for $k = 1, \ldots, m$, it follows that $m \leqslant n$.*

Proof. For Artinian spaces X the smallest neighborhood of a point x, U_x, will exist. To say that $a_{i-1} \in \bar{a}_i$ is the same as saying that $a_i \in U_{a_{i-1}}$. Since U_{a_i} is the smallest open set about a_i, $U_{a_i} \subseteq U_{a_{i-1}}$. X is a T_0-space; thus, U_{a_i} has to be proper in $U_{a_{i-1}}$.

THEOREM 4.6. *If X is an Artinian space, then the following are equivalent*:

(a) *height* $(X) = 0$,

(b) *X is a finite discrete space.*

Proof. This follows easily from Theorem 4.5 since $U_x = \{x\}$ is the only possibility in a space of height 0. The fact that X can not be infinite is clear from the definition of Artinian.

THEOREM 4.7. *For an Artinian space X, the following are equivalent*:

(a) *height* $(X) \leqslant 1$,

(b) *X is a finite space where every point is either open or closed.*

Proof. (a)$\Rightarrow$(b). Suppose $\{x\}$ is neither open nor closed. There exists $y \in U_x$, $y \neq x$. Thus $U_y \subset U_x$. There exists $z \in \bar{x}$, $z \neq x$. Thus z, x, y form a sequence which contradicts the height $(X) \leqslant 1$. To show X is finite we first observe that the set of open points $X_0 \subseteq X$ is finite. For any closed point x which is not open U_x contains, other than x, only open points. If there exists a non-open point $y \in U_x$ we can construct a sequence $U_z \subset U_y \subset U_x$. Therefore $X \setminus X_0$ is a discrete space in the relative topology. Since $X \setminus X_0$ is Artinian, $X \setminus X_0$ is finite. Therefore X is finite.

(b)$\Rightarrow$(a). Assume that the height $(X) \geqslant 2$. There exist three points a, b, c such that $U_a \subset U_b \subset U_c$. The point b is neither open nor closed.

In the two assertions above the word "finite" can be deleted and the theorems are still true. In fact, the following more general statement holds.

THEOREM 4.8. *If X is an Artinian space of finite height, then X is finite.*

Proof. If height $(X) = 0$, then Theorem 4.8 follows from Theorem 4.5. If height $(X) = k \geqslant 1$ and X_0 is the set of all $x \in X$ such that x is open in X, then X_0 is finite. Height $(X \setminus X_0) \leqslant k-1$. Thus, by induction on the height of the space, $X \setminus X_0$ is finite. Therefore, $X = X_0 \cup X \setminus X_0$ is finite.

THEOREM 4.9. *A finite P-space is homeomorphic to some (X, T_p) as defined in Example 2.3 if and only if height $(X) \leqslant 1$. Such a space has height equal to 1 except when it has less than 2 points.*

Remark 4.10. Spaces of infinite height are of some interest and may be subdivided into the class of "short" spaces and "high" spaces. The former spaces do not contain an infinite sequence $a_0, a_1, \ldots, a_n, \ldots$ of distinct points such that $a_i \in \bar{a}_{i+1}$ for all i or $a_{i+1} \in \bar{a}_i$ for all i; the later are the spaces which do contain a sequence of the foregoing types.

Section 5. Universal mappings into p-spaces. A P-space X with topology given by $T_p = \{\{p, q\}\colon q \in X\}$ is called a *p-space*. A topological space X, containing a point p, is a p-space if and only if the family of all closed subsets consists of X and any subset which does not contain p. A topological space X, containing a point p, is a p-space if and only if the family of all non-empty open subsets consists of any subset containing the point p.

PROPOSITION 5.1. *A function f which maps a topological space X into a p-space Y is continuous if and only if $f^{-1}(x)$ is closed for every $x \in Y \setminus \{p\}$ and the family $\{f^{-1}(x)\colon x \in Y \setminus p\}$ is discrete.*

PROPOSITION 5.2. *Let Y be a p-space which contains at least three points. Let $f\colon X \to Y$ be a continuous function, then the following are equivalent*:

(a) *f is not a universal mapping,*

(b) *there exists a discrete family $\{F_y\colon y \in Y \setminus \{p\}\}$ of pairwise disjoint closed subsets of X such that $y \notin f(F_y)$ for all $y \in Y \setminus \{p\}$ and $\bigcup \{F_y\colon y \in Y \setminus \{p\}\}$ contains $f^{-1}(p)$.*

The following theorem is similar to Proposition 5.2 but gives a sharper criterion for non-universality than condition (b) of Proposition 5.2.

THEOREM 5.3. *Let X be a space such that every union of closed single element subsets of X is closed. Let Y be a p-space containing at least three points. Then for any continuous function $f\colon X \to Y$, the following are equivalent*:

(a) *f is not a universal map,*

(b) *there exists a discrete family $\{F_y\colon y \in Y \setminus \{p\}\}$ of pairwise disjoint closed subsets of X such that*

(i) $f^{-1}(p) \subseteq \bigcup \{F_y\colon y \in Y \setminus \{p\}\}$,

(ii) *$y \notin f(F_y)$ for all $y \in Y \setminus \{p\}$,*

(iii) *every closed point of X is an element of $\bigcup \{F_y\colon y \in Y \setminus \{p\}\}$.*

Proof. (b)$\Rightarrow$(a). Define $g\colon X \to Y$ by $g(x) = y$ for all $x \in F_y$ and $g(x) = p$ for all $x \in X \setminus \bigcup \{F_y\colon y \in Y \setminus \{p\}\}$. By Proposition 5.1, g is continuous. $g(x) \neq f(x)$ for all $x \in X$.

(a)$\Rightarrow$(b). There exists a continuous function $g\colon X \to Y$ such that $g(x) \neq f(x)$ for all $x \in X$. Let p, q, r be distinct points of Y. Define the family $\{F_y\colon y \in Y \setminus \{p\}\}$ as follows:

$$F^q = g^{-1}(q) \cup \{x \in X \setminus f^{-1}(q)\colon \{x\} \text{ is closed in } X \text{ and } g(x) = p\},$$
$$F_r = g^{-1}(r) \cup \{x \in f^{-1}(q)\colon \{x\} \text{ is closed in } X \text{ and } g(x) = p\},$$
$$F_y = g^{-1}(y) \text{ for all } y \in Y \setminus \{p, q, r\}.$$

THEOREM 5.4. *If X is an Artinian connected space, Y is a p-space, and $f\colon X\to Y$ is an onto continuous function such that $f^{-1}(p)$ contains all the open points of X, then f is a universal function.*

Proof. Let $g\colon X\to Y$ be a continuous function. $g^{-1}(p)$ is an open set. If $g^{-1}(p)\neq\emptyset$, then $g^{-1}(p)$ contains an open point x such that $g(x)=f(x)$. If $g^{-1}(p)=\emptyset$, then $\{g^{-1}(y)\colon y\in Y\setminus\{p\}\}$ is a collection of clopen sets which cover X. Since X is connected only one $g^{-1}(y)$ is non-empty. Thus, $g(x)=f(x)$ for some $x\in X$.

COROLLARY 5.5. *Let X be a finite connected space containing at least 2 distinct points and let Y be a p-space such that $\operatorname{card}(Y\setminus p)$ is less or equal than the number of closed points in X. Then there exists a universal map of X onto Y.*

The following theorem is a corollary to Theorem 5.3, however, we shall offer a direct proof.

THEOREM 5.6. *Let X be a connected space such that every union of closed single element subsets of X is closed. Let Y be a p-space. If $f\colon X\to Y$ is an onto continuous function such that $f^{-1}(p)$ contains all the non-closed points of X, then f is a universal function.*

Proof. Let $g\colon X\to Y$ be a continuous function. If $g^{-1}(p)\cap f^{-1}(p)\neq\emptyset$, then $g(x)=f(x)$ for some $x\in X$. Assume $g^{-1}(p)\cap f^{-1}(p)=\emptyset$. Since $f^{-1}(p)$ contains all non-closed points, $g^{-1}(p)$ is an open set which contains only closed points. By our assumption on X, the union of the points of $g^{-1}(p)$ is a closed set. Therefore, since X is connected either $g^{-1}(p)$ is empty or $g^{-1}(p)=X$. $g^{-1}(p)\neq X$ since we assumed $g^{-1}(p)\cap f^{-1}(p)=\emptyset$. Therefore $g^{-1}(p)=\emptyset$. $\{g^{-1}(y)\colon\ y\in Y\setminus\{p\}\}$ is a collection of clopen subsets which cover X. Since X is connected only one $g^{-1}(y)$ is non-empty. Thus, there exists $x\in X$ such that $f(x)=g(x)$.

COROLLARY 5.7. *Let X be a connected space containing at least 2 distinct points and such that every union of closed single element subsets of X is closed. Let Y be a p-space such that $[\operatorname{card}(Y\setminus p)]$ is less or equal to the cardinality of the set of closed points of X. Then there exists a universal map of X onto Y.*

Proof. Let F be the set of all closed points of X. Let $f\colon X\to Y$ be an arbitrary function such that $f(F)=Y\setminus\{p\}$ and $f(X\setminus F)\subseteq\{p\}$. Then $\{f^{-1}(y)\colon y\in Y\setminus p\}$ is a discrete family of closed subsets of X. Thus, by Proposition 5.1, f is a continuous function. Since X is connected and has at least 2 distinct points hence $X\setminus F\neq\emptyset$ and f is onto. Thus, by Theorem 5.6 f is universal.

Section 6. Trees-general properties. A *chain* is a finite topological space X which is homeomorphic to a finite universal image space (I_n, T_n) or (I_n, T_n'). (I_n, T_n) and (I_n, T_n') were described explicitly by Theorem 3.3. In a space X, a *path* from a point p to another point q is a set of points $A\subseteq X$ such that $p, q\in A$ and A with the relative topology is a chain such that p and q are the non-cutting points of A. We denote a path A from p to q by pq. For two points p and q, p is *adjacent* to q if $\{p, q\}$ is a path from p to q, i.e., if $p\in\bar{q}$ or $q\in\bar{p}$.

Remark 6.1. Let X be an Artinian space. If $y\in U_x$, $y\neq x$, then there exists a path from x to y. In fact, since y is an open point in the relative topology of $\{x, y\}$, x is adjacent to y.

THEOREM 6.2. *Let X be an Artinian space. Then x and y belong to the same component of X if and only if there exists a path from x to y.*

Proof. Let x_0 be a fixed point of X. Define $C=\{y\in X\colon x_0y \text{ exists}\}\cup\{x_0\}$. We will show that C is the component of X containing x_0. Let $z\in X$ such that either $U_z\cap C\neq\emptyset$ or $\bar{z}\cap C\neq\emptyset$. In either case there exists $y\in C$ such that $\{z, y\}$ is a path from z to y. $y\in C$ implies there exists a path x_0y. Let the points of x_0y be denoted by $x_0=x_1, x_2, \ldots, x_n=y$. Let x_i be the first point of the path x_0y for which either $x_i\in U_z$ or $x_i\in\bar{z}$. Assume first $x_i\in U_z$. If $x_i=x_0$, there is nothing to prove. Let x_i be a closed point in the path x_0x_i. Then $x_{i-1}\in U_{x_i}\subset U_z$, and we have a contradiction to x_i being the first point of the path x_0y. Therefore x_i is an open point in the path x_0x_i. It follows that $x_0x_i\cup x_iz$ is a path from x_0 to z. The other case where $x_i\in\bar{z}$ is handled similarly. Thus, $z\in C$. It follows that C is clopen or that C is the component of X containing x_0.

A *tree* is a finite space X for which there exists a unique path between any two points and for any $p, q, r\in X$,

$$pq\cap qr=\{q\}\Rightarrow pq\cup qr=pr.$$

PROPOSITION 6.3. *A finite space X is a tree iff height of X is $\leqslant 1$ and there exists a unique path pq in X from any point p of X to any point q of X.*

COROLLARY 6.4. *Every tree X is connected and every point of it is closed or open, but unless X is a single-point space, never both.*

Remark 6.5. When U_x exists it contains at most one closed point, namely x. In particular, if x is a point of a space of height 1 then every point of $U_x\setminus x$ is open. Similarly, $\bar{x}$ contains at most one open point, namely x; if x belongs to space of height 1 then every point $\bar{x}\setminus x$ is closed.

Let X be a space of height $\leqslant 1$. Then the *ramification index* $\tau(x)$ of a point x of X is defined as follows:

$$\tau(x)=\begin{cases}\operatorname{card}U_x-1 & \text{if}\quad x \text{ is closed},\\ \operatorname{card}\bar{x}-1 & \text{if}\quad x \text{ is open}.\end{cases}$$

If $\tau(x)=1$ then the number of components of $X\setminus x$ is the same as of X. If $\tau(x)=1$ then we call x an *end-point* of X. If $\tau(x)>2$ we call x a *ramification point* or a *junction point*.

In the case of a tree the ramification index can be characterized as follows:

THEOREM 6.6. *A finite connected space X of height 1 is a tree iff $\tau(x)$ is equal to the number of components of $X\setminus x$ for every point x of X.*

The proof of the above theorem is easy and will be omitted. A similar theorem holds for polyhedra. Moreover, a 1-dimensional connected polyhedra X is a (polyhedral) tree iff $X\setminus x$ is disconnected unless x is an end-point of X. The following example shows however that the last statement fails to be true for finite spaces.

EXAMPLE 6.7. The following 8-point space X of height 1 is not a tree, but $X\setminus x$ is disconnected for every x which is not an end-point (see Fig. 1).

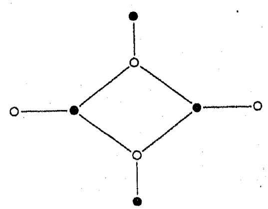

Fig. 1

Thus the analogy between finite polyhedra and finite spaces is not mechanical.

PROPOSITION 6.8. *A tree is a chain iff it does not contain any ramification point.*

If a subspace X of a space Y is a tree then X is called a sub-tree of Y.

PROPOSITION 6.9. *Every connected subspace of a tree X is a sub-tree of X.*

PROPOSITION 6.10. *Every sub-tree of a tree X is a retract of X.*

EXAMPLE 6.11. $X\setminus a$ is a closed subset of and a path in X, but it is not a retract of X (see Fig. 2).

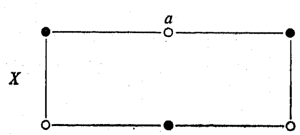

Fig. 2

Thus not only a sub-tree but even a path which is a closed subset of a finite space X of height 1 need not be a retract of X.

The fact that $X\setminus a$ above is not a retract of X follows immediately from the following remark.

Remark 6.12. If function $f\colon X\to Y$ is continuous then it maps every pair of adjacent points into one point or into a pair of adjacent points. Furthermore, if $p\in X$ is closed and $f(p)$ is open in Y, or *vice versa*, then $f(q)=f(p)$ for every $q\in X$ adjacent to p.

Above we had the simplest illustration of non-retraction phenomenon. Below we give the simplest example of a map $f\colon A\to Y$ from a closed subset A of X into a tree (even a chain) which does not admit any continuous extension.

EXAMPLE 6.13. There are certain circumstances when functions can be extended and the following proposition illustrates such a case. Given a subset A of a tree X a *gap* in A is a path pq of X where p and q are not adjacent points and $pq\cap A=\{p,q\}$ (see Fig. 3).

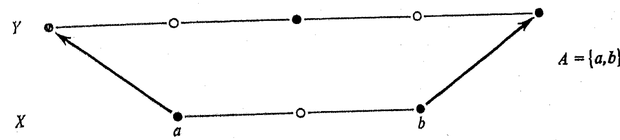

Fig. 3

Remark 6.14. Let X and Y be connected spaces of height 1. $f\colon X\to Y$ is continuous if and only if f preserves adjacency of points (allowing a point to be adjacent to itself).

PROPOSITION 6.15. *Let X be a tree and $A\subseteq X$. If $f\colon A\to Y$ is a continuous map, pq a gap in A such that $f(p)=f(q)$, then f can be extended continuously to $A\cup pq$.*

Proof. It follows from Remark 6.14 that f can be extended continuously to $A\cup pq$ by defining $f(x)=f(p)$ for all $x\in pq$.

It is possible to obtain some more extension theorems from Proposition 6.10.

Section 7. Universal maps into trees. A point q of a tree X is *between* x and y i. $q\in xy$. The point q is strictly between x and y if q is not equal to either x or y. A funcf tion f *preserves betweenness* if when q is between x and y, then $f(q)$ is between $f(x$- and $f(y)$. For points x, y, z, if y is between x and z this will be denoted by $x-y-z$)

Remark 7.1. If x, y are adjacent points of a tree X then $x-y-z$ or $y-x-z$ for every $z\in X$.

THEOREM 7.2. *Let X and Y be trees. If f is a continuous function which is onto Y and preserves betweenness, then f is a universal function.*

Proof. Let $g\colon X\to Y$ be a continuous map such that $g(x)\neq f(x)$ for every $x\in X$. Choose $X_0\subseteq X$ such that $f(X_0)=Y$ and $f|X_0$ is injective. Define inductivey a sequence of points of X as follows:

(a) Choose $x_0\in X$ arbitrarily.

(b) Let $q_n\in X_0$ be the point such that $f(q_n)=g(x_n)$ and let x_{n+1} be the point of the path x_nq_n which is adjacent to x_n but not equal to x_n.

This choice of x_{n+1} is always possible since otherwise $x_n=q_n$ and this contradicts our assumption that $f(x)\neq g(x)$ for all $x\in X$. The following case study will show that x_n is strictly between x_{n-1} and x_{n+1}.

Case 1. $g(x_n)=g(x_{n-1})$. By our choice of X_0 it follows that $q_n=q_{n-1}$. Thus $x_{n-1}-x_n-q_n$ and $x_n-x_{n+1}-q_n$ imply that x_n is between x_{n-1} and x_{n+1}.

Case 2. $g(x_n)\neq g(x_{n-1})$. If $x_{n-1}-x_n-q_n$, then it follows that x_n is between x_{n-1} and x_{n+1}. As was stated in Remark 7.1 the only other possibility is $q_n-x_{n-1}-x_n$.

Since x_n is between x_{n-1} and q_{n-1} the order relationship between the points is $q_n - x_{n-1} - x_n - q_{n-1}$. Since f preserves betweenness $f(q_n) - f(x_{n-1}) - f(x_n) - f(q_{n-1})$ (allowing for equality). Since g is continuous $g(x_n)$ is adjacent to $g(x_{n-1})$. Therefore, $f(q_n)$ is adjacent to $f(q_{n-1})$. The only remaining possibility is for $f(x_{n-1}) = f(q_n) = g(x_n)$ and at the same time $f(x_n) = f(q_{n-1}) = g(x_{n-1})$. Without loss of generality assume x_{n-1} is open and x_n is closed. First assume $f(x_n)$ is open. This contradicts the continuity of f since $f^{-1}(f(x_n))$ is not an open set in X. Conversely assume $f(x_n)$ is closed. This contradicts the continuity of g since $g^{-1}(f(x_n))$ is not a closed set in X.

It has been shown that $\{x_n\}$ is an infinite sequence of distinct points in X which contradicts the fact that X is finite.

COROLLARY 7.3. *A tree has the fixed point property.*

COROLLARY 7.4 *Let X and Y be trees and A a subtree of X. If $f: X \to Y$ is a continuous function such that $f|A: A \to Y$ is onto and preserves betweenness for points of A, then f is a universal function.*

Proof. From Theorem 7.2 it follows that $f|A$ is a universal function from A to Y. It follows that $f: X \to Y$ must also be a universal function.

EXAMPLE 7.5. This example illustrates that it is not always possible, even under the best of circumstances, to get a connected subset of the domain on which the universal map is onto the image space and preserves betweenness in the subset (see Fig. 4).

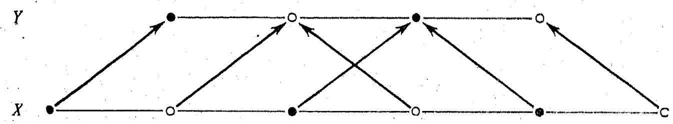

Fig. 4

EXAMPLE 7.6. This particular example is of special interest since it is a universal map of a chain (universal image space) onto a non-chain. Moreover it illustrates that one can not necessarily find a subset of the domain for which the function is onto the image and preserves betweenness in the subset (see Fig. 5).

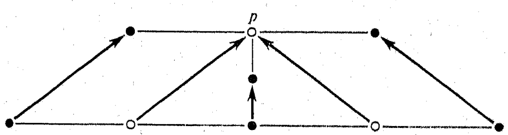

Fig. 5

The image space of Example 7.6 is called the 4-point triod. This is both a tree and a p-space but is not a universal image space as was pointed out by Corollary 3.6. That this space is a universal image for any tree with at least 6 points follows easily from Corollary 5.5.

In reference to a question posed in the introduction of this paper, the following example illustrates that the composition of universal functions between trees need not be universal.

EXAMPLE 7.7 (see Fig. 6).

Let f_1: $K_1 \to K_2$ as given by the diagram.

Let f_2: $K_2 \to K_3$ as given by the diagram.

Let $f = f_2 \circ f_1$. To show that f is not universal define g: $K_1 \to K_3$ such that $g(x_i) = b$ for $i = 1, 2, 3$; $g(x_4) = p$; and $g(x_i) = a$ for $i = 5, 6, 7$.

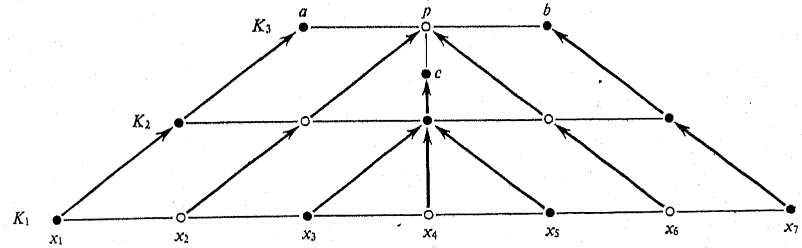

Fig. 6

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Accepté par la Rédaction le 25. 8. 1975