Finite $T_\omega$-spaces and universal mappings

by

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Abstract. A universal map $f: X \to Y$ is a continuous function such that for every continuous function $g: X \to Y$ there exists $x \in X$ such that $f(x) = g(x)$. Universal maps arise naturally as a generalization of the fixed point property. In this paper, we study universal mappings of finite $T_\omega$-spaces. Analogous to defining polyhedral trees we define $T_\omega$-trees using finite $T_\omega$-spaces to play the role of the usual intervals. An examination of universal mappings between finite $T_\omega$-trees is made.

Introduction. A universal map $f: X \to Y$ is a continuous function such that for every continuous function $g: X \to Y$ there exists $x \in X$ such that $f(x) = g(x)$. Universal maps arise naturally as a generalization of the fixed point property (universal mappings appear also in other contexts; see [2] and references given there).

In this paper, we study universal mappings (and the fixed point property) of finite $T_\omega$-spaces. Analogous to defining polyhedral trees we define $T_\omega$-trees. Other topics introduced are Artinian spaces (see Section 1) and $P$-spaces (see Section 2). Section 3 contains a summary of the results from [3], the first general results related to universal mappings into finite spaces, with finite $T_\omega$-spaces playing the role of the usual intervals. In Section 4, the concept of height of a finite $T_\omega$-space is introduced in a manner similar to dimension with respect to polyhedra. In Sections 5, 6, and 7 we discuss trees and universal mappings into trees with relationships between $P$-spaces and trees studied in detail in Section 5 and general properties of trees studied in Section 6. Section 6 also contains examples which illustrate the differences between finite $T_\omega$-trees and polyhedral trees. The primary result of this paper is Theorem 7.2. A more general but also more complicated result is possible, but fairly complete theorems on universal mappings of trees are difficult at the present stage of information (compare A. Wallace [6] and H. Schirmer [5]).

Conventions and notations

1. Everywhere in this paper, with the exception of Section 3, all topological spaces are assumed to be $T_\omega$-spaces.
2. $A \subset B$ means that the set $A$ is contained in $B$ but $A \neq B$.
3. $A \subseteq B$ means that $A \subset B$ or $A = B$. 
4. A point $x$ is called open (closed) if the set $\{x\}$ is open (closed).
5. $\mathcal{A}$ stands for the topological closure of $A$.
6. $\mathcal{X}$ denotes the closure $\{x\}$ of the singleton $\{x\}$.
7. A discrete family of closed sets is a family such that the union of any of its subfamilies is closed.
8. $U_x$ denotes the smallest open set containing the point $x$, provided such exists.
9. Examples of spaces will be given by using "*" for an open point and "**" for a closed point. A thin line will indicate the paths between points, and dark arrows will indicate mappings.

Section 1. Artinian spaces. A topological space is called Noetherian if every decreasing sequence of closed subsets is finite. By analogy a space is Artinian if every decreasing sequence of open subsets is finite; i.e., every family of open subsets has a minimal member.

Proposition 1.1. If $X$ is an Artinian space and $x \in X$, then there exists an open subset $U_x$ such that $U_x$ is the smallest open subset containing the point $x$. That is if $x \in V$, $V$ open in $X$, then $U_x \subseteq V$.

Corollary 1.2. A function $f: X \to Y$ into an Artinian space is continuous if and only if $f^{-1}(U_y)$ is open for each $y \in Y$.

Corollary 1.3. A function $f: X \to Y$ between two Artinian spaces is continuous if and only if $f(U_x) = f(U_{x_0})$ for every $x \in X$. In particular if $\{f(x)\}$ is open, then $f(U_x) = \{f(x)\}$.

Proposition 1.4. If $f: X \to Y$ is a continuous map of an Artinian space $X$ onto an arbitrary space $Y$, then $Y$ is Artinian.

Proposition 1.5. Every subspace of an Artinian space is Artinian.

Proposition 1.6. Every finite space is Artinian. If every proper open subset of a space $X$ is finite, then $X$ is Artinian.

Example 1.7. Let $N$ denote the natural numbers and let $T = \{0, 1, 2, \ldots\} \cup \{N\}$.

Then $(N, T)$ is an Artinian space.

Example 1.8. Let $X$ be an initial interval of ordinal numbers. Let $T = \{\{x: x < y\}: y \in X\} \cup \{X\}$.

Then $(X, T)$ is an Artinian space. If $X$ contains an infinite ordinal $\alpha$, then $X$ contains an infinite proper open subset.

Question 1.9. Is the cartesian product of two Artinian spaces Artinian?

Section 2. $P$-spaces. In this section we study a class of rather simple spaces. They serve as examples for later sections and as test spaces. Any result about $P$-spaces can be considered as the first step toward a more general result.

A topological space $X$ is called a $P$-space if it has exactly one point $p \in X$ such that $\{p\}$ is an open subset of $X$.

Example 2.1. The Alexandroff 2-point space $X = \{p, q\}$ with topology $\{\emptyset, \{p\}, \{p, q\}\}$ is a $P$-space.

Example 2.2. The 3-point space $X = \{p, q, r\}$ with the topology $\{\emptyset, \{p\}, \{p, q\}, \{p, r\}\}$ is a $P$-space.

Example 2.3. Let $X$ be an arbitrary non-empty set. Let $p \in X$. The family $B = \{(p, q): q \in X\}$ is a base for a topology $T_p$ on $X$. The space $(X, T_p)$ is a $P$-space. For $G \subseteq X$, $G \in T_p$ if and only if $G = \emptyset$ or $p \in G$.

The spaces furnished by the above examples give us our most important examples of $P$-spaces. In fact any Artinian $P$-space with topology $\tau$ has the property that the topology $\tau$ can be refined to a $T_\tau$ topology as given in Example 2.3.

Theorem 2.4. Let $(X, T)$ be an Artinian space. The following are equivalent:

(i) $(X, T)$ is a $P$-space,

(ii) $(X, T)$ is a one-to-one continuous image of the space $(X, T_p)$ as described in Example 2.3,

(iii) there exists $p \in X$ such that $\{p\}$ is dense in $(X, T)$.

Proof. In an Artinian space $(X, T)$ every non-empty open set contains a minimal non-empty open set. Since every Artinian space $(X, T)$ is a $T_\tau$-space such a minimal open set must be a singleton $\{p\}$. In a $P$-space $(X, T)$ with an open point $p$ there is exactly one such minimal open set, namely $\{p\}$. This shows that $T \subseteq T_p$ (see Example 2.3). Thus we have proved (i)$\Rightarrow$(ii). The other implications (ii)$\Rightarrow$(iii)$\Rightarrow$(i) are easy.

Corollary 2.5. If $\{p\}$ is the unique open singleton of an Artinian $P$-space $X$ and if $f: X \to Y$ is a continuous map onto a space $Y$, then $Y$ is an Artinian $P$-space and $(f(p))$ is its unique open singleton.

Corollary 2.6. Every Artinian $P$-space is connected.

Proof. The proof follows from property (iii) of the theorem.

Example 2.7. The compact subspace $[0, 1] \cup [1, 2]$ of the real line is a $P$-space, but does not satisfy property (iii) of Theorem 2.4. Thus the assumption that $X$ is Artinian is essential.

Remark 2.8. Properties (ii) and (iii) of Theorem 2.4 are equivalent for arbitrary spaces.

$P$-spaces give simple examples of spaces which have the fixed point property — as is illustrated by the following theorems.

Theorem 2.9. Every finite $P$-space has the fixed point property.

Proof. Let $X$ be a finite $P$-space, and $p \in X$ be the unique open point of $X$.

Let $f: X \to X$ be an arbitrary continuous map. If $p \in f(X)$, then $f^{-1}(p)$ is a non-empty open subset of $X$. Hence, by (ii) of Theorem 2.4, $p \in f^{-1}(p)$, i.e., $p$ is a fixed point of $f$. If $p \notin f(X)$, then consider $f_1 = f|f(X)$. Since $f(X)$ is also a $P$-space...
(see Corollary 1.5) and it has less points than $X$, the theorem is proved by induction on the number of points in $X$. It is clearly true for one point spaces.

**Remark 2.10.** If $(X, T_p)$ is a $P$-space as given by Example 2.3, then $(X, T_p)$ has the fixed point property.

**Proof.** $f(p) = p$ or otherwise $f(X) = f(\{p\}) \supseteq \{f(p)\} = f(p)$ and $f(f(p)) = f(p)$.

**Example 2.11.** The infinite Artinian $P$-space given in Example 1.7 fails to have the fixed point property.

**Remark 2.12.** Consider the category $\text{FinTop}$ of finite topological spaces. As was remarked in [3] the fixed point property (and the property of being a universal map) is invariant under the functor $\text{FinTop} \to \text{FinTop}$ which is the identity on mappings and which replaces open sets by closed sets and replaces closed sets by open sets in every finite topological space. Thus it follows that finite $P'$-spaces, defined as $T_\beta$-spaces which have exactly one closed point, also have the fixed point property. When we consider only finite spaces we can use the functor $\text{FinTop} \to \text{FinTop}$ as defined above to obtain dual theorems for all the above results. It is easy to see that this is impossible for $T_\omega$.

**Section 3. Universal images.** A continuous map $f: X \to Y$ is called universal if for every continuous map $g: X \to Y$ there exists $x \in X$ such that $g(x) = f(x)$.

**Theorem 3.3.** Let $T_\alpha$ (respectively $T_\beta$) be a topology for $I_\alpha = \{1, 2, ..., n\}$ such that the smallest neighborhood of $k$ is $\{k\}$ if $k$ is odd (respectively even), and is $\{k-1, k, k+1\} \cap I_\alpha$ if $k$ is even (respectively odd). Then every finite $n$-point universal image is homeomorphic to either $(I_\alpha, T_\alpha)$ or $(I_\beta, T_\beta)$. Thus, up to a homeomorphism, there exists only one $n$-point universal image when $n$ is even or equal to 1. There are exactly two non-homeomorphic $n$-point universal images when $n$ is odd and greater than 1.

**Remark 3.4.** Two examples of finite universal images were known prior to the above theorem. Alater [4] had observed the two-point Alexandroff space is a universal image and F. Pedersen did the same for a three-point space which turned out to be homeomorphic to $(I_3, T_3)$.

If $f: X \to Y$ is a universal map, then $Y$ has the fixed point property. In particular, every universal image has the fixed point property. Using theorems from this section and examples from Section 2 we produce examples of finite spaces which have the fixed point property and which are not universal images.

**Lemma 3.5.** Let $X$ be a connected space and let $\alpha$ be a linear order in $X$ such that $(*)$ holds. Then

$$\{x \in X : \forall z < x, z \leq y \land \forall z > x, z \geq y\}.$$

**Proof.** Let $a < b < c$. Since $(p \in X : a < b)$ is open in $X \setminus \{b\}$ the closure of $(p \in X : p < b)$ in $X$ is contained in $(p \in X : p < b) \cup \{b\}$. Thus $a \notin \{b\}$. By a similar argument $a \notin \{a\}$. This proves the lemma.

**Corollary 3.6.** If a $P$-space $X$ has at least 4 points, then $X$ is not a universal image. In particular, if $X$ is an Artinian $P$-space with at least 4 points, then $X$ has the fixed point property without being a universal image.

**Remark 3.7.** $(X, T)$ where $X = \{a, b, c\}$ and $T = \{\emptyset, \{a\}, \{a, b\}, X\}$ is the only 3 point $P$-space which is not a universal image.

**Remark 3.8.** Let $X$ be an $n$-point space which is not a universal image. The method developed in [4] can be followed to construct a $n(n-1)$-point connected space and a non-universal continuous map of this space onto $X$. The resulting map will be the first coordinate projection from the deleted square $X^{(2)}$ onto $X$.

**Section 4. Height of a topological space.** All spaces are assumed to be non-empty.

The following notion is analogous to the notion of dimension of a cell-complex.
The height of a space $X$ is the least integer $n$ (or $\infty$) such that if $a_0, a_1, \ldots, a_n$ is a sequence of distinct points with $a_{i-1} \neq a_i$ for $i = 1, \ldots, m$, then $m \leq n$. The following remarks point out some properties of the notion of height.

**Remark 4.1.** The height of a space $X$ is equal to the supremum of the height of the connectivity components of $X$.

**Remark 4.2.** If $X$ is a subspace of $Y$, then height $(X) \leq$ height $(Y)$.

**Remark 4.3.** If $f: X \to Y$ is a bijective continuous map, then height $(X) \leq$ height $(Y)$.

**Remark 4.4.** Height $(X \times Y) = \max(\text{height}(X), \text{height}(Y))$.

**Theorem 4.5.** The height of an Artinian space $X$ is the least integer $n$ such that for every sequence $a_0, a_1, \ldots, a_n \in X$, where $a_k \neq a_{k+1}$, for $k = 0, \ldots, n$, it follows that $n \leq n$.

**Proof.** For Artinian spaces $X$ the smallest neighborhood of a point $x$, $U_x$, will exist. To say that $a_i \neq a_{i+1}$ is the same as saying that $a_i \neq a_{i+1}$. Since $U_n$ is the smallest open set about $a_i$, $U_i \neq a_{i+1}$, $X$ is a $T_0$-space; thus, $U_i$ has to be proper in $U_{i+1}$.

**Theorem 4.6.** If $X$ is an Artinian space, then the following are equivalent:

(a) Height $(X) = 0$.

(b) $X$ is a finite discrete space.

**Proof.** This follows easily from Theorem 4.5 since $U_x = \{ x \}$ is the only possibility in a space of height 0. The fact that $X$ can not be infinite is clear from the definition of Artinian.

**Theorem 4.7.** For an Artinian space $X$, the following are equivalent:

(a) Height $(X) \leq 1$.

(b) $X$ is a finite space where every point is either open or closed.

**Proof.** (a)$\Rightarrow$(b). Suppose $(x)$ is neither open nor closed. There exists $y \in U_x$, $y \neq x$. Thus $U_y = U_x$. There exists $x \in U_x$, $x \neq x$. Thus $x, y$ form a sequence which contradicts the height $(X) \leq 1$. To show $X$ is finite we first observe that the set of open points $X_0 \subseteq X$ is finite. For any closed point $x$ which is not open $U_x$ contains, other than $x$, only open points. If there exists a non-open point $y \in U_x$, we can construct a sequence $U_y = U_x$. Therefore $X \setminus X_0$ is a discrete space in the relative topology. Since $X \setminus X_0$ is Artinian, $X \setminus X_0$ is finite. Therefore $X$ is finite.

(b)$\Rightarrow$(a). Assume that the height $(X) \leq 1$. There exist three points $a, b, c$ such that $U_a = U_b = U_c$. The point $b$ is neither open nor closed.

In the two assertions above the word “finite” can be deleted and the theorems are still true. In fact, the following more general statement holds.

**Theorem 4.8.** If $X$ is an Artinian space of finite height, then $X$ is finite.

**Proof.** If height $(X) = 0$, then Theorem 4.8 follows from Theorem 4.5. If height $(X) = k = 1$ and $X_k$ is the set of all $x \in X$ such that $x$ is open in $X$, then $X_k$ is finite. Height $(X \setminus X_k) \leq k - 1$. Thus, by induction on the height of the space, $X \setminus X_k$ is finite. Therefore, $X = X_k \cup X \setminus X_k$ is finite.

**Theorem 4.9.** A finite $T_0$-space is homeomorphic to some $(X, T_0)$ as defined in Example 2.3 if and only if height $(X) \leq 1$. Such a space has height equal to 1 except when it has less than 2 points.

**Remark 4.10.** Spaces of height 1 are of no interest and may be subdivided into the class of “short” spaces and “high” spaces. The former points do not contain an infinite sequence $a_0, a_1, \ldots, a_n$ of distinct points such that $a_i \neq a_{i+1}$, for all $i$, $0 \leq i < n$. The latter are the spaces which do contain a sequence of the foregoing type.

**Section 5.** Universal mappings into $p$-spaces. A $P$-space $X$ with topology given by $T_p = \{ \{ p, q \}, q \in X \}$ is called a $p$-space. A topological $X$ containing a point $p$, is a $p$-space if and only if the family of all closed subsets consists of $X$ and any subset which does not contain $p$. A topological $X$, containing a point $p$, is a $p$-space if and only if the family of all non-empty open subsets consists of any subset containing the point $p$.

**Proposition 5.1.** A function $f$ which maps a topological space $X$ into a $p$-space $Y$ is continuous if and only if $f^{-1}(x)$ is closed for all $x \in Y \setminus \{ p \}$ and the family $\{ f^{-1}(x) \mid x \in Y \setminus \{ p \} \}$ is discrete.

**Proposition 5.2.** Let $Y$ be a $p$-space which contains at least three points. Let $f: X \to Y$ be a continuous function, then the following are equivalent:

(a) $f$ is not a universal mapping.

(b) There exists a discrete family $\{ F_x \mid x \in Y \setminus \{ p \} \}$ of pairwise disjoint closed subsets of $X$ such that $y \notin f(F_x)$ for all $y \in Y \setminus \{ p \}$ and $\bigcup \{ F_x \mid x \in Y \setminus \{ p \} \}$ contains $f^{-1}(p)$.

The following theorem is similar to Proposition 5.2 but gives a sharper criterion for non-universality than condition (b) of Proposition 5.2.

**Theorem 5.3.** Let $X$ be a space such that every union of closed single element subsets of $X$ is closed. Let $Y$ be a $p$-space containing at least three points. Then for any continuous function $f: X \to Y$, the following are equivalent:

(a) $f$ is not a universal map.

(b) There exists a discrete family $\{ F_x \mid x \in Y \setminus \{ p \} \}$ of pairwise disjoint closed subsets of $X$ such that

(i) $f^{-1}(x) = \{ y \in Y \mid y \notin f(F_x) \}$,

(ii) $y \notin f(F_x)$ for all $y \in Y \setminus \{ p \}$,

(iii) Every closed point of $X$ is an element of $\bigcup \{ F_x \mid x \in Y \setminus \{ p \} \}$.

**Proof.** (b)$\Rightarrow$(a). Define $g: X \to Y$ by $g(x) = y$ for all $x \in F_y$ and $g(x) = p$ for all $x \in X \setminus \bigcup \{ F_x \mid x \in Y \setminus \{ p \} \}$. By Proposition 5.1, $g$ is continuous. $g(x) \neq f(x)$ for all $x \in X$.

(a)$\Rightarrow$(b). There exists a continuous function $g: X \to Y$ such that $g(x) \neq f(x)$ for all $x \in X$. Let $p, q, r$ be distinct points of $Y$. Define the family $\{ F_y \mid y \in Y \setminus \{ p, q, r \} \}$ as follows:

- $F_y = g^{-1}(y) \cup \{ x \in X \setminus f^{-1}(y) \}$ if $x$ is closed in $X$ and $g(x) = p$,
- $F_y = g^{-1}(y) \cup \{ x \in f^{-1}(y) \}$ if $x$ is closed in $X$ and $g(x) = r$,
- $F_y = g^{-1}(y)$ if $x \in Y \setminus \{ p, q, r \}$.
Theorem 5.4. If \( X \) is an Artinian connected space, \( Y \) is a p-space, and \( f: X \to Y \) is an onto continuous function such that \( f^{-1}(p) \) contains all the open points of \( X \), then \( f \) is a universal function.

Proof. Let \( g: X \to Y \) be a continuous function. \( g^{-1}(p) \) is an open set. If \( g^{-1}(p) \not= \emptyset \), then \( g^{-1}(p) \) contains an open point \( x \) such that \( g(x) = f(x) \). If \( g^{-1}(p) = \emptyset \), then \( (g^{-1}(y) : y \in Y \setminus \{p\}) \) is a collection of clopen sets which cover \( X \). Since \( X \) is connected only one \( g^{-1}(y) \) is non-empty. Thus, \( g(x) = f(x) \) for some \( x \in X \).

Corollary 5.5. Let \( X \) be a finite connected space containing at least 2 distinct points and let \( Y \) be a p-space such that \( \text{card}(Y \setminus \{p\}) \) is less or equal than the number of closed points in \( X \). Then there exists a universal map of \( X \) onto \( Y \).

The following theorem is a corollary to Theorem 5.3, however, we shall offer a direct proof.

Theorem 5.6. Let \( X \) be a connected space such that every union of closed single element subsets of \( X \) is closed. Let \( Y \) be a p-space. Then any \( f: X \to Y \) is an onto continuous function such that \( f^{-1}(p) \) contains all the non-closed points of \( X \), then \( f \) is a universal function.

Proof. Let \( g: X \to Y \) be a continuous function. If \( g^{-1}(p) \cap f^{-1}(p) \not= \emptyset \), then \( g(x) = f(x) \) for some \( x \in X \). Assume \( g^{-1}(p) \cap f^{-1}(p) = \emptyset \). Since \( f^{-1}(p) \) contains all non-closed points, \( g^{-1}(p) \) is an open set which contains only closed points. By our assumption on \( X \), the union of the points of \( g^{-1}(p) \) is closed. Therefore, since \( X \) is connected either \( g^{-1}(p) \) is empty or \( g^{-1}(p) = X \). \( g^{-1}(p) \not= X \) since we assumed \( g^{-1}(p) \cap f^{-1}(p) = \emptyset \). Therefore \( g^{-1}(p) = \emptyset \). \( (g^{-1}(y) : y \in Y \setminus \{p\}) \) is a collection of clopen subsets which cover \( X \). Since \( X \) is connected only one \( g^{-1}(y) \) is non-empty. Thus, there exists \( x \in X \) such that \( f(x) = g(x) \).

Corollary 5.7. Let \( X \) be a connected space containing at least 2 distinct points and such that every union of closed single element subsets of \( X \) is closed. Let \( Y \) be a p-space such that \( \text{card}(Y \setminus \{p\}) \) is less or equal than the cardinality of the set of closed points of \( X \). Then there exists a universal map of \( X \) onto \( Y \).

Proof. Let \( F \) be the set of all closed points of \( X \). Let \( f: X \to Y \) be an arbitrary function such that \( f(F) = Y \setminus \{p\} \) and \( f(X \setminus F) = \{p\} \). Then \( f \) is a discrete family of closed subsets of \( X \). Thus, by Proposition 5.1, \( f \) is a continuous function. Since \( X \) is connected and has at least 2 distinct points hence \( X \setminus F \not= \emptyset \) and \( f \) is onto. Thus, by Theorem 5.6 \( f \) is universal.

Section 6. Trees-general properties. A chain is a finite topological space \( X \) which is homeomorphic to a finite universal image space \( (L_u, T_u) \) or \( (L_v, T_v) \). \( (L_u, T_u) \) and \( (L_v, T_v) \) were described explicitly by Theorem 3.5. In a space \( X \), a path from a point \( p \) to another point \( q \) is a set of points \( A \subseteq X \) such that \( p, q \in A \) and \( A \) with the relative topology is a chain such that \( p \) and \( q \) are the non-cutting points of \( A \). We denote a path \( A \) from \( p \) to \( q \) by \( pq \). For two points \( p \) and \( q \), \( p \) is adjacent to \( q \) if \( \{p, q\} \) is a path from \( p \) to \( q \), i.e., if \( p \in A \) or \( q \in A \).

Remark 6.1. Let \( X \) be an Artinian space. If \( y \in U_x \), \( y \not= x \), then there exists a path from \( x \) to \( y \). In fact, since \( y \) is an open point in the relative topology of \( \{x, y\} \), \( x \) is adjacent to \( y \).

Theorem 6.2. Let \( X \) be an Artinian space. Then \( x \) and \( y \) belong to the same component of \( X \) if and only if there exists a path from \( x \) to \( y \).

Proof. Let \( x_0 \) be a fixed point of \( X \). Define \( C = \{y \in X : x_0 \text{ exists} \} \cup \{x_0\} \).

We will show that \( C \) is the component of \( X \) containing \( x_0 \). Let \( x \in X \) such that either \( U_x \cap C \not= \emptyset \) or \( x \not\in C \). In either case there exists \( y \in C \) such that \( \{x, y\} \) is a path from \( x \) to \( y \). \( y \in C \) implies there exists a path \( x_0 \) from \( x_0 \) to \( x \). Let the points of \( x_0 \) be denoted by \( x_0 = x_1, x_2, \ldots, x_{n-1} = y \). Let \( x_1 \) be the first point of the path \( x_0 \) for which either \( x_1 \not\in X \) or \( x_1 \not\in E \). Assume first \( x_1 \not\in X \). If \( x_1 = x_0 \), there is nothing to prove. Let \( x_1 \) be a closed point in the path \( x_0 \). Then \( x_{n-1} \in E \). And we have a contradiction to \( x_1 \) being the first point of the path \( x_0 \). Therefore \( x_1 \) is an open point in the path \( x_0 \). It follows that \( x_0 \) is an open path from \( x_0 \) to \( x \). The other case where \( x_1 \not\in E \) is handled similarly. Thus, \( \langle x \rangle \). It follows that \( C \) is clopen or that \( C \) is the component of \( X \) containing \( x_0 \).

A tree is a finite space \( X \) for which there exists a unique path between any two points and for any \( p, q, r \in X \),

\[
(pq \cap qr) = (pr) \Rightarrow (pq \cup qr = pr).
\]

Proposition 6.3. A finite space \( X \) is a tree if and only if there exists a unique path \( pq \) in \( X \) from any point \( p \) of \( X \) to any point \( q \) of \( X \).

Corollary 6.4. Every tree \( X \) is connected and every point of it is closed or open, but unless \( X \) is a single-point space, never both.

Remark 6.5. When \( U_x \) exists it contains at most one closed point, namely \( x \). In particular, if \( x \) is a point of a space of height 1 then every point of \( U_x \) is open. Similarly, \( X \) contains at most one open point, namely \( x \); if \( x \) belongs to space of height 1 then every point of \( X \) is closed.

Let \( X \) be a space of height \( \leq 1 \). Then the ramification index \( \tau(x) \) of a point \( x \) of \( X \) is defined as follows:

\[
\tau(x) = \begin{cases} 
\text{card} U_x - 1 & \text{if } x \text{ is closed;} \\
\text{card} U_x - 1 & \text{if } x \text{ is open.}
\end{cases}
\]

If \( \tau(x) = 1 \) then the number of components of \( X \) is the same as of \( X \) if \( \tau(x) = 2 \) then we call \( x \) an end-point of \( X \). If \( \tau(x) > 2 \) we call \( x \) a ramification point or a junction point.

In the case of a tree the ramification index can be characterized as follows:

Theorem 6.6. A finite connected space \( X \) of height is a tree if \( \tau(x) \) is equal to the number of components of \( X \) for every point \( x \) of \( X \).

The proof of the above theorem is easy and will be omitted. A similar theorem holds for polyhedra. However, a 1-dimensional connected polyhedra \( X \) is a (poly-
The analogy between finite polyhedra and finite spaces is not mechanical.

**Proposition 6.8.** A tree is a chain if and only if it contains any ramification point.

If a subspace $X$ of a space $Y$ is a tree, then $X$ is called a sub-tree of $Y$.

**Proposition 6.9.** Every connected subspace of a tree $X$ is a sub-tree of $X$.

**Proposition 6.10.** Every sub-tree of a tree $X$ is a retract of $X$.

**Example 6.11.** $X \setminus a$ is a closed subset of and a path in $X$, but it is not a retract of $X$ (see Fig. 2).

Thus not only a sub-tree but even a path which is a closed subset of a finite space $X$ of height 1 need not be a retract of $X$.

The fact that $X \setminus a$ above is not a retract of $X$ follows immediately from the following remark.

**Remark 6.12.** If function $f: X \to Y$ is continuous, then it maps every pair of adjacent points into one point or into a pair of adjacent points. Furthermore, if $p \in X$ is closed and $f(p)$ is open in $Y$, or vice versa, then $f(q) = f(p)$ for every $q \in X$ adjacent to $p$.

Above we had the simplest illustration of non-retraction phenomenon. Below we give the simplest example of a map $f: A \to Y$ from a closed subset $A$ of $X$ into a tree (even a chain) which does not admit any continuous extension.

**Example 6.13.** There are certain circumstances when functions can be extended and the following proposition illustrates such a case. Given a subset $A$ of a tree $X$, a gap in $A$ is a path $pq$ of $X$ where $p$ and $q$ are not adjacent points and $pq \cap A = \{p, q\}$ (see Fig. 3).

**Remark 6.14.** Let $X$ and $Y$ be connected spaces of height 1. $f: X \to Y$ is continuous if and only if $f$ preserves adjacency of points (allowing a point to be adjacent to itself).

**Proposition 6.15.** Let $X$ be a tree and $A \subseteq X$. If $f: A \to Y$ is a continuous map, $pq$ a gap in $A$ such that $f(p) = f(q)$, then $f$ can be extended continuously to $A \cup pq$.

**Proof.** It follows from Remark 6.14 that $f$ can be extended continuously to $A \cup pq$ by defining $f(q) = f(p)$ for all $x \in pq$.

It is possible to obtain some more extension theorems from Proposition 6.10.

**Section 7. Universal maps into trees.** A point $q$ of a tree $X$ is between $x$ and $y$ if and only if $q \in xy$. The point $q$ is strictly between $x$ and $y$ if $q$ is not equal to either $x$ or $y$. A function $f$ preserves betweenness if when $q$ is between $x$ and $y$, then $f(q)$ is between $f(x)$ and $f(y)$. For points $x, y, z, v$, if $x$ is between $y$ and $z$, then $x = y \sim z$ or $y = x \sim z$ for every $z \in X$.

**Theorem 7.2.** Let $X$ and $Y$ be trees. If $f$ is a continuous function which is onto $Y$ and preserves betweenness, then $f$ is a universal function.

**Proof.** Let $g: X \to Y$ be a continuous map such that $g(x) \neq f(x)$ for every $x \in X$. Choose $X_0 \subseteq X$ such that $f(X_0) = Y$ and $f|X_0$ is injective. Define inductively a sequence of points of $X$ as follows:

(a) Choose $x_0 \in X$ arbitrarily.

(b) Let $x_n \in X_0$ be the point such that $f(x_n) = g(x_n)$ and let $x_{n+1}$ be the point of the path $x_nq_n$ which is adjacent to $x_n$ but not equal to $x_n$.

This choice of $x_{n+1}$ is always possible since otherwise $x_n = q_n$ and this contradicts our assumption that $f(x) \neq g(x)$ for all $x \in X$. The following case study will show that $x_n$ is strictly between $x_{n-1}$ and $x_{n+1}$.

**Case 1.** $g(x_n) = g(x_{n-1})$. By our choice of $X_0$, it follows that $q_n = q_{n-1}$. Thus $x_{n-1} - x_n - q_n$, and $x_{n+1} - x_n - q_n$ imply that $x_n$ is between $x_{n-1}$ and $x_{n+1}$.

**Case 2.** $g(x_n) \neq g(x_{n-1})$. If $x_{n-1} - x_n - q_n$, then it follows that $x_n$ is between $x_{n-1}$ and $x_{n+1}$. As was stated in Remark 7.1 the only other possibility is $q_n - x_{n-1} - x_n$.
Since $x_n$ is between $x_{n-1}$ and $q_{n-1}$ the order relationship between the points is $q_n - x_{n-1} - x_n - q_{n-1}$. Since $f$ preserves betweenness $f(q_n) - f(x_{n-1}) = f(x_n) - f(q_{n-1})$ (allowing for equality). Since $g$ is continuous $g(x_n)$ is adjacent to $g(x_{n-1})$. Therefore, $f(q_n)$ is adjacent to $f(q_{n-1})$. The only remaining possibility is for $f(x_{n-1}) = f(q_n)$ and at the same time $f(x_n) = f(q_{n-1}) = g(x_{n-1})$. Without loss of generality assume $x_{n-1}$ is open and $x_n$ is closed. First assume $f(x_n)$ is open. This contradicts the continuity of $f$ since $f^{-1}(f(x_n))$ is not an open set in $X$. Conversely assume $f(x_n)$ is closed. This contradicts the continuity of $g$ since $g^{-1}(f(x_n))$ is not a closed set in $X$.

It has been shown that $\{x_n\}$ is an infinite sequence of distinct points in $X$ which contradicts the fact that $X$ is finite.

**Corollary 7.3.** A tree has the fixed point property.

**Corollary 7.4** Let $X$ and $Y$ be trees and $A$ a subtree of $X$. If $f: X \to Y$ is a continuous function such that $f|A: A \to Y$ is onto and preserves betweenness for points of $A$, then $f$ is a universal function.

**Proof.** From Theorem 7.2 it follows that $f|A$ is a universal function from $A$ to $Y$. It follows that $f: X \to Y$ must also be a universal function.

**Example 7.5.** This example illustrates that it is not always possible, even under the best of circumstances, to get a connected subset of the domain on which the universal map is onto the image space and preserves betweenness in the subset (see Fig. 4).

![Diagram](image4.png)

**Example 7.6.** This particular example is of special interest since it is a universal map of a chain (universal image space) onto a non-chain. Moreover it illustrates that one can not necessarily find a subset of the domain for which the function is onto the image and preserves betweenness in the subset (see Fig. 5).

![Diagram](image5.png)

The image space of Example 7.6 is called the 4-point tried. This is both a tree and a p-space but is not a universal image space as was pointed out by Corollary 3.6. That this space is a universal image for any tree with at least 6 points follows easily from Corollary 5.5.

In reference to a question posed in the introduction of this paper, the following example illustrates that the composition of universal functions between trees need not be universal.

**Example 7.7** (see Fig. 6).

Let $f_1: K_1 \to K_2$ as given by the diagram.

Let $f_2: K_2 \to K_3$ as given by the diagram.

Let $f = f_2 \circ f_1$. To show that $f$ is not universal define $g: K_1 \to K_3$ such that $g(x_i) = b$ for $i = 1, 2, 3$; $g(x_4) = p$; and $g(x_5) = a$ for $i = 5, 6, 7$.

![Diagram](image6.png)

**References**


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