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Axiomatic foundations for Nonstandard Analysis

by

Karel Hrbacek (New York, N. Y.)

Abstract. We propose an axiomatic system for nonstandard set theory, which can be used to formalize nonstandard mathematics in much the same way axiomatic set theory has been used to formalize standard mathematics. It turns out that the axioms of Power Set and Replacement cannot hold simultaneously in the universe of external sets. This leads to several variants of the system; some of them are conservative extensions of ZFC, and others are essentially stronger.

The usual mathematical foundations for Nonstandard Analysis consist in the use of higher-order structures and their enlargements. There are two main disadvantages of this approach: different enlargements are needed for different problems, and the work with higher-order structures involves the type-theoretic language repugnant to most mathematicians. We attempt to remedy both of these faults by setting down a simple axiomatic system which is a conservative extension of Zermelo–Fraenkel set theory ZFC, whose intuitive interpretation is easy to grasp, and in which all results of, say, Robinson's [8] can be naturally formulated and proved.

Consideration of axiomatic systems for Nonstandard Analysis was initiated by Kreisel in [6] for philosophical reasons (see also Parikh [7]). Kreisel asks:

(1) Is there a simple formal system (...) in which existing practice of nonstandard analysis can be codified? And if the answer is positive:

(2) Is this formal system a conservative extension of the current system of analysis (in which the existing practice of standard analysis has been codified)?

As the methods of nonstandard "analysis" found fruitful applications in general topology, abstract measure theory and functional analysis, we will consider questions (1) and (2) for set theory rather than analysis proper. Two conservative extensions of ZFC, $\mathfrak{NS}_1$ and $\mathfrak{NS}_2$, will be formulated; the extensions differ in properties of "external sets", but both of them provide practically satisfactory positive answer to (1). We will also consider a strengthening of $\mathfrak{NS}_2$ which is a nonconservative extension of ZFC.

The theories $\mathfrak{NS}_1$, $\mathfrak{NS}_2$ and $\mathfrak{NS}_3$ and the conservation results are formulated in § 1. Examples 1 and 2 and Lemma 2 show how simple nonstandard results can be proved from the axioms. The proofs of the main theorems are in § 3. The proof of Theorem 1 uses saturated models and methods similar to those of Chang [1],

[2], [3]. The necessary definitions and technical lemmas can be found in § 2; we hope that these methods can find other applications.

The following intuitive interpretation underlies the whole work:

The sets from the usual set-theoretic universe V are called standard sets. Standard sets may have additional, ideal, nonstandard elements. The universe of internal (both standard and nonstandard) sets is an elementary, "ON-saturated" extension of V. One may picture internal sets as elements of a limit ultraproduct of V. Finally, we extend the universe by collections of internal sets which are not themselves internal (such as the collection of all nonstandard natural numbers). The only technically nontrivial problem is to show that this can be done so that enough of the axioms of set theory will be satisfied.

§ 1. Axioms for nonstandard set theory. Let $\mathfrak{T}$ be an axiomatic set theory, i.e., a theory formalized in a language having a single binary predicate $\in$ and such that the Zermelo–Fraenkel axioms, including the axioms of Regularity (AR) and Choice (AC) are provable in $\mathfrak{T}$. The language of the *nonstandard extension* of $\mathfrak{T}$, $\mathfrak{NS}(\mathfrak{T})$, contains $\in$ and unary predicates $\mathfrak{S}(.)$ and $\mathfrak{I}(.)$. Boldface types $\boldsymbol{x}, \boldsymbol{A}, \ldots$ will denote variables of $\mathfrak{NS}(\mathfrak{T})$; intuitively, they range over the "universe of discourse" of $\mathfrak{NS}(\mathfrak{T})$ consisting of *external sets*. $\boldsymbol{x} \in \boldsymbol{A}$ reads: $\boldsymbol{x}$ belongs to $\boldsymbol{A}$; $\mathfrak{S}(\boldsymbol{x})$ reads: $\boldsymbol{x}$ is a *standard set*; intuitively, standard sets should be identified with the members of the "universe of discourse" of $\mathfrak{T}$. Lightface letters $a, A, \ldots$ will denote variables ranging over standard sets. $\mathfrak{I}(\boldsymbol{x})$ reads: $\boldsymbol{x}$ is an *internal set*. Variables ranging over internal sets will be denoted by Greek letters $\xi, \eta, \ldots$ It is assumed throughout that different letters denote different variables.

If $\Phi(v_1, \ldots, v_n)$ is a formula of $\mathfrak{T}$, $\bar{\Phi}$ is a formula of $\mathfrak{NS}(\mathfrak{T})$ obtained by replacing all variables of Φ by variables of $\mathfrak{NS}(\mathfrak{T})$ (in a one-to-one way). $\Phi^{\mathfrak{S}}$ ($\Phi^{\mathfrak{I}}$, resp.) is obtained from $\bar{\Phi}$ by replacing all bound variables by variables ranging over standard sets (internal sets, resp.).

We will consider three groups of axioms for $\mathfrak{NS}(\mathfrak{T})$:

(A) $\Phi^{\mathfrak{S}}$ is an axiom of $\mathfrak{NS}(\mathfrak{T})$ whenever the sentence Φ is an axiom of $\mathfrak{T}$.

(B1) $(\forall x)\ \mathfrak{I}(x)$.

All standard sets are internal.

(B2) $(\forall \boldsymbol{x})(\forall \xi)(\boldsymbol{x} \in \xi \rightarrow \mathfrak{I}(\boldsymbol{x}))$

The universe of internal sets is transitive.

(B3) (*The Axiom Schema of Embedding*)

Let Φ be a formula of the language of $\mathfrak{T}$.

$$(\forall x_1, \ldots, x_n)\left(\Phi^{\mathfrak{S}}(x_1, \ldots, x_n) \equiv \Phi^{\mathfrak{I}}(x_1, \ldots, x_n)\right).$$

The universe of internal sets is an elementary extension of the universe of standard sets.

(B4) (*The Axiom Schema of Saturation*)

Let Φ be a formula of the language of $\mathfrak{T}$.

$$(\forall x_1, \ldots, x_n)(\forall A)[(\forall a)(a \subseteq A \ \&\ a \text{ is finite} \rightarrow$$
$$(\exists b)(\forall x \in a)\Phi^{\mathfrak{S}}(x, b, A, x_1, \ldots, x_n)) \rightarrow (\exists \beta)(\forall x \in A)\Phi^{\mathfrak{I}}(x, \beta; A, x_1, \ldots, x_n)].$$

An explanation of the notation is in order. There are three possible definitions of inclusion between standard sets:

$$a \subseteq A \equiv (\forall \boldsymbol{x})(\boldsymbol{x} \in a \rightarrow \boldsymbol{x} \in A),$$
$$a \subseteq^{\mathfrak{I}} A \equiv (\forall \xi)(\xi \in a \rightarrow \xi \in A),$$
$$a \subseteq^{\mathfrak{S}} A \equiv (\forall x)(x \in a \rightarrow x \in A).$$

However, $a \subseteq^{\mathfrak{S}} A \equiv a \subseteq^{\mathfrak{I}} A$ by (B3) and $a \subseteq^{\mathfrak{I}} A \equiv a \subseteq A$ by (B1) and (B2). Similarly, a is $\mathfrak{S}$-finite $\equiv a$ is $\mathfrak{I}$-finite $\equiv a$ is finite (see Lemma 1 for general discussion and Lemma 2 for the proof of the last equivalence.).

A stronger version of (B4) is often useful in practice. An external set $\boldsymbol{A}$ *has standard size*, $\mathfrak{SS}(\boldsymbol{A})$, if there is a standard set A and a function $\boldsymbol{f}$ such that $\boldsymbol{x} \in \boldsymbol{A}$ iff $\boldsymbol{x} = \boldsymbol{f}(x)$ for some $x \in A$.

(B4$^+$) Let Φ be a formula of the language of $\mathfrak{T}$.

$$(\forall \eta_1, \ldots, \eta_n)(\forall \boldsymbol{A} \mathfrak{SS}(\boldsymbol{A}))[(\forall \boldsymbol{a})(\boldsymbol{a} \subseteq \boldsymbol{A} \ \&\ \boldsymbol{a} \text{ is finite} \rightarrow$$
$$(\exists \beta)(\forall \xi \in \boldsymbol{a})\,\Phi^{\mathfrak{I}}(\xi, \beta, \eta_1, \ldots, \eta_n)) \rightarrow (\exists \beta)(\forall \xi \in \boldsymbol{A})\,\Phi^{\mathfrak{I}}(\xi, \beta, \eta_1, \ldots, \eta_n)].$$

(C0) (*The Axiom of Transfer*)

$$(\forall \boldsymbol{A})(\exists A)(\forall x)(x \in \boldsymbol{A} \equiv x \in A).$$

For every external set $\boldsymbol{A}$ there is a standard set $A^* = \{x \in \boldsymbol{A} |\ \mathfrak{S}(x)\}$, the *standard kernel* of $\boldsymbol{A}$, having the same standard elements as $\boldsymbol{A}$. The Axiom of Transfer permits unlimited use of external sets in constructions of standard sets.

(C1) (*The Axiom of Extensionality*)

$$(\forall \boldsymbol{X}, \boldsymbol{Y})[\boldsymbol{X} = \boldsymbol{Y} \equiv (\forall \boldsymbol{u})(\boldsymbol{u} \in \boldsymbol{X} \equiv \boldsymbol{u} \in \boldsymbol{Y})].$$

(C2) (*The Axiom of Pairing*)

$$(\forall \boldsymbol{X}, \boldsymbol{Y})(\exists \boldsymbol{Z})(\boldsymbol{X} \in \boldsymbol{Z} \ \&\ \boldsymbol{Y} \in \boldsymbol{Z}).$$

(C3) (*The Axiom of Union*)

$$(\forall \boldsymbol{A})(\exists \boldsymbol{S})(\forall \boldsymbol{u})(\boldsymbol{u} \in \boldsymbol{A} \rightarrow \boldsymbol{u} \subseteq \boldsymbol{S}).$$

(C4) (*The Axiom Schema of Comprehension*)

Let Φ be a formula of the language of $\mathfrak{NS}(\mathfrak{T})$.

$$(\forall \boldsymbol{A})(\exists \boldsymbol{B})(\forall \boldsymbol{x})\left(\boldsymbol{x} \in \boldsymbol{B} \equiv \boldsymbol{x} \in \boldsymbol{A} \ \&\ \Phi(\boldsymbol{x}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)\right).$$

The *nonstandard extension of* $\mathfrak{T}$, $\mathfrak{NS}(\mathfrak{T})$, has axioms (A), (B1)-(B3), (B4$^+$), (C0)-(C4). Before considering further strengthening of $\mathfrak{NS}(\mathfrak{T})$, we prove several easy results.

EXAMPLE 1. *Every standard infinite set A has nonstandard elements.* (A set is *nonstandard* if it is internal, but not standard.)

Proof. For every finite $a \subseteq A$ there is b such that, for all $x \in a$, $x \neq b$ & $b \in A$. By (B4) there exists β such that, for all $x \in A$, $x \neq \beta$ & $\beta \in A$. β is a nonstandard element of A. ∎

EXAMPLE 2. *For every standard set A there is an internally finite set β such that* $(\forall x)(x \in A \to x \in \beta)$, i.e., all standard elements of A belong to β.

Proof. For every finite $a \subseteq A$ there exists b such that, for all $x \in a$, $x \in b$ & b if $\mathfrak{S}$-finite. (Let $b = a$.) By (B4) there exists β such that, for all $x \in A$, $x \in \beta$ & β is $\mathfrak{I}$-finite. ∎

LEMMA 1. *Let Φ be a bounded formula of $\mathfrak{T}$. Then*

$$\mathfrak{NS}(\mathfrak{T}) \vdash \Phi^{\mathfrak{I}}(\xi_1, \ldots, \xi_n) \equiv \bar{\Phi}(\xi_1, \ldots, \xi_n)\,.$$

In particular, $\mathfrak{NS}(\mathfrak{T}) \vdash \Phi^{\mathfrak{S}}(a_1, \ldots, a_n) \equiv \Phi^{\mathfrak{I}}(a_1, \ldots, a_n) \equiv \bar{\Phi}(a_1, \ldots, a_n)$.

Proof. Immediate from (B2) and (B3). The details of the proof of absoluteness of bound formulas in transitive classes can be found e.g. in [5], p. 22. ∎

Notice also that the universe of internal sets is closed under pairing, i.e., $\mathfrak{NS}(\mathfrak{T}) \vdash (\forall \xi)(\forall \eta)(\exists \zeta)(\xi \in \zeta \;\&\; \eta \in \zeta)$ (The Axiom of Pairing is provable in $\mathfrak{T}$ and we have (A) and (B3)). Therefore, if Φ is one of the formulas (1)-(22) on pp. 22–23 in [5], $\Phi^{\mathfrak{S}} \equiv \Phi^{\mathfrak{I}} \equiv \bar{\Phi}$ is provable in $\mathfrak{NS}(\mathfrak{T})$.

Although $\Phi^{\mathfrak{S}}(x) \equiv \bar{\Phi}(x)$ does not always hold (examples: $\Phi(v_0) \equiv v_0$ is an ordinal, $\Phi(v_0, v_1) \equiv v_0$ is the power set of v_1), we have

LEMMA 2. *Let $\Phi_1(v) \equiv v$ is a natural number; $\Phi_2(v) \equiv v$ is finite. The following is provable in* $\mathfrak{NS}(\mathfrak{T})$:

a) $\bar{\Phi}_1(\boldsymbol{n}) \equiv \mathfrak{S}(\boldsymbol{n}) \;\&\; \Phi_1^{\mathfrak{S}}(\boldsymbol{n})$,

b) $\Phi_2^{\mathfrak{S}}(\boldsymbol{x}) \;\&\; \mathfrak{S}(\boldsymbol{x}) \equiv \bar{\Phi}_2(\boldsymbol{x}) \;\&\; (\forall \boldsymbol{y} \in \boldsymbol{x})\mathfrak{S}(\boldsymbol{y})$.

In words, the standard natural numbers coincide with the external natural numbers (but not with the internal natural numbers). The external finite collections of standard sets coincide with standard finite collections of standard sets. In part., all elements of a standard finite set are standard.

Proof. a) We first prove $\bar{\Phi}_1(\boldsymbol{n}) \to (\mathfrak{S}(\boldsymbol{n}) \;\&\; \Phi_1^{\mathfrak{S}}(\boldsymbol{n}))$. Let $\boldsymbol{n}_0$ be the first $\boldsymbol{n}$ (in the wellordering of external natural numbers by $\in$) for which it fails. Surely $\boldsymbol{n}_0 \neq \varnothing$; thus $\boldsymbol{n}_0 = \boldsymbol{m}_0 \cup \{\boldsymbol{m}_0\}$ for some $\boldsymbol{m}_0 \in \boldsymbol{n}_0$ s.t. $\bar{\Phi}_1(\boldsymbol{m}_0)$. We have $\mathfrak{S}(\boldsymbol{m}_0) \;\&\; \Phi_1^{\mathfrak{S}}(\boldsymbol{m}_0)$ by assumption. Let $n_0 = \boldsymbol{m}_0 \cup^{\mathfrak{S}} \{\boldsymbol{m}_0\}^{\mathfrak{S}}$; then $\Phi_1^{\mathfrak{S}}(n_0)$ holds and $(\forall x)(x \in n_0 \equiv x \in \boldsymbol{m}_0 \vee x = \boldsymbol{m}_0)$ is (equivalent to) a bounded formula; by Lemma 1, $(\forall \boldsymbol{x})(\boldsymbol{x} \in n_0 \equiv \boldsymbol{x} \in \boldsymbol{m}_0 \vee \boldsymbol{x} = \boldsymbol{m}_0)$; i.e., $n_0 = \boldsymbol{m}_0 \cup \{\boldsymbol{m}_0\} = \boldsymbol{n}_0$. So $\mathfrak{S}(\boldsymbol{n}_0) \;\&\; \Phi_1^{\mathfrak{S}}(\boldsymbol{n}_0)$, a contradiction.

To prove that, conversely, $(\mathfrak{S}(\boldsymbol{n}) \;\&\; \Phi_1^{\mathfrak{S}}(\boldsymbol{n})) \to \bar{\Phi}_1(\boldsymbol{n})$, we first notice that $\{\boldsymbol{n} |\ \mathfrak{S}(\boldsymbol{n}) \;\&\; \Phi_1^{\mathfrak{S}}(\boldsymbol{n})\} \subseteq \omega_0^{\mathfrak{S}}$ is wellordered (as an external set) by $\in$. Indeed, if $\varnothing \neq \boldsymbol{X} \subseteq \{\boldsymbol{n} |\ \mathfrak{S}(\boldsymbol{n}) \;\&\; \Phi_1^{\mathfrak{S}}(\boldsymbol{n})\}$, $\varnothing \neq \boldsymbol{X}^* \subseteq \omega_0^{\mathfrak{S}}$, so $(\exists n \in \boldsymbol{X}^*)(\forall k \in \boldsymbol{X}^*)(n \in k \vee n = k)$. But then $n \in \boldsymbol{X}$ is the first element of $\boldsymbol{X}$. Let now $\boldsymbol{n}_0$ be the first $\boldsymbol{n}$ for which the above fails. Surely $\boldsymbol{n}_0 \neq \varnothing$, so $\boldsymbol{n}_0 = \boldsymbol{m}_0 \cup^{\mathfrak{S}} \{\boldsymbol{m}_0\}^{\mathfrak{S}}$ where $\mathfrak{S}(\boldsymbol{m}_0) \;\&\; \Phi_1^{\mathfrak{S}}(\boldsymbol{m}_0) \;\&\; \bar{\Phi}_1(\boldsymbol{m}_0)$. As before, $\boldsymbol{n}_0 = \boldsymbol{m}_0 \cup \{\boldsymbol{m}_0\}$, so $\bar{\Phi}_1(\boldsymbol{n}_0)$, a contradiction.

b) The proof is similar; use the definition v is finite $\equiv$ there is a one-to-one mapping of v onto a natural number. ∎

Let us now consider three additional axioms for external sets:

(APE) (*The Axiom of Power Set for External Sets*)

$$(\forall \boldsymbol{A})(\exists \boldsymbol{P})(\forall \boldsymbol{a})(\boldsymbol{a} \subseteq \boldsymbol{A} \to \boldsymbol{a} \in \boldsymbol{P})\,.$$

(ARE) (*The Axiom Schema of Replacement for External Sets*)

Let Φ be a formula of the language of $\mathfrak{NS}(\mathfrak{T})$.

$$(\forall \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)(\forall \boldsymbol{A})(\exists \boldsymbol{B})(\forall \boldsymbol{x} \in \boldsymbol{A})[(\exists \boldsymbol{y})\Phi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \to (\exists \boldsymbol{y} \in \boldsymbol{B})\,\Phi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)]\,.$$

(ACE) (*The Axiom of Choice for External Sets*)

$$(\forall \boldsymbol{A})(\exists \boldsymbol{W})\ (\boldsymbol{W} \text{ wellorders } \boldsymbol{A})\,.$$

THEOREM 1. $\mathfrak{NS}_1(\mathfrak{T}) = \mathfrak{NS}(\mathfrak{T}) + \text{(ARE)}$ *is a conservative extension of* $\mathfrak{T}$. (I.e., $\mathfrak{T} \vdash \Phi$ iff $\mathfrak{NS}_1(\mathfrak{T}) \vdash \Phi^{\mathfrak{S}}$ holds for any sentence Φ of $\mathfrak{T}$.)

THEOREM 2. $\mathfrak{NS}_2(\mathfrak{T}) = \mathfrak{NS}(\mathfrak{T}) + \text{(APE)} + \text{(ACE)}$ *is a conservative extension of* $\mathfrak{T}$.

THEOREM 3. a) $\mathfrak{NS}(\mathfrak{T}) + (\Sigma_1\text{-ARE}) + \text{(APE)}$ *is inconsistent.*

b) $\mathfrak{NS}(\mathfrak{T}) + (\Sigma_1\text{-ARE}) + \text{(ACE)}$ *is inconsistent.*

$(\Sigma_1\text{-ARE})$ is obtained by restricting (ARE) to Σ_1-formulas of $\mathfrak{NS}(\mathfrak{T})$.

A weak vestige of (ARE) is compatible with (APE). Let ($\mathfrak{SS}$-ARE) be obtained by restricting all instances of (ARE) to external sets of standard size, i.e., by replacing $(\forall \boldsymbol{A})\ldots$ with $(\forall \boldsymbol{A})(\mathfrak{SS}(\boldsymbol{A}) \to \ldots$

THEOREM 4. a) $\mathfrak{NS}(\mathfrak{T}) + (\mathfrak{SS}\text{-ARE}) + \text{(APE)}$ *is a nonconservative extension of* $\mathfrak{T}$.

b) *If* ZFC$+(\exists\lambda)$ (λ *is strongly inaccessible and* $V_\lambda \vDash \mathfrak{T}$) *is consistent, then* $\mathfrak{NS}_3(\mathfrak{T}) = \mathfrak{NS}(\mathfrak{T}) + (\mathfrak{SS}\text{-ARE}) + \text{(APE)} + \text{(ACE)}$ *is consistent.*

Summarily, we can postulate that the external sets satisfy either the axioms of Zermelo–Fraenkel set theory without Power set or the axioms of Zermelo set theory with Choice, but further strengthenings are inconsistent. Both theories seem sufficient for formalization of the usual proofs and constructions of nonstandard mathematics; when nonstandard sets are investigated for their own sake (e.g., one considers external Dedekind cuts in the ordering of internal real numbers), one theory is sometimes more convenient than the other.

We will prove Theorems 3 and 4a) now and the rest in § 3.

For every A let $A^0 = \{x \in A |\ \mathfrak{S}(x)\}$. Notice that $A^0 = A$ for finite standard A, but otherwise $A^0 \subsetneq A$; however, $(A^0)^* = A$ because A^0 and A have the same standard elements.

Proof of Theorem 3. For every finite $a \subseteq A$ there is a 1-1 mapping f s.t. $a \subseteq \mathrm{dom}(f)$, $\mathrm{ran}(f) \subseteq \omega$. Using (B4) we conclude that there is a 1-1 (internal) mapping f s.t. $A^0 \subseteq \mathrm{dom}(f)$, $\mathrm{ran}(f) \subseteq \omega$. Then $f \upharpoonright A^0$ is a 1-1 (external) mapping of A^0 into ω. We use this to show the existence of a set S containing arbitrarily large standard ordinals. If then $S = \bigcup S^*$, S is a standard ordinal, $S \in \bigcup S^* = S$, a contradiction with $(\mathrm{AR})^{\mathfrak{S}}$.

a) (APE) implies existence of P s.t. $X \subseteq \omega \rightarrow X \in P$. Let

$$\varphi(X, Y) \equiv Y \text{ is a standard ordinal } \& (\exists f)\ (f \text{ is a 1-1 mapping of } Y^0 \text{ onto } X).$$

(Σ_1-ARE) implies existence of S s.t. $(\forall X \in P)[(\exists Y)\varphi(X, Y) \rightarrow (\exists Y \in S)\varphi(X, Y)]$. Let A be any standard ordinal; let $f \upharpoonright A^0$ be as above. Then $X = \mathrm{ran}(f \upharpoonright A^0) \subseteq \omega$, so $X \in P$ & $(\exists Y)\varphi(X, Y)$, namely, $Y = A$. Pick $B \in S$ s.t. $\varphi(X, B)$ (B is a standard ordinal) and let g be a 1-1 mapping of B^0 onto X. Then $g^{-1} \circ f$ is a 1-1 mapping of A^0 onto B^0, so $(g^{-1} \circ f)^*$ is a standard 1-1 mapping of A onto B. The standard ordinal $B \in S$ and the standard cardinality of B equals the standard cardinality of A.

b) (ACE) implies the existence of a wellordering W of ω. Let

$$\varphi(X, Y) \equiv Y \text{ is a standard ordinal } \& (\exists f)\ (f \text{ is an isomorphism between } (Y^0, \in \upharpoonright Y^0) \text{ and the initial segment of } W \text{ determined by } X).$$

(Σ_1-ARE) implies the existence of S s.t. $(\forall X \in \omega)((\exists Y)\varphi(X, Y) \rightarrow (\exists Y \in S)\varphi(X, Y))$. Let A be any standard ordinal; let $f \upharpoonright A^0$ be as before. Thus there is a 1-1 mapping between $A^0 = \mathrm{dom}(f \upharpoonright A^0)$ and $\tilde{X} \subseteq \omega$ where $\tilde{X} = \mathrm{ran}(f \upharpoonright A^0)$; but there is no 1-1 mapping between A^0 and ω; also, $(A^0, \in \upharpoonright A^0)$ and (ω, W) are wellordered sets. A standard set-theoretic argument, using only the axioms satisfied by external sets (in part., Σ_1-replacement, but not Power set), shows that $(A^0, \in \upharpoonright A^0)$ is isomorphic to an initial segment of (ω, W) determined by some $X \in W$, i.e., $\varphi(X, A)$ holds. If $B \in S$ is such that $\varphi(X, B)$, the argument used in conclusion of part a) will again show that the standard cardinality of B equals the standard cardinality of A (actually, $B = A$ in this case). ■

Proof of Theorem 4a). Let $\mathfrak{KM}(\mathfrak{T})$, the Kelley-Morse extension of $\mathfrak{T}$, be the theory whose language has variables for sets and *classes* and whose axioms include $\mathfrak{T}$ and axioms of Kelley–Morse set theory, particularly the Full Comprehension Schema $(\forall X_1, \ldots, X_n)(\exists X)(\forall x)(x \in X \equiv \varphi(x, X_1, \ldots, X_n))$ where φ is any formula of $\mathfrak{KM}(\mathfrak{T})$.

It is well known that $\mathfrak{KM}(\mathfrak{T})$ is a nonconservative extension of $\mathfrak{T}$ (because satisfaction for set-theoretic formulas can be defined in it). It will thus suffice to give an interpretation of $\mathfrak{KM}(\mathfrak{T})$ in $\mathfrak{NS}'(\mathfrak{T}) = \mathfrak{NS}(\mathfrak{T}) + (\mathrm{APE}) + (\mathfrak{SS}\text{-ARE})$.

Working in $\mathfrak{NS}'(\mathfrak{T})$, let

$$\psi(A, R) \equiv R \subseteq \omega^2 \ \& (\exists f)\ (f \text{ is an isomorphism between } \in \upharpoonright (\mathrm{TC}^{\mathfrak{S}}\{A\})^0 \text{ and } R)$$

($\mathrm{TC}^{\mathfrak{S}}\{A\}$ is the standard transitive closure of $\{A\}$). The argument at the beginning of the proof of Theorem 3 shows $(\forall A)(\exists R)\psi(A, R)$. Set

$$R_1 \approx R_2 \equiv (\exists A)(\psi(A, R_1) \ \& \ \psi(A, R_2))$$

and denote the equivalence class of R in $\approx$ by $\bar{R}$. Finally, put

$$V_0 = \{\bar{R} |\ (\exists A)\psi(A, R)\}; \quad V_0 \in P(P(P(\omega^2))),$$
$$E_0 = \{(\bar{R}_1, \bar{R}_2) |\ (\exists A_1, A_2)(A_1 \in A_2 \ \& \ \psi(A_1, R_1) \ \& \ \psi(A_2, R_2))\}.$$

If we let $\bar{\psi}(A, \bar{R}) \equiv \psi(A, R)$, the formula $\bar{\psi}$ describes an isomorphic correspondence between $\in$ on standard sets and (V_0, E_0).

It is now trivial to extend this interpretation, by adding classes, to a model of $\mathfrak{KM}(\mathfrak{T})$: let

$$Cl = \{X |\ X \subseteq V_0\},$$
$$E = \{(X, Y) \in Cl^2 |\ (\exists Z \in V_0)[(\forall U \in V_0)(U \in X \equiv (U, Z) \in E_0) \ \& \ Z \in Y]\},$$
$$S = \{X \in Cl |\ (\exists Z \in V_0)(\forall U \in V_0)(U \in X \equiv (U, Z) \in E_0)\}.$$

We interpret classes and sets as elements of Cl, S resp., and the membership relation as E. The verification of the validity of the axioms for $\mathfrak{KM}(\mathfrak{T})$ in (Cl, S, E) is routine. The Full Comprehension Schema follows from (C4). The Axiom of Replacement for $\mathfrak{KM}(\mathfrak{T})$ follows from ($\mathfrak{SS}$-ARE) via the fact that $\{U \in V_0 |\ (U, Z) \in E_0\}$ is a standard size external set. ■

§ 2. $\varkappa$-constructible sets over structures. In this paragraph we will work in ZFC and use standard set-theoretic and model-theoretic notation.

Let $\mathfrak{W} = (W; E, V)$ be a structure where E is a binary relation, V is a unary relation and $\mathfrak{W}$ is *extensional*:

If $X, Y \in W \ \& \ (\forall Z \in W)((Z, X) \in E \equiv (Z, Y) \in E)$ then $X = Y$.

We will extend $\mathfrak{W}$ to an interpretation for $\mathrm{ZF}^0 = \mathrm{ZF} -$ Axiom of Regularity so that E is the membership relation on W. The construction slightly generalizes Chang's $\varkappa$-constructible sets [3].

The infinitary language $L_{\infty\infty}$ has a binary predicate symbol $\in$, a unary predicate symbol $\mathfrak{V}$ and variables $u_\iota, v_\iota, w_\iota, \ldots$ for all ι. Formulas are obtained from the atomic ones by applications of negations, infinitary conjunctions $\bigwedge_{\iota \in I}$, disjunctions $\bigvee_{\iota \in I}$ and quantifiers $(\exists v_\iota)_{\iota \in I}$ and $(\forall v_\iota)_{\iota \in I}$. Equality $v_1 = v_2$ is considered an abbreviation for $(\forall u)\ (u \in v_1 \equiv u \in v_2)$. If $\varkappa \leqslant \lambda$ are regular cardinals, $L_{\lambda,\varkappa}$ consists of formulas of $L_{\infty\infty}$ having $\bigwedge_{\iota \in I}$ and $\bigvee_{\iota \in I}$ for $|I| < \lambda$, $(\exists v_\iota)_{\iota \in I}$, $(\forall v_\iota)_{\iota \in I}$ for $|I| < \varkappa$ and fewer than $\varkappa$ free variables. If $\varphi(v_\iota)_{\iota \in I}$ is a formula of $L_{\infty\infty}$ with all of its free variables among v_ι, $\iota \in I$, and $(a_\iota)_{\iota \in I}$ is a sequence of elements from W, $\mathfrak{W} \models \varphi(\underline{a}_\iota)_{\iota \in I}$ will mean: the sequence $(a_\iota)_{\iota \in I}$ satisfies φ in $\mathfrak{W}$; similarly for other structures of the same type as $\mathfrak{W}$. $A^{<\varkappa}$ is the set of all functions from $I \subseteq \varkappa$, $|I| < \varkappa$, into A.

Fix a regular cardinal $\varkappa$. Define $\mathfrak{Z}(\alpha) = (Z(\alpha);\ E(\alpha), V)$ inductively:
if $\alpha = 0$, $Z(\alpha) = W$, $E(\alpha) = E$;
if α is limit, $Z(\alpha) = \bigcup_{\beta\in\alpha} Z(\beta)$, $E(\alpha) = \bigcup_{\beta\in\alpha} E(\beta)$;
if $\alpha = \beta+1$, $A \in Z(\alpha)$ iff $A \in Z(\beta)$ or there is a formula $\varphi(u, v_\iota)_{\iota\in I} \in L_{\varkappa,\varkappa}$ and a sequence $s(A) = (B_\iota)_{\iota\in I} \in Z(\beta)^{<\varkappa}$ such that

(1) $$A = \{X \in Z(\beta)|\ \mathfrak{Z}(\beta) \models \varphi(\underline{X}, \underline{B}_\iota)_{\iota\in I}\}\,.$$

$E(\alpha)$ and $I(\alpha)$ are defined simultaneously as the smallest subsets of $Z(\alpha)^2$ such that
a) $E(\beta) \subseteq E(\alpha)$,
b) if $X \in Z(\beta)$, $A \in Z(\alpha)$ and $\mathfrak{Z}(\beta) \models \varphi(\underline{X}, \underline{B}_\iota)_{\iota\in I}$ then $(X, A) \in E(\alpha)$ (where A is as in (1)),
c) if $A_1, A_2 \in Z(\alpha)$ and, for all $X \in Z(\beta)$, $(X, A_1) \in E(\alpha) \equiv (X, A_2) \in E(\alpha)$ then $(A_1, A_2) \in I(\alpha)$,
d) if $A_1, A_2 \in Z(\alpha)$ and for some $X \in Z(\beta)$ $(A_1, X) \in I(\alpha)\ \&\ (X, A_2) \in E(\alpha)$ then $(A_1, A_2) \in E(\alpha)$.

Let $Z(\infty) = \bigcup_{\alpha\in ON} Z(\alpha)$, $E(\infty) = \bigcup_{\alpha\in ON} E(\alpha)$, $\mathfrak{Z} = (Z(\infty);\ E(\infty), V)$. We will write $Z(\alpha; \varkappa)$ etc. if we wish to stress the dependence on $\varkappa$. If $W = (\varnothing;\ \varnothing, \varnothing)$, $Z(\infty)$ is the class of $\varkappa$-constructible sets $L_\varkappa$ as defined in [3].

We will say that a structure $(Z_1;\ E_1, V_1)$ is an *initial segment* of $(Z_2;\ E_2, V_2)$ if $(Z_1;\ E_1, V_1) \subseteq (Z_2;\ E_2, V_2)$ and for all $A \in Z_1$, $X \in Z_2$

$$(X, A) \in E_2 \quad \text{iff} \quad (\exists B \in Z_1)\big((B, A) \in E_1\ \&\ (Z_2;\ E_2, V_2) \models \underline{X} = \underline{B}\big)\,.$$

i embeds $(Z_1;\ E_1, V_1)$ into $(Z_2;\ E_2, V_2)$ as an initial segment if i is an isomorphism between $(Z_1;\ E_1, V_1)$ and an initial segment of $(Z_2;\ E_2, V_2)$.

LEMMA 1. a) *For all $A \in W$ there is $A' \in Z(1) \setminus W$ such that $\mathfrak{Z}(1) \models \underline{A} = \underline{A'}$; $s(A') = (A)$.*
b) *For $\alpha_1 < \alpha_2$, $\mathfrak{Z}(\alpha_1)$ is an initial segment of $\mathfrak{Z}(\alpha_2)$.*

Proof. a) Let $A' = \{X \in Z(0)|\ \mathfrak{Z}(0) \models \underline{X} \in \underline{A}\}$.
b) Straightforward induction. ∎

LEMMA 2. $(Z(\infty), E(\infty))$ *is an interpretation for* ZF^0.

Notice that the axioms of Regularity and Choice need not hold.

Proof. Entirely standard. The Axiom of Extensionality is verified by induction. The Axiom of Comprehension for bound formulas is gotten from (1). The remaining axioms follow from "almost universality" of $\mathfrak{Z}$: if $Y \subseteq Z(\infty)$ then $Y \subseteq Z(\alpha)$ for some α; so $(\forall X \in Y)[(X, Z(\alpha)) \in E(\alpha+1)]$. ∎

LEMMA 3. $\mathfrak{Z}$ *is closed under subsets of cardinality $<\varkappa$; i.e., if $B \subseteq Z(\alpha)$, $|B| < \varkappa$, then there is $A \in Z(\alpha+1)$ such that $X \in B$ implies $(X, A) \in E(\alpha+1)$; $(X, A) \in E(\alpha+1)$ implies $(\exists Y \in B)(\mathfrak{Z}(\alpha+1) \models \underline{X} = \underline{Y})$.*

Proof. Let $A = \{X \in Z(\alpha)|\ \mathfrak{Z}(\alpha) \models \bigvee_{\iota\in B} \underline{X} = \underline{B}_\iota\}$ where $B_\iota = \iota$.

LEMMA 4. *Let $\sup_{\xi<\varkappa_1} 2^\xi < \varkappa_2$ and let i_0 be an embedding of $\mathfrak{W}_1$ into $\mathfrak{W}_2$ as an initial segment. For all $\alpha \leqslant \varkappa_2$ there exists an embedding i_α of $Z_1(\alpha; \varkappa_1)$ into $Z_2(\alpha; \varkappa_2)$ as an initial segment such that $\alpha_1 < \alpha_2$ implies $i_{\alpha_1} \subseteq i_{\alpha_2}$.*

Proof. i_0 is given and $i_\alpha = \bigcup_{\beta<\alpha} i_\beta$ for limit α. If $\alpha = \beta+1$, we define $i_\alpha(A) = i_\beta(A)$ for $A \in Z_1(\beta)$. Next,

$$i_\alpha(Z_1(\beta)) = \{X \in Z_2(\beta)|\ \mathfrak{Z}_2(\beta) \models \psi(\underline{X}, \underline{i_\beta(Z_1(\delta))})_{\delta<\beta}\}$$

where

$$\psi(u, v_\delta)_{\delta<\beta} \equiv \bigvee_{\delta<\beta}\ \bigvee_{\Phi\in L_{\varkappa_1\varkappa_1}} (\exists t_\iota \in v_\delta)_{\iota\in I}(\forall r)\big(r \in u \equiv r \in v_\delta\ \&\ \Phi^{v_\delta}(u, t_\iota)_{\iota\in I}\big)$$

and Φ^{v_δ} is the formula obtained from Φ by restricting all quantifiers to v_δ; i.e., by replacing $(\exists s_\nu)_{\nu\in N}$ with $(\exists s_\nu)_{\nu\in N}\big(\bigwedge_{\nu\in N}(s_\nu \in v_\delta)\ \&\ \ldots\big)$, etc.

Finally, if $A \in Z_1(\alpha) \setminus (Z_1(\beta) \cup \{Z_1(\beta)\})$, say,

$$A = \{X \in Z_1(\beta)|\ \mathfrak{Z}_1(\beta) \models \varphi(\underline{X}, \underline{B}_\iota)_{\iota\in I}\}\,,$$

we let

$$i_\alpha(A) = \{X \in Z_2(\beta)|\ \mathfrak{Z}_2(\beta) \models \psi(\underline{X}, \underline{i_\beta(Z_1(\delta))})_{\delta<\beta}\ \&$$
$$\&\ \varphi^\psi(\underline{X}, \underline{i_\beta(B_\iota)}, \underline{i_\beta(Z_1((\delta))})_{\iota\in I,\ \delta<\beta}\}$$

where φ^ψ is the formula obtained from φ by replacing $(\exists s_\nu)_{\nu\in N}$ with

$$(\exists s_\nu)_{\nu\in N}\big(\bigwedge_{\nu\in N} \psi(s_\nu, v_\delta)_{\delta<\beta}\ \&\ \ldots\big) \text{ etc.}$$

(i.e., by "restricting" all quantifiers to $i_\alpha(Z_1(\beta))$).

The initial segment property is verified inductively. ∎

We could prove Lemma 4 for all $\alpha \in ON$ and under weaker assumption $\varkappa_1 \leqslant \varkappa_2$, but we will not need it. The proof could be based on the "absoluteness" of sets of cardinality $<\varkappa_1$, $L_{\varkappa_1\varkappa_1}$, and the entire construction of $\mathfrak{Z}_1(\alpha; \varkappa_1)$, in $\mathfrak{Z}_2$.

The main technical result, contained in Lemmas 5 and 6, shows that $L_{\infty\varkappa}$-statements about $\mathfrak{Z}(\alpha)$ are equivalent to $L_{\infty\varkappa}$-statements about $\mathfrak{W}$.

We define an auxiliary language $L^*_{\infty\varkappa}$ having next to unranked variables $u_\iota, v_\iota, \ldots$ also variables of rank α, $u^\alpha_\iota, v^\alpha_\iota, \ldots$ for all $\alpha \in ON$. Ranked terms and formulas are defined inductively:

Formulas of rank α are gotten from the atomic formulas of the form $u^\alpha \in v^\alpha$, $u^\alpha \in t(u_\mu)_{\mu\in M}$, $t(u_\mu)_{\mu\in M} \in u^\alpha$, $t_1(u_\mu)_{\mu\in M} \in t_2(u_\nu)_{\nu\in N}$, $\mathfrak{W}(u^\alpha)$, $\mathfrak{W}(t(u_\mu)_{\mu\in M})$, where t, t_1, t_2 are terms of rank $\leqslant\alpha$, by closure under $\sim$, $\bigwedge_{\iota\in I}$, $\bigvee_{\iota\in I}$ for all I and $(\exists u_\iota)_{\iota\in I}$ and $(\forall u_\iota)_{\iota\in I}$ for $|I| < \varkappa$. (u_ι are unranked variables; notice that they can occur in a ranked formula only as free variables of some ranked term.) Let $L^*_{\alpha\varkappa}$ be the set of formulas of rank α in which $\bigwedge_{\iota\in I}$, $\bigvee_{\iota\in I}$ occur also only for $|I| < \varkappa$, which have $<\varkappa$ free variables and u^α as the only ranked variable. If $\varphi(u^\alpha, v_\iota)_{\iota\in I} \in L^*_{\alpha\varkappa}$, $\{u^\alpha|\ \varphi(u^\alpha, v_\iota)_{\iota\in I}\}$ is a term of rank $\alpha+1$.

In the intended interpretation the unranked variables and the variables of rank 0 range over W; the variables of rank $\alpha>0$ range over $Z(\alpha)\setminus W$ and the terms of rank α denote elements of $Z(\alpha)\setminus W$ (there are no terms of rank α if α is limit).

We now define values of ranked terms and, simultaneously, assign to each ranked formula $\varphi(v_\iota)$ without free ranked variables a formula $\widetilde{\varphi(v_\iota)}$ of $L_{\infty\varkappa}$. We will write

$$\models^* \varphi(\underline{a}_\iota)_{\iota\in I} \quad \text{in place of} \quad \mathfrak{W} \models \widetilde{\varphi(\underline{a}_\iota)}_{\iota\in I}$$

and show in Lemma 5 that $\models^*$ interprets the ranked formulas in the intended way.

If $\varphi(v_\iota)$ has rank 0, let $\widetilde{\varphi(v_\iota)}$ be the formula obtained from φ by replacing all variables of rank 0 with unranked variables (in a one-to-one way which does not conflict with v_ι); φ has no terms.

If $\varphi(u^0, u_\iota)_{\iota\in I} \in L^*_{0\varkappa}$, define the value of the term $\{u^0|\ \varphi(u^0, u_\iota)_{\iota\in I}\}$ at $(a_\iota)_{\iota\in I}$ by $\{u^0|\ \varphi(u^0, \underline{a}_\iota)\} = \{X\in W|\ \mathfrak{W}\models \widetilde{\varphi(\underline{X}, \underline{a}_\iota)}\}$.

If $\varphi \equiv t_1(u_\mu)\in t_2(v_\nu)$ where $t_1 = \{u^0|\ \varphi_1(u^0, u_\mu)\}$, $t_2 = \{u^0|\ \varphi_2(u^0, v_\nu)\}$, let

$$\tilde\varphi \equiv (\exists u)[\varphi_2(u, v_\nu)\ \&\ (\forall v)(v\in u \equiv \varphi_1(v, u_\mu))]\,.$$

If $\varphi \equiv \mathfrak{V}(t_1(u_\mu))$, let $\tilde\varphi \equiv (\exists u)[\mathfrak{V}(u)\ \&\ (\forall v)(v\in u\equiv \varphi_1(v, u_\mu)]$.

If $\varphi(v_\iota)$ is of the form $\sim\psi(v_\iota)$, $\bigwedge_{\nu\in N}\psi_\nu(v_\iota)$, $(\exists w_\nu)_{\nu\in N}\psi(v_\iota, w_\nu)$ etc., let $\tilde\varphi$ be $\sim\widetilde{\psi(v_\iota)}$, $\bigwedge_{\nu\in N}\widetilde{\psi_\nu(v_\iota)}$, $(\exists w_\nu)_{\nu\in N}\widetilde{\psi(v_\iota, w_\nu)}$, etc.

If $\varphi(u^\alpha, u_\iota)\in L^*_{\alpha\varkappa}$ for $\alpha>0$, let

$$\{u^\alpha|\ \varphi(u^\alpha, \underline{a}_\iota)\} = \{t(\underline{b}_\nu)|\ t(v_\nu) \text{ is a term of rank } \leqslant\alpha,\ (b_\nu)\in W^{<\varkappa}$$
$$\text{and } \mathfrak{W}\models \widetilde{\varphi(t(\underline{b}_\nu), \underline{a}_\iota)}\}\,.$$

If $t_1(u_\mu) = \{u^{\beta_1}|\ \varphi_1(u^{\beta_1}, u_\mu)\}$, $t_2(v_\nu) = \{u^{\beta_2}|\ \varphi_2(u^{\beta_2}, v_\nu)\}$, are terms of ranks β_1, β_2 where $\max(\beta_1, \beta_2) = \alpha+1$ for $\alpha>0$, define $\tilde\varphi$ for $\varphi\equiv t_1\in t_2$ as follows:
if $\beta_1<\beta_2 = \alpha+1$, $\tilde\varphi \equiv \widetilde{\varphi_2(t_1(u_\mu), v_\nu)}$;
if $\beta_2\leqslant\beta_1 = \alpha+1$,

$$\tilde\varphi \equiv \bigvee_{d(s_\sigma)\in T^*_{\alpha\varkappa}} (\exists s_\sigma)_{\sigma\in S}\Big(\widetilde{d(s_\sigma)\in t_2(v_\nu)}\ \&\ \bigwedge_{f(r_\varrho)\in T^*_{\alpha\varkappa}} (\forall r_\varrho)\big(\widetilde{f(r_\varrho)\in t_1(u_\mu)} \equiv \widetilde{f(r_\varrho)\in d(s_\sigma)}\big)\Big)$$

where $T^*_{\alpha\varkappa}$ is the set of terms of rank $\leqslant\alpha$.

Notice that $t(\underline{a}_\iota)\in Z(\alpha)\setminus W$ if t has rank $\leqslant\alpha$ and $(a_\iota)\in W^{<\varkappa}$. Also, $\tilde\varphi\in L_{\infty\varkappa}$, but $\varphi\in L^*_{\alpha\varkappa}$ does not imply $\tilde\varphi\in L_{\alpha\varkappa}$.

Finally, if $\varphi\equiv\mathfrak{V}(t_1(u))$ where t_1 has rank $\alpha+1$, define

$$\tilde\varphi \equiv \bigvee_{d(s_\sigma)\in T^*_{\alpha\varkappa}} (\exists s_\sigma)\big(\widetilde{t_1(u) = d(s_\sigma)}\ \&\ \widetilde{\mathfrak{V}(d(s_\sigma))}\big)\,.$$

Lemma 5. *For every $A\in Z(\alpha)\setminus W$ there is a term $\bar A(u_\mu)_{\mu\in M}$ of rank $<\alpha$ and a sequence $\bar s(A) = (a_\mu)_{\mu\in M}\in W^{<\varkappa}$ such that, for all $B\in Z(\alpha)\setminus W$,*

$$(+) \qquad (B, A)\in E(\alpha) \quad \textit{iff} \quad \models^* \bar B(\underline{b}_\nu)_{\nu\in N}\in \bar A(\underline{a}_\mu)_{\mu\in M}\,.$$

For every formula $\varphi(u_\iota)_{\iota\in I}$ of $L_{\infty\varkappa}$ there is a formula $\bar\varphi^\alpha(u^\alpha_\iota)_{\iota\in I}$ of rank α such that, for all $(A_\iota)_{\iota\in I}\in (Z(\alpha)\setminus W)^{<\varkappa}$

$$(++) \qquad (Z(\alpha), E(\alpha))\models\varphi(\underline{A}_\iota)_{\iota\in I} \quad \textit{iff} \quad \models^* \bar\varphi^\alpha(\bar A_\iota(\underline{a}_{\iota\mu})_{\mu\in M_\iota})_{\iota\in I}\,.$$

Proof. By induction on α.

$\alpha = 1$: $A\in Z(1)\setminus W$ implies $A = \{X\in W|\ \mathfrak{W}\models \Phi(\underline{X}, \underline{A}_\iota)_{\iota\in I}\}$ for $\Phi(u, v_\iota)\in L$, $(A_\iota)\in W^{<\varkappa}$. Set $\bar s(A) = (A_\iota)$, $\bar A(v_\iota) = \{u^0|\ \Phi(u^0, v_\iota)\}$; this is a term of rank 0. (+) follows immediately. $\bar\varphi^1$ is defined as in the next case.

$\alpha = \beta+1$: If $A\in Z(\beta)$, $\bar A$ was already defined. If $A\in Z(\alpha)\setminus Z(\beta)$, A as in (1), we have already defined $\bar\Phi(u^\beta, w^\beta_\iota)$, $\bar B_\iota(w_{\iota\mu})_{\mu\in M_\iota}$, $\bar s(B_\iota)\in W^{<\varkappa}$. Set

$$\bar A(w_{\iota\mu})_{\mu\in M_\iota,\ \iota\in I} = \{u^\beta|\ \bar\Phi(u^\beta, \bar B_\iota(w_{\iota\mu})_{\mu\in M_\iota})_{\iota\in I}\}$$

and $\bar s(A) =$ the concatenation of $\bar s(B_\iota)_{\iota\in I}$. The inductive assumption shows immediately that (+) holds for A and for all $B\in Z(\beta)$.

We next define $\bar\varphi^\alpha$ for $\varphi\in L_{\infty\varkappa}$ by induction on logical complexity.

a) $\varphi\equiv u_1\in u_2$, $\bar\varphi^\alpha\equiv u_1^\alpha\in u_2^\alpha$,
b) $\varphi\equiv\mathfrak{V}(u)$, $\bar\varphi^\alpha\equiv\mathfrak{V}(u^\alpha)$,
c) $\overline{\sim\varphi}^\alpha\equiv\sim\bar\varphi^\alpha$, $\overline{\bigwedge_{\iota\in I}\varphi_\iota}^\alpha \equiv \bigwedge_{\iota\in I}\bar\varphi^\alpha_\iota$,
d) $\varphi(u_\iota)\equiv(\exists v_\mu)_{\mu\in M}\Psi(v_\mu, u_\iota)$,

$$\bar\varphi^\alpha(u^\alpha_\iota)\equiv \bigvee_{(t_\mu)_{\mu\in M}\in(T^*_{\alpha\varkappa})^{<\varkappa}} (\exists v_{\mu\nu})_{\nu\in N_\mu,\ \mu\in M}\bar\Psi^\alpha(t_\mu(v_{\mu\nu})_{\nu\in N}, u^\alpha_\iota)\,.$$

(+) for all $A, B\in Z(\alpha)\setminus W$ and (++) follow immediately.

α limit: We have only to define $\bar\varphi^\alpha(u^\alpha_\iota)$; this is done exactly as in the previous case. ■

Lemma 6. *An $L_{\infty\varkappa}$-elementary embedding j_0 of $\mathfrak{W}^1$ into $\mathfrak{W}^2$ can be extended to an $L_{\infty\varkappa}$-elementary embedding j_α of $\mathfrak{Z}^1(\alpha;\varkappa)$ into $\mathfrak{Z}^2(\alpha;\varkappa)$ so that $\alpha<\alpha'$ implies $j_\alpha = j_{\alpha'}\upharpoonright Z(\alpha)$.*

Proof. We will use superscripts 1 and 2 to distinguish concepts over $\mathfrak{Z}^1$ and $\mathfrak{Z}^2$. By induction on α define j_α and prove

(i) for $A\in Z^1(\alpha)\setminus W$, $\bar A^1(u_\mu) = \overline{j_\alpha(A)}^2(u_\mu)$ and $\bar s^2(j_\alpha(A)) = j_0(\bar s^1(A))$;
(ii) $\bar\varphi^{\alpha,1}(u^\alpha_\iota)\equiv\bar\varphi^{\alpha,2}(u^\alpha_\iota)$ for all $\varphi\in L_{\infty\varkappa}$

as follows:

$\alpha = 1$: If $A = \{X \in W^1 |\ \mathfrak{W}^1 \models \Phi(\underline{X}, \underline{a_\iota})\}$, set

$$j_1(A) = \{X \in W^2 |\ \mathfrak{W}^2 \models \Phi(\underline{X}, \underline{j_0(a_\iota)})\}.$$

It is obvious that i) and ii) hold; particularly, note that the formula $\bar{\varphi}^\alpha$ is independent of $\mathfrak{W}$ even in case d) in the proof of the previous lemma.

$\alpha = \beta+1$: If $A \in Z^1(\beta)$, $j_\alpha(A) = j_\beta(A)$. Otherwise,

$$A = \{X \in Z^1(\beta) |\ \mathfrak{Z}^1(\beta) \models \Phi(\underline{X}, \underline{B_\iota})\}.$$

Let

$$j_\alpha(A) = \{X \in Z^2(\beta) |\ \mathfrak{Z}^2(\beta) \models \Phi(\underline{X}, \underline{j_\beta(B_\iota)})\}.$$

i) and ii) follow from the inductive assumptions.

α limit: Set $j_\alpha = \bigcup_{\beta<\alpha} j_\beta$ and check ii).

To prove that j_α is an $L_{\infty\varkappa}$-elementary embedding, we have

$$\mathfrak{Z}^1(\alpha) \models \varphi(\underline{A_\iota}) \quad \text{iff} \quad \models^* \bar{\varphi}^{\alpha,1}(\bar{A}^1_\iota(\underline{a_{\iota\mu}})) \quad \text{iff} \quad \mathfrak{W}^1 \models \overline{\bar{\varphi}^{\alpha,1}(\bar{A}^1_\iota(\underline{a_{\iota\mu}}))} \quad \text{iff} \quad \mathfrak{W}^1 \models \chi^1(\underline{a_{\iota\mu}});$$

$$\mathfrak{Z}^2(\alpha) \models \varphi(\underline{j_\alpha(A_\iota)}) \quad \text{iff} \quad \models^* \bar{\varphi}^{\alpha,2}\overline{(j_\alpha(A_\iota)}^2(\underline{j_0(a_{\iota\mu})})) \quad \text{iff}$$

$$\mathfrak{W}^2 \models \overline{\bar{\varphi}^{\alpha,2}(\overline{j_\alpha(A_\iota)}^2(\underline{j_0(a_{\iota\mu})}))} \quad \text{iff} \quad \mathfrak{W}^2 \models \chi^2(\underline{j_0(a_{\iota\mu})}),$$

for certain $\chi^1, \chi^2 \in L_\infty$.

But $\bar{\varphi}^{\alpha,1} \equiv \bar{\varphi}^{\alpha,2}$ by ii), $\bar{A}^1_\iota \equiv \overline{j_\alpha(A_\iota)}^2$ by i), so $\chi^1 \equiv \chi^2$. Since j_0 is an elementary embedding of $\mathfrak{W}^1$ into $\mathfrak{W}^2$, we have

$$\mathfrak{W}^1 \models \chi^1(\underline{a_{\iota\mu}}) \quad \text{iff} \quad \mathfrak{W}^2 \models \chi^2(\underline{j_0(a_{\iota\mu})}). \quad \blacksquare$$

§ 3. **Proofs of the conservation theorems.** Let $\mathfrak{T}$ be an axiomatic set theory; we can conservatively extend $\mathfrak{T}$ by adding a global form of the Axiom of Choice (see [4] for the proof and other references). The resulting theory $\mathfrak{T}^{\mathfrak{F}}$ has an additional unary function symbol $\mathfrak{F}$; its axioms are those of $\mathfrak{T}$ with the addition of

i) axioms of comprehension and replacement for formulas with $\mathfrak{F}$,

ii) $(\forall x \neq \emptyset)(\mathfrak{F}(x) \in x)$.

The proof can be formulated entirely in terms of forcing, i.e., finitistically.

We will further extend the theory $\mathfrak{T}^{\mathfrak{F}}$ by adding a constant V and the following axioms:

0) $(\forall x \in V)(x \subseteq V)$,

1) $(\forall x \in V)(P(x) \in V)$,

2) Φ^V whenever Φ is an axiom of $\mathfrak{T}^{\mathfrak{F}}$; Φ^V is obtained by restricting all quantifiers in Φ into V,

3) axioms of comprehension and replacement for formulas with V (and $\mathfrak{F}$).

The resulting theory $\mathfrak{T}^{\mathfrak{F},V}$ is a standard device for simulating transitive models of set theory (see e.g. [9], pp. 279–281). It is well known that $\mathfrak{T}^{\mathfrak{F},V}$ is a conservative extension of $\mathfrak{T}^{\mathfrak{F}}$. Moreover, if Φ is a formula of $\mathfrak{T}^{\mathfrak{F}}$, $\mathfrak{T}^{\mathfrak{F}} \vdash \Phi$ iff $\mathfrak{T}^{\mathfrak{F},V} \vdash \Phi^V$. These facts are immediate consequences of the Reflection Principle and their proof is finitistic.

Theorems 1 and 2 will follow if we give an interpretation of $\mathfrak{NS}_1(\mathfrak{T})$ and $\mathfrak{NS}_2(\mathfrak{T})$ in $\mathfrak{T}^{\mathfrak{F},V}$ such that $\mathfrak{S}(x)$ is interpreted as $x \in V$. Indeed; if, say, $\mathfrak{NS}_1(\mathfrak{T}) \vdash \Phi^{\mathfrak{S}}$ for some formula Φ of $\mathfrak{T}$, the interpretation shows $\mathfrak{T}^{\mathfrak{F},V} \vdash \Phi^V$; consequently, $\mathfrak{T}^{\mathfrak{F}} \vdash \Phi$ and $\mathfrak{T} \vdash \Phi$.

All formulas and theories considered in this paragraph will be objects of the metalanguage and will be treated finitistically. If Φ, $A(u_1, \dots, u_n)$, $E(u, v, u_1, \dots, u_n)$ are formulas of $\mathfrak{T}^{\mathfrak{F},V}$, $(A, E) \models \Phi$ will stand for the formula obtained from Φ by replacing each occurrence of $s \in t$ by $E(s, t, u_1, \dots, u_n)$ (s, t are variables or terms) and each occurrence of $(\forall w)$ or $(\exists w)$ by

$$(\forall w)(A(w, u_1, \dots, u_n) \to \dots) \quad \text{or} \quad (\exists w)(A(w, u_1, \dots, u_n) \,\&\, \dots),$$

resp. We find this more convenient than the usual notation $\Phi^{(A,E)}$; it should not lead to confusion as we will never employ the concept of satisfaction for mathematical formulas. However, the statement "j is an elementary embedding of $\mathfrak{M}_1$ into $\mathfrak{M}_2$" has its usual model-theoretic meaning.

From now till the end of the proof of Theorem 2 all mathematical work will be done in $\mathfrak{T}^{\mathfrak{F},V}$. We will first construct a suitably saturated elementary extension of V and interpret internal sets as its elements. Subsequently, we will extend the interpretation to external sets. This is easy in case of $\mathfrak{NS}_2(\mathfrak{T})$, but will require results of § 2 in case of $\mathfrak{NS}_1(\mathfrak{T})$.

Let V_0 = the set of all hereditarily finite sets, $V_{\alpha+1} = P(V_\alpha)$, $V_\alpha = \bigcup_{\delta<\alpha} V_\delta$ for limit α; notice that $V = V_\lambda$ for a limit ordinal λ.

Let $(U_\beta |\ \beta \in \varkappa)$, $\varkappa \in ON$, be a sequence of regular ultrafilters. The iterated ultraproducts of V_α and their canonical elementary embeddings are defined inductively for $\beta \leqslant \varkappa$ and $\alpha < \lambda$:

$$(W^0_\alpha, E^0_\alpha) = (V_\alpha, \in \restriction V_\alpha), \quad j^{0,0}_\alpha = \text{identity},$$

$(W^{\beta+1}_\alpha, E^{\beta+1}_\alpha)$ = the ultraproduct of $(W^\beta_\alpha, E^\beta_\alpha)$ over U_β, $j^{\beta,\beta+1}_\alpha$ is the canonica, elementary embedding of $(W^\beta_\alpha, E^\beta_\alpha)$ into the ultraproduct, $j^{\beta',\beta+1}_\alpha = j^{\beta,\beta+1}_\alpha \circ j^{\beta',\beta}_\alpha$, $(W^\beta_\alpha, E^\beta_\alpha)$ for β limit is the direct limit of $(W^{\beta'}_\alpha, E^{\beta'}_\alpha)$ with resp. to the embeddings $j^{\beta',\beta''}_\alpha$, $j^{\beta',\beta}_\alpha$ are the canonical elementary embeddings from the definition of the direct limit.

For $\alpha < \alpha'$ there is a natural embedding $i^\beta_{\alpha,\alpha'}$ of $(W^\beta_\alpha, E^\beta_\alpha)$ into $(W^\beta_{\alpha'}, E^\beta_{\alpha'})$ as an initial segment so that the following diagram commutes:

$$\text{(D)} \qquad \begin{array}{ccc} (W^\beta_{\alpha'}, E^\beta_{\alpha'}) & \xrightarrow{\ j^{\beta,\beta'}_{\alpha'}\ } & (W^{\beta'}_{\alpha'}, E^{\beta'}_{\alpha'}) \\ {\scriptstyle i^\beta_{\alpha,\alpha'}} \uparrow & & \uparrow {\scriptstyle i^{\beta'}_{\alpha,\alpha'}} \\ (W^\beta_\alpha, E^\beta_\alpha) & \xrightarrow{\ j^{\beta,\beta'}_\alpha\ } & (W^{\beta'}_\alpha, E^{\beta'}_\alpha) \end{array}$$

We will identify x and $i^\beta_{\alpha,\alpha'}(x)$ when no confusion is possible. Let

$$W^\beta = \bigcup_{\alpha<\lambda} W^\beta_\alpha, \quad E^\beta = \bigcup_{\alpha<\lambda} E^\beta_\alpha, \quad i^\beta_\alpha = \bigcup_{\alpha'<\lambda} i^\beta_{\alpha,\alpha'}, \quad j^{\beta,\beta'} = \lim_{\alpha<\lambda} j^{\beta,\beta'}_\alpha$$

in the obvious sense. Finally, let $W = W^\varkappa$, $E = E^\varkappa$, $i_\alpha = i^\varkappa_\alpha$, $j^\beta = j^{\beta,\varkappa}$. j^β embeds W^β into W, but it is not immediately obvious that j^β is an elementary embedding.

LEMMA 1. *Let Φ be a formula of $\mathfrak{T}$. Then*

$$\mathfrak{T}^{\mathfrak{F},V} \vdash (\forall\beta)(\forall\vec{x}\in W^\beta)[(W^\beta, E^\beta) \models \Phi(\vec{x}) \equiv (W, E)\models \Phi(\vec{x})].$$

In particular,

$$\mathfrak{T}^{\mathfrak{F},V} \vdash (\forall\vec{x}\in V)[(V, \in\restriction V)\models \Phi(\vec{x}) \equiv (W, E)\models \Phi(j^0(\vec{x}))]$$

$(\vec{x} = (x_1, \ldots, x_n)$, $j(\vec{x}) = (j(x_1), \ldots, j(x_n)))$.

The lemma follows from

i) $j^{\beta,\beta'}_\alpha$ *is an elementary embedding of* $(W^\beta_\alpha, E^\beta_\alpha)$ *into* $(W^{\beta'}_\alpha, E^{\beta'}_\alpha)$ *for* $\beta<\beta'<\varkappa$.

ii) (*Reflection Principle*): *If* $(\forall\vec{x}\in V_\alpha)[(V_\alpha, \in\restriction V_\alpha)\models\Phi(\vec{x}) \equiv (V, \in\restriction V)\models\Phi(\vec{x})]$ *then* $(\forall\vec{x}\in W^\beta_\alpha)[(W^\beta_\alpha, E^\beta_\alpha)\models\Phi(\vec{x}) \equiv (W^\beta, E^\beta)\models\Phi(\vec{x})]$ *for all* $\beta<\varkappa$ *and all* $\alpha<\lambda$.

Proof of ii) is by induction on complexity of Φ. In the only nontrivial case $\Phi(\vec{x}) \equiv (\exists y)\Psi(\vec{x}, y)$ proceed by induction on β. If $x\in W^{\beta+1}_\alpha$ and $(W^{\beta+1}, E^{\beta+1}) \models (\exists y)\Psi(\vec{x}, y)$, pick $\alpha'>\alpha$ s.t. α' satisfies ii) for Ψ and $(\exists y\in W^{\beta+1}_{\alpha'})(W^{\beta+1}, E^{\beta+1}) \models \Psi(\vec{x}, y)$; i.e., $(W^{\beta+1}_{\alpha'}, E^{\beta+1}_{\alpha'})\models(\exists y)\Psi(\vec{x}, y)$. If $f_x\colon \bigcup U_\beta\to W^\beta_\alpha$ is such that x is the equivalence class of f_x in the ultraproduct over U_β, we have

$$\{\iota\in\bigcup U_\beta \mid (W^\beta_{\alpha'}, E^\beta_{\alpha'})\models(\exists y)\,\Psi(\overrightarrow{f_x(\iota)}, y)\}\in U_\beta.$$

The inductive assumptions imply that α' satisfies ii) for Ψ and α satisfies ii) for Φ and β, so $(W^\beta_\alpha, E^\beta_\alpha)\models(\exists y)\Psi(\overrightarrow{f_x(\iota)}, y)$. Thus

$$\{\iota\in\bigcup U_\beta \mid (W^\beta_\alpha, E^\beta_\alpha)\models(\exists y)\Psi(\overrightarrow{f_x(\iota)}, y)\}\in U_\beta;$$

using AC, pick $g\colon \bigcup U_\beta\to W^\beta_\alpha$ such that

$$\{\iota\in\bigcup U_\beta \mid (W^\beta_\alpha, E^\beta_\alpha)\models\Psi(\overrightarrow{f_x(\iota)}, g(\iota))\}\in U_\beta;$$

and let y be the equivalence class of g in $W^{\beta+1}_\alpha$; then

$$(W^{\beta+1}_\alpha, E^{\beta+1}_\alpha)\models\Psi(x, y). \ ■$$

If one interprets $\mathfrak{S}(x)$ as $x\in j^0[V]$, $\mathfrak{I}(x)$ as $x\in W$, $x\in y$ as $(x, y)\in E$, the axioms (A), (B1) and (B3) immediately follow. We now proceed to the heart of the matter and extend the interpretation to external sets so that the remaining axioms hold. This is done differently for $\mathfrak{NS}_1$ and $\mathfrak{NS}_2$.

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Proof of Theorem 1. Choose $\varkappa = \lambda$ and let U_β be the regular ultrafilter on $\mathfrak{b}_\beta = |V_\beta|$ chosen with the help of the universal choice function $\mathfrak{F}$. We do this in order to ensure that W, E and all other concepts related to the interpretation are definable over $(V, \in\restriction V, \mathfrak{F}\restriction V)$. Let $\mathfrak{W}^\beta_\alpha = (W^\beta_\alpha, E^\beta_\alpha, V_\alpha)$ etc. Define $\mathfrak{Z}^\beta_\alpha(\xi; \mathfrak{b}^+_\alpha)$ for $\xi<\lambda$ using methods of § 2. Let

(*) $$B_\alpha = \{\beta<\lambda \mid \mathrm{cf}(\beta)>\mathfrak{b}_\alpha \ \&\ (\forall\xi<\beta)(\mathfrak{b}_\xi<\beta)\}.$$

Well-known properties of ultraproducts over regular ultrafilters and of their limits then imply:

a) $(W^\beta_\alpha, E^\beta_\alpha)$ is β-saturated, in part., $\mathfrak{b}^+_\alpha$-saturated, for $\beta\in B_\alpha$ (see [0]).

b) Consequently, if $\beta, \beta'\in B_\alpha$, $\beta<\beta'$, $j^{\beta,\beta'}_\alpha$ is an $L_{\infty\mathfrak{b}^+_\alpha}$-elementary embedding of $(W^\beta_\alpha, E^\beta_\alpha)$ into $(W^{\beta'}_\alpha, E^{\beta'}_\alpha)$ (see [2]).

c) V_α is definable over $(W^\beta_\alpha, E^\beta_\alpha)$ by a formula of $L_{\infty\mathfrak{b}^+_\alpha}$:

$$V_\alpha = \{x\in W^\beta_\alpha \mid (W^\beta_\alpha, E^\beta_\alpha)\models \bigvee_{a\in V_\alpha}\!\!\bigvee (\underset{\sim}{x} = \underset{\sim}{a})\}$$

and $j^{\beta,\beta'}_\alpha\restriction V_\alpha$ = identity. Therefore, for $\beta, \beta'\in B_\alpha$, $\beta<\beta'$, $j^{\beta,\beta'}_\alpha$ is an $L_{\infty\mathfrak{b}^+_\alpha}$-elementary embedding of $\mathfrak{W}^\beta_\alpha = (W^\beta_\alpha, E^\beta_\alpha, V_\alpha)$ into $\mathfrak{W}^{\beta'}_\alpha = (W^{\beta'}_\alpha, E^{\beta'}_\alpha, V_\alpha)$.

Lemma 6 § 2 shows that $j^{\beta,\beta'}_\alpha$ can be canonically extended to an $L_{\infty\mathfrak{b}^+_\alpha}$-elementary embedding $j^{\beta,\beta'}_\alpha(\xi)$ of $\mathfrak{Z}^\beta_\alpha(\xi; \mathfrak{b}^+_\alpha)$ into $\mathfrak{Z}^{\beta'}_\alpha(\xi; \mathfrak{b}^+_\alpha)$.

If $\alpha<\alpha'<\lambda$, $i^\beta_{\alpha,\alpha'}$ embeds $\mathfrak{W}^\beta_\alpha$ into $\mathfrak{W}^\beta_{\alpha'}$ as an initial segment. By Lemma 4 of § 2, there is an embedding $i^\beta_{\alpha,\alpha'}(\xi)$ of $\mathfrak{Z}^\beta_\alpha(\xi; \mathfrak{b}^+_\alpha)$ into $\mathfrak{Z}^\beta_{\alpha'}(\xi; \mathfrak{b}^+_{\alpha'})$ as an initial segment (for all $\xi\leqslant\mathfrak{b}^+_{\alpha'}$). The diagram (D) can now be extended to

(D+) $$\begin{array}{ccc} \mathfrak{Z}^\beta_{\alpha'}(\xi; \mathfrak{b}_{\alpha'}) & \xrightarrow{\ j^{\beta,\beta'}_{\alpha'}(\xi)\ } & \mathfrak{Z}^{\beta'}_{\alpha'}(\xi; \mathfrak{b}_{\alpha'}) \\ \uparrow{\scriptstyle i^\beta_{\alpha,\alpha'}(\xi)} & & \uparrow{\scriptstyle i^{\beta'}_{\alpha,\alpha'}(\xi)} \\ \mathfrak{Z}^\beta_\alpha(\xi; \mathfrak{b}_\alpha) & \xrightarrow[\ j^{\beta,\beta'}_\alpha(\xi)\ ]{} & \mathfrak{Z}^{\beta'}_\alpha(\xi; \mathfrak{b}_\alpha) \end{array}$$

for $\alpha<\alpha'<\lambda$ $(\xi\leqslant\mathfrak{b}^+_{\alpha'})$.

We let the interpretation for external sets to be the direct limit of this system of structures. More precisely, let

$$\mathfrak{Z}^\beta_\alpha = \mathfrak{Z}^\beta_\alpha(\mathfrak{b}^+_\alpha; \mathfrak{b}^+_\alpha), \quad i^\beta_{\alpha,\alpha'} = i^\beta_{\alpha,\alpha'}(\mathfrak{b}^+_\alpha), \quad j^{\beta,\beta'}_\alpha = j^{\beta,\beta'}_\alpha(\mathfrak{b}^+_\alpha),$$

$\mathfrak{Z}_\alpha$ = the limit of $(\mathfrak{Z}^\beta_\alpha, \beta\in B_\alpha)$ with resp. to $j^{\beta,\beta'}_\alpha$,

j^β_α = the canonical embedding of $\mathfrak{Z}^\beta_\alpha$ into $\mathfrak{Z}_\alpha$.

There is no conflict with the previous notation because $i^\beta_{\alpha,\alpha'}\restriction W^\beta_\alpha$, $j^{\beta,\beta'}_\alpha\restriction W^\beta_\alpha$ are the embeddings denoted $i^\beta_{\alpha,\alpha'}$, $j^{\beta,\beta'}_\alpha$ resp. in (D). Notice also that $j^\beta_\alpha\restriction W^\beta_\alpha = j^{\beta,\varkappa}_\alpha$ as defined in (D).

For $\alpha<\alpha'$ we construct an embedding $i_{\alpha,\alpha'}$ of $\mathfrak{Z}_\alpha$ into $\mathfrak{Z}_{\alpha'}$ as an initial segment by taking the limit of $(i^\beta_{\alpha,\alpha'}, \beta\in B_{\alpha'})$; $i_{\alpha,\alpha'}\restriction W^\lambda_\alpha = i^\lambda_{\alpha,\alpha'}$ as defined in (D). The limit of $\mathfrak{Z}_\alpha$ with resp. to $i_{\alpha,\alpha'}$ will be denoted $\mathfrak{Z}(\infty)$ and the corresponding embeddings i_α. Then i_α embeds $\mathfrak{Z}_\alpha$ into $\mathfrak{Z}(\infty)$ as an initial segment. Finally, $i = \bigcup_{\alpha<\lambda} i_\alpha \restriction W_\alpha^\lambda$ embeds $\mathfrak{W}$ into $\mathfrak{Z}$. We will identify $\mathfrak{W}$ with $i(\mathfrak{W})$, $\mathfrak{Z}_\alpha$ with $i_\alpha(\mathfrak{Z}_\alpha)$ and $\mathfrak{Z}_\alpha^\beta$ with both $j_\alpha^\beta(\mathfrak{Z}_\alpha^\beta)$ and $i_\alpha(j_\alpha^\beta(\mathfrak{Z}_\alpha^\beta))$ if it cannot lead to misunderstanding.

Proof of the remaining axioms:

(B1) is obvious. ■

(B4$^+$): Let φ be a formula of $\mathfrak{T}$.

(1) If $\vec{\eta} \in W_\alpha^\alpha$, $B \subseteq W_\alpha^\alpha$ and for every finite $b \subseteq B$

$$(\exists \beta \in W)(\forall x \in b)\mathfrak{W} \models \varphi(x, \beta, \vec{\eta}) \quad \text{then} \quad (\exists \beta \in W)(\forall x \in B)\mathfrak{W} \models \varphi(x, \beta, \vec{\eta}).$$

Proof. Pick $\delta > \alpha$ s.t. i) W_δ^δ is $|B|^+$-saturated,

ii) $(\forall x, \beta, \vec{\eta} \in V_\delta)[(V, \varepsilon) \models \varphi(x, \beta, \vec{\eta}) \equiv (V_\delta, \varepsilon) \models \varphi(x, \beta, \vec{\eta})]$,

iii) for every finite $b \subseteq B$ $(\exists \beta \in W_\delta^\delta)(\forall x \in b)\mathfrak{W} \models \varphi(x, \beta, \vec{\eta})$.

By Lemma 1 and ii) following it, $(\forall x \in b)\mathfrak{W}_\delta^\delta \models \varphi(x, \beta, \vec{\eta})$. i) now implies the existence of $\beta \in W_\delta^\delta$ s.t. $(\forall x \in B)\mathfrak{W}_\delta^\delta \models \varphi(x, \beta, \vec{\eta})$. Using Lemma 1 again, we get $(\forall x \in B)\mathfrak{W} \models \varphi(x, \beta, \vec{\eta})$ for this β. ■

(2) If $\mathfrak{Z} \models \mathfrak{S}\mathfrak{S}(A)$ then there is $\alpha < \lambda$, $B \subseteq Z_\alpha^\alpha$ s.t.

$$\mathfrak{Z} \models x \in A \equiv (\exists y \in B)\mathfrak{Z} \models (x = y \;\&\; y \in A).$$

Proof. We need the fact that the construction of $\mathfrak{Z}$ is definable over $(V, \varepsilon, \mathfrak{F})$. In part., there is a formula χ such that

$$(V, \in, \mathfrak{F}) \models \chi(x, f, \gamma) \text{ iff } \mathfrak{Z} \models f \text{ is a function } \& \; f(x) \in Z_\gamma^\gamma.$$

Let A, f be s.t.

$$\mathfrak{Z} \models \mathfrak{S}(A) \;\&\; (\forall x)[x \in A \equiv (\exists x \in A)(\mathfrak{S}(x) \;\&\; f(x) = x)].$$

Then $(V, \in, \mathfrak{F}) \models (\forall x \in A)(\exists \gamma)\chi(x, f, \gamma)$. The Axiom of Replacement holds in $(V, \in, \mathfrak{F})$ (see 2) in the formulation of $\mathfrak{T}^{\mathfrak{F},V}$), so there is $\alpha < \lambda$ s.t.

$$(V, \in, \mathfrak{F}) \models (\forall x \in A)(\exists \gamma < \alpha)\chi(x, f, \gamma).$$

Let $B = \{x \in Z_\alpha^\alpha |\; \mathfrak{Z} \models x \in A\}$.

(3) (B4$^+$) now follows from (1), (2) and the obvious "absoluteness" of finiteness: $\mathfrak{Z} \models \boldsymbol{a}$ is finite iff there is a finite set a s.t.

$$\mathfrak{Z} \models x \in \boldsymbol{a} \equiv (\exists y \in a)\mathfrak{Z} \models (x = y \;\&\; y \in \boldsymbol{a}). \;■$$

(C0): if $A \in Z$, $A \in Z_\alpha^\alpha$ for some $\alpha < \lambda$; let $A = \{x \in V_{\alpha+\alpha} |\; \mathfrak{Z} \models x \in A\}$. ■

(C1), (C2), (C3) and (C4) for bound formulas hold in each $\mathfrak{Z}_\alpha^\beta$ by Lemma 1 of § 2, and they are existential formulas. Since $\mathfrak{Z}_\alpha^\beta$ is embedded into $\mathfrak{Z}$ as an initial segment and existential formulas are preserved by such embeddings, they hold in $\mathfrak{Z}$.

The full (C4) and (ARE) will follow immediately from the Reflection Principle stated in Lemma 2 (notice $\breve{Z}_\tau^\sigma \in Z$):

Let $\tau < \lambda$ be limit. $\breve{\mathfrak{Z}}_\tau^\beta$ will denote the limit of $(\mathfrak{Z}_\alpha^\beta |\; \alpha < \tau)$ with resp. to the embeddings $i_{\alpha,\alpha'}^\beta$. Similarly, $\breve{\mathfrak{Z}}_\tau$ will be the limit of $(\mathfrak{Z}_\alpha |\; \alpha < \tau)$ with resp. to $i_{\alpha,\alpha'}$. In the usual identification,

$$\breve{\mathfrak{Z}}_\tau^\beta = \bigcup_{\alpha<\tau} \mathfrak{Z}_\alpha^\beta, \qquad \breve{\mathfrak{Z}}_\tau = \bigcup_{\alpha<\tau} \mathfrak{Z}_\alpha.$$

Lemma 2. *Let Φ be a formula of $\mathfrak{N}\mathfrak{S}(\mathfrak{T})$. There is a closed unbounded set $C_\Phi \subseteq \lambda$ such that for all $\tau \in C_\Phi$, $\sigma \in B_\tau$ (see (*)), $X \in \breve{Z}_\tau^\sigma$*

$$(+) \qquad \breve{\mathfrak{Z}}_\tau^\sigma \models \Phi(X) \equiv \breve{\mathfrak{Z}}_\tau \models \Phi(X) \equiv \mathfrak{Z} \models \Phi(X).$$

Proof. By induction on Φ. If Φ is bound, let $C_\Phi = \{\tau < \lambda |\; \tau \text{ limit}\}$. If $X \in \breve{Z}_\tau^\sigma$, $X \in Z_\alpha^\sigma$ for some $\alpha < \tau$. $\mathfrak{Z}_\alpha^\sigma$ is elementarily embedded into $\mathfrak{Z}_\alpha$ and $\mathfrak{Z}_\alpha^\sigma$, $\mathfrak{Z}_\alpha$ are initial segments of $\breve{\mathfrak{Z}}_\tau^\sigma$, $\breve{\mathfrak{Z}}_\tau$ resp., so

$$\breve{\mathfrak{Z}}_\tau^\sigma \models \Phi(X) \quad \text{iff } \mathfrak{Z}_\alpha^\sigma \models \Phi(X) \quad \text{iff } \mathfrak{Z}_\alpha \models \Phi(X) \quad \text{iff } \breve{\mathfrak{Z}}_\tau \models \Phi(X).$$

Also, $\breve{\mathfrak{Z}}_\tau \models \Phi(X)$ iff $\mathfrak{Z} \models \Phi(X)$ ($\breve{\mathfrak{Z}}_\tau$ is an initial segment of $\mathfrak{Z}$).

The remaining nontrivial case is $\Phi(X) \equiv (\exists Y)\Psi(X, Y)$.

The rest of the proof is done inside $(V, \in, \mathfrak{F})$; this is possible because the entire construction of $\breve{Z}_\tau^\sigma$, $\breve{Z}_\tau$, Z and C_χ for subformulas χ of Φ is definable in $(V, \in, \mathfrak{F})$.

Define: $\tau \in C_\Phi$ iff $\tau \in C_\Psi$, $(\forall \alpha < \tau)(\mathfrak{b}_\alpha < \tau)$ and

(+ +) for all $X \in \breve{Z}_\tau^\tau$, if $\mathfrak{Z} \models (\exists Y)\Psi(X, Y)$ then there are $\alpha, \beta < \tau$, $\alpha \in C_\Psi$, $\beta \in B_\alpha$, and $Y \in \breve{Z}_\alpha^\beta$ s.t. $\breve{\mathfrak{Z}}_\alpha^\beta \models \Psi(X, Y)$.

It is clear that C_Φ is a nonempty closed unbounded subset of λ definable in $(V, \in, \mathfrak{F})$. Let $\tau \in C_\Phi$; the only nontrivial step in the proof of (+) is to show that, for fixed $\sigma \in B_\tau$, $X \in Z_\tau^\sigma$,

$$\mathfrak{Z} \models (\exists Y)\Psi(X, Y) \rightarrow \breve{\mathfrak{Z}}_\tau^\sigma \models (\exists Y)\Psi(X, Y).$$

Assume $\mathfrak{Z} \models (\exists Y)\Psi(X, Y)$; pick $\gamma, \delta > \max(\sigma, \tau)$, $\gamma \in C_\Psi$, $\delta \in B_\gamma$ s.t. $\mathfrak{Z} \models \Psi(X, Y)$ for some $Y \in \breve{Z}_\gamma^\delta$; then also $\breve{\mathfrak{Z}}_\gamma^\delta \models \Psi(X, Y)$.

Claim 1. *There is an automorphism k^* of $\breve{\mathfrak{Z}}_\gamma^\delta$ s.t. $k^*(X) \in \breve{Z}_\tau^\tau$.*

Claim 1 implies $\breve{\mathfrak{Z}}_\gamma^\delta \models \Psi(k^*(X), k^*(Y))$, i.e., $\mathfrak{Z} \models (\exists Y)\Psi(k^*(X), Y)$. From (+ +) we get $\alpha, \beta < \tau$, $\alpha \in C_\Psi$, $\beta \in B_\alpha$, $Y^{**} \in \breve{Z}_\alpha^\beta$ s.t. $\breve{\mathfrak{Z}}_\alpha^\beta \models \Psi(k^*(X), Y^{**})$. As $\alpha, \tau \in C_\Psi$, we conclude $\mathfrak{Z} \models \Psi(k^*(X), Y^{**})$, $\breve{\mathfrak{Z}}_\tau^\sigma \models \Psi(k^*(X), Y^{**})$ and finally $\breve{\mathfrak{Z}}_\tau^\sigma \models (\exists Y)\Psi(k^*(X), Y)$.

Claim 2. *There is an automorphism l^* of $\breve{\mathfrak{Z}}_\tau^\sigma$ s.t. $l^*(k^*(X)) = X$.*

Claim 2 implies $\breve{\mathfrak{Z}}_\tau^\sigma \models (\exists Y)\Psi(X, Y)$ and completes the proof of Lemma 2. ■

Proof of Claim 1. Pick $\pi < \tau$ s.t. $X \in Z_\pi^\sigma$. Let $\bar{s}(X) \in (W_\pi^\sigma)^{<\mathfrak{b}_\pi^+}$ be the sequence assigned to X by Lemma 5 § 2 and let $(\tilde{W}, \tilde{E})$ be $\mathfrak{b}_\pi^+$-saturated elementary submodel of $(W_\pi^\sigma, E_\pi^\sigma)$ s.t. $\bar{s}(X) \subseteq \tilde{W}$, $V_\alpha \subseteq \tilde{W}$, $|\tilde{W}| = \mathfrak{b}_{\pi+1}$. Notice that $\varrho = \mathfrak{b}_{\pi+1}^+ < \tau$, $(W_\pi^\varrho, E_\pi^\varrho)$ is a ϱ-saturated model elementarily equivalent to $(\tilde{W}, \tilde{E})$. Thus there is an elementary embedding k of $(\tilde{W}, \tilde{E})$ ~~into $(W_\pi^\varrho$~~, $E_\pi^\varrho)$ s.t. $k \restriction V_\pi =$ identity.

Let $(\varrho_\iota, \iota<\mathrm{cf}(\delta))$ be an increasing sequence of regular cardinals $\varrho_0 = \varrho$, $\varrho_1 > \mathfrak{b}_\gamma$, $\varrho_{\iota+1} > \mathfrak{b}_{\varrho_\iota}$, $\lim \varrho_\iota = \delta$. $(W_\gamma^{\varrho_1}, E_\gamma^{\varrho_1})$ is ϱ_1-saturated, $\varrho_1 > |W_\gamma^{\varrho_0}|$. We can thus find an elementary embedding k_0 of $(W_\gamma^{\varrho_0}, E_\gamma^{\varrho_0})$ into $(W_\gamma^{\varrho_1}, E_\gamma^{\varrho_1})$ s.t. $k_0 \restriction V_\gamma = \text{identity}$, $k^{-1} \subseteq k_0$. Notice that for $\alpha<\gamma$, $X \in W_\alpha^{\varrho_0}$ iff $(W_\gamma^{\varrho_0}, E_\gamma^{\varrho_0}) \models x \in V_\alpha$ iff $(W_\gamma^{\varrho_1}, E_\gamma^{\varrho_1}) \models k_0(X) \in k_0(V_\alpha)$ iff $(W_\gamma^{\varrho_1}, E_\gamma^{\varrho_1}) \models k_0(X) \in V_\alpha$ iff $k_0(X) \in W_\alpha^{\varrho_1}$.

The usual zig-zag iteration of this construction by induction on ι will produce a sequence of elementary embeddings k_ι s.t. $\bar{k} = \bigcup_{\iota<\mathrm{cf}(\delta)} k_\iota$ is an automorphism of $(W_\gamma^\delta, E_\gamma^\delta)$, $\bar{k} \restriction V_\gamma = \text{identity}$, $k^{-1} \subseteq \bar{k}$ and for all $\alpha<\gamma$, $X \in W_\alpha^\delta$ iff $\bar{k}(X) \in W_\alpha^\delta$.

Lemma 6 § 2 provides a canonical extension of $\bar{k}$ to an automorphism k^* of $\tilde{\mathfrak{Z}}_\gamma^\delta$ s.t. for all $\alpha<\gamma$, $X \in Z_\alpha^\delta$ iff $k^*(X) \in Z_\alpha^\delta$. Thus $k^* \restriction Z_\gamma^\delta = \bigcup_{\alpha<\gamma} Z_\alpha^\delta$ is an automorphism of $\tilde{\mathfrak{Z}}_\gamma^\delta$. The claim i) in the proof of Lemma 6 shows that $\bar{s}(k^*(X)) = k(\bar{s}(X)) \subseteq W_\pi^\varrho$, so $k^*(X) \in Z_\pi^\varrho \subseteq \tilde{Z}_\pi^\tau$. ■

Proof of Claim 2. We have $\bar{s}(k^*(X)) = k(\bar{s}(X))$; let l be an elementary embedding of $\mathfrak{W}_\pi^\varrho$ into $\mathfrak{W}_\pi^\sigma$ s.t. $l \circ f = \text{identity}$, $l \restriction V_\pi = \text{identity}$. The procedure used in the proof of Claim 1 will again extend l to an automorphism l^* of $\tilde{\mathfrak{Z}}_{\tau_1}^\sigma$. We have $\bar{s}(l^*(k^*(X))) = l(k(\bar{s}(X))) = \bar{s}(X)$, so $l^*(k^*(X)) = X$. ■

Proof of Theorem 2. This is much easier. We choose $\varkappa = \lambda^+$ $(= \mathfrak{b}_\lambda^+)$ where $V = V_\lambda$, and let U_β be a regular ultrafilter on $\mathfrak{b}_\beta$. Let $\mathfrak{W}_\alpha = \mathfrak{W}_\alpha^\varkappa$ and define $\mathfrak{Z}_\alpha(\xi) = (Z_\alpha(\xi), E_\alpha(\xi), V_\alpha)$ for $\xi<\lambda$ inductively:

if $\xi = 0$, $Z_\alpha(\xi) = W_\alpha$, $E_\alpha(\xi) = E_\alpha$,

if ξ is limit, $Z_\alpha(\xi) = \bigcup_{\eta<\xi} Z_\alpha(\eta)$, $E_\alpha(\xi) = \bigcup_{\eta<\xi} E_\alpha(\eta)$,

if $\xi = \eta+1$, $A \in Z_\alpha(\xi) \equiv A \in Z_\alpha(\eta) \vee A \subseteq Z_\alpha(\eta)$.

$E_\alpha(\xi)$ and $I_\alpha(\xi)$ are defined inductively:

a) $E_\alpha(\eta) \subseteq E_\alpha(\xi)$,

b) if $X \in E_\alpha(\eta)$, $A \subseteq E_\alpha(\eta)$ and $X \in A$ then $(X, A) \in E_\alpha(\xi)$,

c) if $A_1, A_2 \in E_\alpha(\xi)$ and, for all $X \in E_\alpha(\eta)$, $(X, A_1) \in E_\alpha(\xi) \equiv (X, A_2) \in E_\alpha(\xi)$ then $(A_1, A_2) \in I_\alpha(\xi)$,

d) if $A_1, A_2 \in Z_\alpha(\xi)$ and for some $X \in Z_\alpha(\eta)$ $(A_1, X) \in I_\alpha(\xi)$ & $(X, A_2) \in E_\alpha(\xi)$ then $(A_1, A_2) \in E_\alpha(\xi)$.

Let $Z_\alpha = Z_\alpha(\lambda)$, $Z = \bigcup_{\alpha<\lambda} Z_\alpha$, $E = \bigcup_{\alpha<\lambda} E_\alpha$, $W = \bigcup_{\alpha<\lambda} W_\alpha$, $\mathfrak{Z} = (Z, E, W, V)$. We have to show that $\mathfrak{Z}$ satisfies (B2), (B4$^+$), (C0)-(C4), (APE) and (ACE).

(B2) and (C0) are as in the proof of Theorem 1.

(B4$^+$): Let $A \in Z_\alpha(\xi)$, $\mathfrak{Z} \models \mathfrak{S}\mathfrak{S}(A)$; there is $A \subseteq Z_\alpha(\xi)$ s.t. $|A|<\lambda$ and

$$\mathfrak{Z} \models X \in A \quad \text{iff} \quad (\exists Y \in A)\mathfrak{Z} \models (X = Y \;\&\; Y \in A).$$

As $\mathrm{cf}(\varkappa) > \lambda \geq |A|$, there is $\beta<\varkappa$ s.t. $A \cap W_\alpha^\beta = A \cap W$. The result then follows from $|A \cap W_\alpha^\beta|^+$-saturatedness of $\mathfrak{W}$ and "absoluteness" of the concept of finiteness.

The remaining axioms follow trivially from the fact that for every $B \subseteq Z_\alpha(\xi)$ there is $A \in Z_\alpha(\xi+1)$ s.t.

$$\mathfrak{Z} \models X \in A \quad \text{iff} \quad (\exists Y \in B)\mathfrak{Z} \models X = Y. \;■$$

Proof of Theorem 4b). We will work in ZFC+ "λ is strongly inaccessible" $+ V_\lambda \models \mathfrak{T}$. Set $\varkappa = \lambda$, choose regular ultrafilters U_β on $\mathfrak{b}_\beta$ and construct $\mathfrak{W}_\alpha^\beta$, $\alpha<\lambda$, $\beta \leq \varkappa$ as before; let $\mathfrak{W}_\alpha = \mathfrak{W}_\alpha^\varkappa$. Construct $\mathfrak{Z}_\alpha(\xi)$, $\mathfrak{Z}_\alpha$, $\mathfrak{Z}$ for $\xi<\lambda$ as in the proof of Theorem 2.

The proof of validity of the axioms (B2), (C0)-(C4), (APE) and (ACE) in $\mathfrak{Z}$ is the same as for Theorem 2. The only change needed in the proof of (B4$^+$) is the observation that $|A|<\lambda \to |A|<\mathrm{cf}(\varkappa)$ follows from the inaccessibility of $\varkappa = \lambda$.

Finally, to prove ($\mathfrak{S}\mathfrak{S}$-ARE), let $\mathfrak{Z} \models \mathfrak{S}\mathfrak{S}(A)$ & $(\forall X \in A)(\exists Y)\varphi(X, Y)$. Again there is $A \subseteq Z_\alpha(\xi)$, $|A|<\lambda$, s.t. $\mathfrak{Z} \models X \in A$ iff $(\exists Y \in A)\mathfrak{Z} \models (X = Y \;\&\; Y \in A)$. As $|A|<\lambda$, λ inaccessible, there is $\zeta<\lambda$ s.t. $(\forall X \in A)(\exists Y \in Z_\zeta(\xi))\mathfrak{Z} \models \varphi(X, Y)$. Let $B = Z_\zeta(\xi)$; then obviously $\mathfrak{Z} \models (\forall X \in A)[(\exists Y)\varphi(X, Y) \to (\exists Y \in B)\varphi(X, Y)]$. ■

Note (added August 1, 1977). Some of the recent constructions in Nonstandard Analysis can be better formalized in $\mathfrak{NS}_2$ or $\mathfrak{NS}_3$ than in $\mathfrak{NS}_1$. This is true e.g. for the Loeb–Anderson approach to measure theory (which turns out to be closely connected with the external Dedekind cuts in the ordering of internal real numbers mentioned in the remark following Theorem 4). An expository paper dealing with some of these topics from the point of view of $\mathfrak{NS}_2$ (but not requiring any knowledge of logic) is in preparation. On the other hand, we feel that $\mathfrak{NS}_1$ might show to be more suitable for applications of nonstandard methods to the study of set theory.

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