Complements of solenoids in S^3 are m-spaces *

by

G. J. Michaelides (Tampa, Fla.)

Abstract. An m-space is a pair (X, m), where X is a topological space and m, called an *n-mean on X*, is a map from X^n (the n-fold cartesian product of X, $n>1$) to X satisfying the two conditions:

1. $m(x, x, \ldots, x) = x$ for every $x \in X$,

2. $m(x_1, x_2, \ldots, x_n) = m(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for every n-tuple $(x_1, x_2, \ldots, x_n) \in X^n$, and every permutation $\sigma \in S_n$, where S_n is the symmetric group of n elements.

It is known that compact finite polyhedra are m-spaces if and only if they are contractible. In this paper we examine the existence of a *mean* on a class of non-compact, infinite polyhedra, namely the complements of solenoids Σ_p in S^3, and we show that they are indeed m-spaces. As a corollary to this and from some properties of the homotopy groups of m-spaces we deduce that the fundamental group of $S^3 - \Sigma_p$ is isomorphic to the p-adic rationals.

An n-mean on a space X is a continuous function (map) m from X^n, the n-fold cartesian product of X, to X satisfying the following two conditions:

1. $m(x, \ldots, x) = x$ for every $x \in X$,

2. $m(x_1, \ldots, x_n) = m(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for every n-tuple $(x_1, \ldots, x_n) \in X^n$, and every permutation $\sigma \in S_n$, where S_n denotes the symmetric group of n elements.

A space that admits an n-mean will be called an *m-space*. We shall assume that $n \geqslant 2$ since, for $n = 1$, every space becomes an m-space with m being the identity map.

Examples of means are the arithmetic n-mean on the real line and the geometric n-mean on the non-negative real line. Investigation of the existence of n-means on topological spaces and some consequences was done in [3], [6], [7]. For example in [7] it was shown that compact connected finite polyhedra admit a mean if and only if they are contractible. In this paper we examine the existence of a mean on some non-compact polyhedra (infinite), namely, the complements of p-adic solenoids, Σ_p, in S^3 and show that these are m-spaces. Since Σ_p is itself an m-space [3] one may ask whether complements of compact m-spaces in S^n (the n-dimensional sphere) are necessarily m-spaces. That the existence of a mean on a compact space is neither necessary nor sufficient condition for the existence of a mean on its complement can be seen from the following examples.

---

* This is a part of the author's dissertation written at the University of Georgia under the direction of R. B. Sher.

1. Let K be a Fox–Artin arc ([1] Ex. 1.1) in S^3 whose complement S^3-K is not simply connected. K is homeomorphic to I, hence an m-space, but S^3-K does not admit a mean since its fundamental group, $\pi(S^3-K)$ is not abelian, a necessary condition for the existence of a mean [6].

2. Let $Y=\{(x,y)\colon 0<x\leqslant 1,\ y=\sin(1/x)\}\cup\{(0,y)\colon -1\leqslant y\leqslant 1\}$ be a subset of S^2. Then Y does not admit a mean [4], but S^2-Y does because Y is cellular and S^2-Y is homeomorphic to S^2-p, p being a point of S^2, [5], which in turn is homeomorphic to R^2 (the Euclidean plane).

From the last example we can make the general observation that if K is cellular in S^n, then S^n-K is an m-space since S^n-K is homeomorphic to R^n.

A p-adic solenoid, Σ_p, where p is a positive integer greater than, or equal to 2, can be described in two different ways.

1. As the inverse limit of the inverse sequence

$$S^1\overset{\varphi}{\leftarrow}S^1\overset{\varphi}{\leftarrow}\dots S^1\overset{\varphi}{\leftarrow}S^1\overset{\varphi}{\leftarrow}\dots$$

where S^1 is the multiplicative group of all complex numbers z with $|z|=1$ and $\varphi(z)=z^p$, and

2. As the intersection $\bigcap_{i=0}^{\infty} T_i$ of solid tori, where $T_{i+1}\subset\operatorname{Int}T_i$ and T_{i+1} "wraps around p times" in $\operatorname{Int}T_i$. For the precise meaning of "wrapping around p times" see [8], p. 230 exc.4.

Lemma 1. *Let S^3 be the one-point compactification of R^3, i.e. $S^3=\{(z,s)\in C\times C\colon |z|^2+|s|^2=1\}$, where C denotes the complex numbers. Then S^3 can be written as the union of two closed subsets*

$$T=\{(x_1,x_2,x_3,x_4)\in S^3\colon x_1^2+x_2^2\geqslant x_3^2+x_4^2\}$$

and

$$T'=\{(x_1,x_2,x_3,x_4)\in S^3\colon x_1^2+x_2^2\leqslant x_3^2+x_4^2\}$$

such that T and T' are solid tori and their intersection is a torus.

For the proof of this well-known Lemma see [9], p. 138 and also [2].

The torus $T\cap T'$ of Lemma 1 corresponds to a torus imbedded in R^3 in the standard way. Since $T\cap T'$ is the boundary of T and T', T corresponds to the standard solid torus in R^3. If a solid torus in the interior of the standard solid torus goes around p times, then this will correspond to a solid torus $T_1\subset\operatorname{Int}T$ that goes around p times (see above).

In the following lemma a solid torus in S^3 corresponding to the standard solid torus in R^3 will be referred to also as the *standard torus.*

Definitions. By a *spine* of a solid torus we mean a subpolyhedron of lower dimension to which the torus collapses.

A simple closed curve is said to be *unknotted* in R^3 (equivalently S^3) if there is a homeomorphism $h\colon R^3\to R^3$ ($h\colon S^3\to S^3$) which sends the simple closed curve to the circle $x^2+y^2=1$, $z=0$.

Lemma 2. *Let T_0 be the standard solid torus in S^3 and let $T_1\subset\operatorname{Int}T_0$ be a solid torus that goes around p times in the interior of T_0. Then $\operatorname{Cl}(S^3-T_1)=\overline{S^3-T_1}$ is a solid torus and there is a homeomorphism $h\colon S^3\to S^3$ such that $h(\overline{S^3-T_1})=T_0$ and $h(\overline{S^3-T_0})=T_1$, i.e. $\overline{S^3-T_0}$ is imbedded in $\overline{S^3-T_1}$ the same way T_1 is imbedded in T_0.*

Proof. By Lemma 1, $\overline{S^3-T_0}$ is a solid torus. To show that $\overline{S^3-T_1}$ is a solid torus we shall show that there is a homeomorphism $h\colon S^3\to S^3$ such that $h(\overline{S^3-T_1})=T_0$.

Let Σ and Σ' be spines of $\overline{S^3-T_0}$ and T_1, respectively. In this case Σ and Σ' are both unknotted simple closed curves hence there is a homeomorphism h_1 of S^3 onto itself such that $h_1(\Sigma)=\Sigma'$ and $h_1(\Sigma')=\Sigma$. But

$$\overline{S^3-T_0}\cup T_1 \quad\text{and}\quad h_1(\overline{S^3-T_0}\cup T_1)$$

are both regular neighborhoods of $\Sigma\cup\Sigma'$ in S^3, therefore there is a homeomorphism $h_2\colon S^3\to S^3$ such that $h_2|\Sigma\cup\Sigma'$ is the identity on $\Sigma\cup\Sigma'$,

$$h_2h_1(\overline{S^3-T_0}\cup T_1)=\overline{S^3-T_0}\cup T_1,\quad h_2h_1(\overline{S^3-T_0})=T_1\ \text{and}\ h_2h_1(T_1)=\overline{S^3-T_0}.$$

If we let $h_2h_1=h$, then $h(\overline{S^3-T_0})=T_1$ and $h(T_1)=\overline{S^3-T_0}$ which implies that $h(\overline{S^3-T_1})=T_0$.

The complement of Σ_p in S^3 is connected, but $S^3-\Sigma_p$ is not simply connected since its singular homology $H_1(S^3-\Sigma_p;Z)$ is isomorphic to the p-adic rationals (the set of rationals of the form k/p^n, where k is an integer and n is a non-negative integer). To see that $H_1(S^3-\Sigma_p;Z)$ is isomorphic to the p-adic rationals, we view Σ_p as the inverse limit of the sequence

$$S^1\overset{\varphi}{\leftarrow}S^1\overset{\varphi}{\leftarrow}S^1\overset{\varphi}{\leftarrow}\dots$$

Then $H^1(S^1;Z)$ (the singular cohomology) is isomorphic to Z so $H^1(\Sigma_p;Z)$ (the Čech cohomology) is the direct limit of the sequence

$$Z\overset{\varphi_*}{\to}Z\overset{\varphi_*}{\to}Z\overset{\varphi_*}{\to}\dots$$

where $\varphi_*(m)=pm$, by the continuity of Čech cohomology ([8], Thm 3.1, p. 261). But this direct limit is the p-adic rationals. By [10], Thm. 17, p. 296 we have an isomorphism

$$H_2(S^3,S^3-\Sigma_p;Z)\approx H^1(\Sigma_p;Z) \tag{1}$$

and, by using the exact homology sequence of the pair $(S^3,S^3-\Sigma_p)$

$$\dots\to H_2(S^3;Z)\to H_2(S^3,S^3-\Sigma_p;Z)\to H_2(S^3-\Sigma_p;Z)\to H_1(S^3;Z)\to\dots$$

we obtain $H_2(S^3,S^3-\Sigma_p;Z)\approx H_1(S^3-\Sigma_p;Z)$ since $H_2(S^3;Z)=H_1(S^3;Z)=0$. From the last isomorphism and (1) we have $H_1(S^3-\Sigma_p;Z)\approx H^1(\Sigma_p;Z)$.

In order to prove the main result we shall view Σ_p as the intersection of solid tori, i.e. $\Sigma_p = \bigcap_{i=0}^{\infty} T_i$ (see 2, above). If we let $Y_i = \overline{S^3 - T_i}$, $i = 0, 1, 2, \ldots$ then

$$X = S^3 - \Sigma_p = \bigcup_{i=0}^{\infty} (\overline{S^3 - T_i}) = \bigcup_{i=0}^{\infty} Y_i .$$

Each Y_i is a solid torus and is imbedded in Y_{i+1} the same way T_{i+1} is imbedded in T_i, by Lemma 2.

THEOREM. *The complement of the p-adic solenoid in S^3, $S^3 - \Sigma_p$, admits a mean.*

Proof. We shall prove the existence of a p-mean on $S^3 - \Sigma_p$ by induction on k. We have $S^3 - \Sigma_p = \bigcup_{i=0}^{\infty} Y_i$, where $\Sigma_p = \bigcap_{i=0}^{\infty} T_i$, $Y_i = \overline{S^3 - T_i}$ and $Y_i \subset \operatorname{Int} Y_{i+1}$, $i = 1, 2, \ldots$ Each Y_i is a solid torus, hence homeomorphic to $S^1 \times D^2$ (D^2 is the 2-dimensional disc) so we can assign to each point of Y_i coordinates (t, x), where $t \in [0, 1]$ and assume that the diagram

(1)
$$\begin{array}{ccc} Y_k & \xrightarrow{i} & Y_{k+1} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ S^1 & \xrightarrow[g]{} & S^1 \end{array}$$

commutes. Here i denotes the inclusion map, π_1 is the projection onto the first factor and g is defined by $g(t) = pt \bmod 1$, where 1 denotes the length of S^1 normalized to unit length.

Letting $Y_{-1} = \emptyset$ (empty set) we assume that

$$h_k\colon Y_k^p \to S^3 - \Sigma_p$$

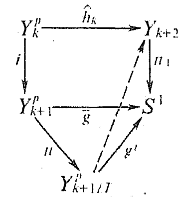
has been defined for some k such that

1. $h_k(z, \ldots, z) = z$ for every $z \in Y_k$,
2. $h_k(z_1, \ldots, z_p) = h_k(z_{\sigma(1)}, \ldots, z_{\sigma(p)})$, for every p-tuple $(z_1, \ldots, z_p) \in Y_k^p$, and every permutation $\sigma \in S_p$,
3. $h_k(Y_k^p) \subset Y_{k+1}$, and
4. if $\hat{h}_k\colon Y_k^p \to Y_{k+2}$ is defined by $\hat{h}_k(z_1, \ldots, z_p) = h_k(z_1, \ldots, z_p)$, then the diagram

(2)
$$\begin{array}{ccc} Y_k^p & \xrightarrow{\hat{h}_k} & Y_{k+2} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ Y_{k+1}^p & \xrightarrow[\bar{g}]{} & S^1 \end{array}$$

commutes if $\bar{g}$ is defined by $\bar{g}\big((t_1, x_1), \ldots, (t_p, x_p)\big) = (t_1 + t_2 + \ldots + t_p) \bmod 1$.

We shall show that with these assumptions on h_k we can extend h_k to $h_{k+1}\colon Y_{k+1}^p \to S^3 - \Sigma_p$ so that h_{k+1} satisfies the above four conditions. Toward that end we let T be a relation on $(S^3 - \Sigma_p)^p$ such that $(z_1, \ldots, z_p)$ is T-related to $(z_1', \ldots, z_p')$ if there is a permutation $\sigma \in S_p$ such that $(z_{\sigma(1)}, \ldots, z_{\sigma(p)}) = (z_1', \ldots, z_p')$; then T is easily shown to be an equivalence relation and for each k we can form the quotient space Y_k^p/T. We let $\pi\colon Y_k^p \to Y_k^p/T$ denote the quotient map and consider the following diagram

(3)


where $g' = \bar{g}\pi^{-1}$. That g' is well-defined can be seen from the following argument. Let $[(t_1, x_1), \ldots, (t_p, x_p)]$ be a point in Y_{k+1}^p/T, and let $\big((t_{\sigma(1)}, x_{\sigma(1)}), \ldots, (t_{\sigma(p)}, x_{\sigma(p)})\big)$ be a point in the equivalence class of $\big((t_1, x_1), \ldots, (t_p, x_p)\big)$. Then

$$\bar{g}\big((t_{\sigma(1)}, x_{\sigma(1)}), \ldots, (t_{\sigma(p)}, x_{\sigma(p)})\big) = (t_{\sigma(1)} + \ldots + t_{\sigma(p)}) \bmod 1 = \bar{g}\big((t_1, x_1), \ldots, (t_p, x_p)\big) .$$

Thus g' is well-defined.

To show that g' is continuous, let U be an open subset of S^1. Then $(g')^{-1}(U) = \pi(\bar{g})^{-1}(U)$ and since $\bar{g}$ is continuous, $(\bar{g})^{-1}(U) = V$ is open in Y_{k+1}^p. But $\pi^{-1}(\pi V) = V$, and since π is a quotient map, hence an identification, $\pi(V)$ is open, therefore $\pi(\bar{g})^{-1}(U)$ is open and g' is continuous. Thus g' is well-defined and continuous and, by the way it was defined, the lower part of diagram (3) commutes. Let ΔY_{k+i} be the diagonal of Y_{k+i}^p, $i = 1, 2$, and define $\varphi\colon [\pi i(Y_k^p)] \cup \Delta Y_{k+1}/T \to Y_{k+2}$ so that $\pi_1 \varphi = g'$. The restriction $\varphi|_{\pi i(Y_k^p)} = \hat{h}_k i^{-1} \pi^{-1}|_{\pi i(Y_k^p)}$ is single-valued, thus well-defined. If $[(t, x), \ldots, (t, x)] \in \Delta Y_{k+1}/T$, $\varphi\big((t, x), \ldots, (t, x)\big) = (pt, x')$, so φ is well-defined.

To show that φ is continuous let A be a closed subset of Y_{k+2}. Then $\varphi^{-1}(A) = \pi i \hat{h}_k^{-1}(A)$. Now $\hat{h}_k^{-1}(A)$ is closed in Y_k^p and $i(\hat{h}_k^{-1}(A)) = V$ is a closed subset of Y_{k+1}^p since Y_k^p is imbedded in Y_{k+1}^p as a closed subset. Since $\pi^{-1}(\pi(V)) = V$ and π is an identification, we conclude that $\pi(V)$ is closed, therefore $\varphi^{-1}(A)$ is closed and φ is continuous.

If $\pi_2\colon Y_{k+2} \to D^2$ is the projection of Y_{k+2} onto its second factor, we have the map

$$\pi_2 \varphi\colon [\pi i(Y_k^p)] \cup \Delta Y_{k+1}/T \to D^2$$

defined on a closed subset of Y_{k+1}^p/T. Since D^2 is an AR (absolute retract), $\pi_2 \varphi$ can be extended to $\hat{\varphi}$ over all of Y_{k+1}^p/T. If we let φ^* be defined by

$$\varphi^*([(t_1, x_1), \ldots, (t_p, x_p)]) = \big(g'([(t_1, x_1), \ldots, (t_p, x_p)]),\ \hat{\varphi}([(t_1, x_1), \ldots, (t_p, x_p)])\big) ,$$

then $\varphi^*\colon Y_{k+1}^p/T \to Y_{k+2}$ is an extension of φ, and we shall show that diagram (3), with the dotted line now filled in, commutes. To do this it will suffice to show that

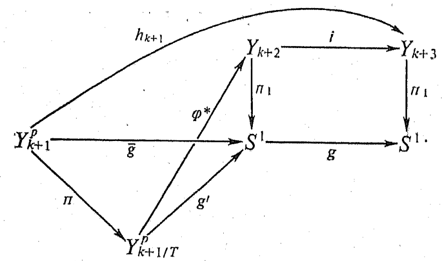
$\pi_1\varphi^* = g'$, since the rest of the diagram is commutative. But this is so because

$$\pi_1\varphi^*([(t_1, x_1), \dots, (t_p, x_p)]) = \pi_1\big(g'([(t_1, x_1), \dots, (t_p, x_p)]),\ \phi([(t_1, x_1), \dots, (t_p, x_p)])\big)$$
$$= g'([(t_1, x_1), \dots, (t_p, x_p)]) .$$

To complete the inductive step we let $h_{k+1}(y) = \varphi^*\pi(y)$ for every $y \in Y^p_{k+1}$, $\hat{h}_{k+1} = ih_{k+1} = i\varphi^*\pi$, and verify that the diagram below

$$\text{(4)} \qquad \begin{array}{ccc} Y^p_{k+1} & \xrightarrow{\hat{h}_{k+1}} & Y_{k+3} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ Y^p_{k+1} & \xrightarrow[\bar{g}]{} & S^1 \end{array}$$

commutes. From the definition $\hat{h}_{k+1}$, the following diagram



is commutative, and we have $\pi_1\hat{h}_{k+1} = \pi_1 ih_{k+1}$ and $\pi_1\hat{h}_{k+1} = g\pi_1 h_{k+1} = g\bar{g}$. Now let $((t_1, x_1), \dots, (t_p, x_p)) \in Y^p_{k+1}$. Then

$$\pi_1\hat{h}_{k+1}((t_1, x_1), \dots, (t_p, x_p)) = g((t_1 + \dots + t_p) \bmod 1) = p(t_1 + \dots + t_p) \bmod 1 ,$$

because of the last equality above. But

$$\bar{g}i((t_1, x_1), \dots, (t_p, x_p)) = \bar{g}((pt_1, x_1'), \dots, (pt_p, x_p')) = (pt_1 + \dots + pt_p) \bmod 1$$
$$= p(t_1 + \dots + t_p) \bmod 1 .$$

Thus $\pi_1\hat{h}_{k+1} = \bar{g}i$, and diagram (4) commutes.

This completes the inductive step and it is seen that each h_{k+1} is an extension of h_k. We now define inductively $h_1, h_2, \dots$ and set $m = \bigcup_{i=1}^{\infty} h_i$. It is easily verified that m is a p-mean on $S^3 - \Sigma_p$.

COROLLARY. *The fundamental group of $S^3 - \Sigma_p$ is isomorphic to the p-adic rationals.*

Proof. We have seen earlier that $H_1(S^3 - \Sigma_p; Z)$ is isomorphic to the p-adic rationals. By the Theorem $S^3 - \Sigma_p$ is an m-space so $\pi(S^3 - \Sigma_p)$ is abelian, therefore $\pi(S^3 - \Sigma_p) \approx H_1(S^3 - \Sigma_p; Z)$.

References

[1] E. Artin and R. H. Fox, *Some wild cells in spheres in three-dimensional space*, Ann. of Math. 49 (1948), pp. 979–990.
[2] J. W. Alexander, *On the subdivision of 3-space by a polyhedron*, Proc. Nat. Acad. Sci. USA 10 (1924), pp. 8–10.
[3] G. Aumann, *Über Räume mit Mittelbildungen*, Math. Ann. 119 (1943), pp. 210–215.
[4] P. Bacon, *An acyclic continuum that admits no mean*, Fund. Math. 67 (1970), pp. 11–13.
[5] M. Brown, *A proof of the generalized Schoenflies Theorem*, Bull. Amer. Math. Soc. 66 (2) (1960), pp. 74–76.
[6] B. Eckmann, *Räume mit Mittelbildungen*, Comm. Math. Helv. 28 (1954), pp. 329–340.
[7] T. Ganea and P. Hilton, *Generalized Means*, Studies in Mathematical Analysis and Related Topics, Stanford Univ. Press, 1962.
[8] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton Univ. Press, 1952.
[9] W. S. Massey, *Algebraic Topology: An Introduction*, Harcourt, Brace and World, Inc., 1967.
[10] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, 1966.

UNIVERSITY OF SOUTH FLORIDA
DEPARTMENT OF MATHEMATICS
Tampa, Florida

Accepté par la Rédaction le 7. 4. 1975