On c-continuous fundamental groups

by

David B. Gauld (Private Bag, Auckland)

Abstract. The notion of c-continuous fundamental group is introduced by Gentry and Hoyle in their paper "c-continuous fundamental groups", Fund. Math. 76 (1972), pp. 9-17. In this paper we prove this group in a more natural setting, finding that it is in fact a subgroup of the usual fundamental group after an adjustment of the topology on the space in question.

The notion of a c-continuous fundamental group is introduced in [1]. In this paper the group is placed in a more natural setting thereby simplifying both the description of the group and the proofs of various properties. Throughout, $I = [0, 1]$ and $J = [0, 1]$ denote respectively the closed and half-open unit interval, each with the usual topology.

Following [1], we say that a function $f: X \rightarrow Y$ between topological spaces is c-continuous at $x \in X$ if for each open set $U \subset Y$ containing $f(x)$ for which $Y - U$ is compact, there is an open set $V$ in $X$ containing $x$ such that $f(V) \subset U$. The function $f$ is c-continuous if it is c-continuous at each point of $X$. Let $T$ denote the topology on $Y$. Let $T^*$ be the topology on $Y$ having as basis

$$\{U \in T \mid Y - U \text{ is compact in } T\}.$$

Call this basis the standard basis for $T^*$.

Theorem 1. $f: X \rightarrow (Y, T)$ is c-continuous if and only if $f: X \rightarrow (Y, T^*)$ is continuous.

Proof. Suppose $f: X \rightarrow (Y, T)$ is c-continuous and $U$ is a member of the standard basis for $T^*$. Then for each $x \in f^{-1}(U)$ there is an open set $V$ in $X$ containing $x$ such that $f(V) \subset U$, i.e. $V \subset f^{-1}(U)$. Hence $f^{-1}(U)$ is open so $f: X \rightarrow (Y, T^*)$ is continuous.

Conversely suppose $f: X \rightarrow (Y, T^*)$ is continuous, $x \in X$ and $U \in T$ is such that $f(x) \not\in U$ and $Y - U$ is compact in $T$. Then $U \in T^*$ so $V = f^{-1}(U) \in X$. Since $x \in V$ and $f(V) \subset U$, we see that $f$ is c-continuous. $\square$

Let $C(Y, y_0)$ be as in [1], i.e. $C(Y, y_0)$ consists of all loops in $Y$ based at $y_0$ using the topology $T$ on $Y$. In [1] an equivalence relation $\sim$ is defined on $C(Y, y_0)$ as follows: if $f, g \in C(Y, y_0)$ then $f \sim g$ if and only if there is a c-continuous function $F: I \times I \rightarrow (Y, T)$ such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$, $F(0, t) = y_0 = F(1, t)$ for each $x, t \in I$. Since $T^* \subset T$, $f$ and $g$ are also loops in $(Y, T^*)$. By Theorem 1,
Theorem 2 motivates the following as an example of a non-compact, path-connected space having non-trivial $c$-continuous fundamental group.

**Example.** Let $S^1$ denote the unit circle and $L$ the long line; $L$ is obtained from the ordinal space $[0, \omega]$ by connecting each ordinal with its successor by a copy of the unit interval. Let $X$ be the space obtained from the disjoint union of $S^1$ and $L$ by identifying the point $0 \in L$ with some point of $S^1$. Then $X$ is a non-compact, path-connected space.

**Claim.** $X$ has non-trivial $c$-continuous fundamental group; in fact the group is $\mathbb{Z}$.

**Proof.** It suffices to show that any $c$-continuous homotopy

$$H: I \times I \rightarrow X$$

is actually continuous.

Suppose not: say $(s_0, t_0) \in I \times I$ is a point at which $H$ is not continuous. Choose $a \in L$ such that

$$H(s_0, t_0) \notin S^1 \cup [0, a] .$$

Now $V$ vicinity neighbourhood $V$ of $(s_0, t_0)$,

$$H(V) \notin S^1 \cup [0, a] ,$$

since $S^1 \cup [0, a]$ is compact, and if $H(V)$ did lie in this set then $c$-continuity of $H$ in $V$ would imply continuity of $H$ in $V$. Thus there is a sequence $(s_n, t_n) \in I \times I$ satisfying:

(i) $(s_n, t_n)$ converges to $(s_0, t_0)$;

(ii) $H(s_n, t_n) \notin S^1 \cup [0, a]$, $\forall n = 1, 2, ...$

Since $\{H(s_n, t_n)\}$ is countable, $\exists b \in L$ so that

$$H(s_n, t_n) \in [a, b], \quad \forall n = 1, 2, ...$$

Consider $S^1 \cup [0, a] \cup (b, \omega) = U$ say. Since $U$ is open in $X$ and $X - U = [a, b]$ is compact, $c$-continuity of $H$ implies that $H^{-1}(U)$ is open in $I \times I$. But this contradicts our choice of the sequence $(s_n, t_n)$ since $(s_n, t_n) \notin H^{-1}(U)$ for $n > 0$.

Thus if $H$ is $c$-continuous, $H$ is also continuous. ■

**Reference**


**THE UNIVERSITY OF AUCKLAND**

New Zealand

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