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DRUKARNIA UNIWERSYTETU JAGIELLONSKIEGO W KRAKOWIE

On c-continuous fundamental groups

by

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Abstract. The notion of c-continuous fundamental group is introduced by Gentry and Hoyle in their paper "c-continuous fundamental groups", Fund. Math. 76 (1972), pp. 9–17. In this paper we place this group in a more natural setting, finding that it is in fact a subgroup of the usual fundamental group after an adjustment of the topology on the space in question.

The notion of a c-continuous fundamental group is introduced in [1]. In this paper the group is placed in a more natural setting thereby simplifying both the description of the group and the proofs of various properties.

Throughout, $I = [0, 1]$ and $J = [0, 1)$ denote respectively the closed and half-open unit interval, each with the usual topology.

Following [1], we say that a function $f\colon X\to Y$ between topological spaces is *c-continuous* at $x \in X$ if for each open set $U \subset Y$ containing $f(x)$ for which $Y-U$ is compact, there is an open set V in X containing x such that $f(V)\subset U$. The function f is *c-continuous* if it is c-continuous at each point of X. Let T denote the topology on Y. Let T^c be the topology on Y having as basis

$$\{U\in T|\ Y-U \text{ is compact (in } T)\}\,.$$

Call this basis the *standard basis* for T^c.

THEOREM 1. *$f\colon X\to (Y,T)$ is c-continuous if and only if $f\colon X\to (Y,T^c)$ is continuous.*

Proof. Suppose $f\colon X\to (Y,T)$ is c-continuous and U is a member of the standard basis for T^c. Then for each $x\in f^{-1}(U)$ there is an open set V in X containing x such that $f(V)\subset U$, i.e. $V\subset f^{-1}(U)$. Hence $f^{-1}(U)$ is open so $f\colon X\to (Y,T^c)$ is continuous.

Conversely suppose $f\colon X\to (Y,T^c)$ is continuous, $x\in X$ and $U\in T$ is such that $f(x)\in U$ and $Y-U$ is compact in T. Then $U\in T^c$ so $V=f^{-1}(U)$ is open in X. Since $x\in V$ and $f(V)\subset U$, we see that f is c-continuous. ■

Let $C(Y, y_0)$ be as in [1], i.e. $C(Y, y_0)$ consists of all loops in Y based at y_0 using the topology T on Y. In [1] an equivalence relation $\overset{c}{\underset{y_0}{\sim}}$ is defined on $C(Y, y_0)$ as follows: if $f, g\in C(Y, y_0)$ then $f \overset{c}{\underset{y_0}{\sim}} g$ if and only if there is a c-continuous function $F\colon I\times I\to (Y,T)$ such that $F(x,0)=f(x)$, $F(x,1)=g(x)$, $F(0,t)=y_0=F(1,t)$ for each $x, t\in I$. Since $T^c\subset T$, f and g are also loops in (Y,T^c). By Theorem 1,

$f \overset{c}{\underset{y_0}{\sim}} g$ if and only if f is homotopic modulo end points to g when Y is topologised by T^c. Thus if we let $\Lambda(Y, y_0)$ denote loops in (Y, T^c) based at y_0 we see that $C(Y, y_0) \subset \Lambda(Y, y_0)$ and that $\overset{c}{\underset{y_0}{\sim}}$ is the restriction to $C(Y, y_0)$ of the usual notion of homotopic loops in $\Lambda(Y, y_0)$. The operation $*$ on $C(Y, y_0)$ given in Definition 3 of [1] is the usual notion of composition of loops so the operation $*$ on $C_1(Y, y_0) = C(Y, y_0)| \overset{c}{\underset{y_0}{\sim}}$ is well-defined. $C_1(Y, y_0)$ is clearly closed under this operation: if $f, g\colon I \to (Y, T)$ are loops then $f * g\colon I \to (Y, T)$ is also a loop. Moreover if $f \in C(Y, y_0)$ then the reverse of f is also in $C(Y, y_0)$. Thus $C(Y, y_0)$ is a subgroup of the fundamental group $\pi(Y, y_0; T^c)$. In fact,

$$C_1(Y, y_0) = \{\alpha \in \pi(Y, y_0;\ T^c)|\ \text{there is a loop } f \in \alpha \text{ for which } f\colon I \to (Y, T) \text{ is continuous}\}.$$

Theorem 5 of [1] is now a consequence of the fact that if $H\colon (Y_1, T_1) \to (Y_2, T_2)$ is a homeomorphism then so is $H\colon (Y_1, T_1^c) \to (Y_2, T_2^c)$. The isomorphism from $\pi(Y_1, y_1; T_1^c)$ to $\pi(Y_2, y_2; T_2^c)$ induced by H carries $C_1(Y_1, y_1)$ to $C_1(Y_2, y_2)$.

Theorem 7 of [1] is an immediate consequence of the fact that if (Y, T) is compact then $T^c = T$.

THEOREM 2. *If (Y, T) is non-compact and there is a proper path $\pi\colon J \to (Y, T)$ then $C_1(Y, \pi(0)) = 0$.*

Proof. "Proper" means for any compact $K \subset Y$, $\pi^{-1}(K)$ is compact. Let $\lambda\colon I \to (Y, T)$ be any loop based at $\pi(0)$. Define $F\colon I \times I \to Y$ by

$$F(x, t) = \begin{cases} \lambda(x) & \text{if } t = 0, \\ \pi(4tx^2 - 4tx + 1) & \text{if } 0 < t \leqslant \tfrac{1}{2} \text{ and } 0 < x < 1, \\ \pi(4(1-t)x^2 - 4(1-t)x + 1) & \text{if } \tfrac{1}{2} \leqslant t < 1 \text{ and } 0 < x < 1, \\ \pi(0) & \text{if } t = 1 \text{ or } x = 0 \text{ or } x = 1. \end{cases}$$

Note that $F(x, 0) = \lambda(x)$ and $F(x, 1) = \pi(0)$ so if we can show that $F\colon I \times I \to (Y, T^c)$ is continuous then we can deduce that $C_1(Y, \pi(0)) = 0$.

Suppose U is in the standard basis of T^c. Then U is open in T and $Y - U$ is compact, so $\pi^{-1}(Y - U) = J - \pi^{-1}(U)$ is compact. Thus there is a real number $a \in J$ so that $J - \pi^{-1}(U) \subset [0, a]$, so that $(a, 1) \subset \pi^{-1}(U)$.

Suppose $(x, t) \in F^{-1}(U)$. If $0 < t < 1$ and $0 < x < 1$ then (x, t) is clearly in the interior of $F^{-1}(U)$. If $t = 0$, then $(x, 0) \in \lambda^{-1}(U) \times [0, 1 - a)$. Moreover,

$$\lambda^{-1}(U) \times [0, 1 - a) \subset F^{-1}(U),$$

so $(x, 0)$ is in the interior of $F^{-1}(U)$. Similarly one shows that (x, t) is in the interior of $F^{-1}(U)$ if $t = 1$ or $x = 0$ or $x = 1$. Thus $F^{-1}(U)$ is open so that $F\colon I \times I \to (Y, T^c)$ is continuous.

The effect of F is to pull the loop λ onto the "end" of $\pi(J)$ then to push the loop back to $\pi(0)$. ∎

Theorem 2 motivates the following as an example of a non-compact, path-connected space having non-trivial c-continuous fundamental group.

EXAMPLE. Let S^1 denote the unit circle and L the long line; L is obtained from the ordinal space $[0, \Omega)$ by connecting each ordinal with its successor by a copy of the unit interval I. Let X be the space obtained from the disjoint union of S^1 and L by identifying the point $0 \in L$ with some point of S^1. Then X is a non-compact, path-connected space.

CLAIM. *X has non-trivial c-continuous fundamental group: in fact the group is Z.*

Proof. It suffices to show that any c-continuous homotopy

$$H\colon I \times I \to X$$

is actually continuous.

Suppose not: say $(s_0, t_0) \in I \times I$ is a point at which H is not continuous. Choose $a \in L$ such that

$$H(s_0, t_0) \in S^1 \cup [0, a).$$

Now $\forall$ neighbourhood V of (s_0, t_0),

$$H(V) \not\subset S^1 \cup [0, a],$$

since $S^1 \cup [0, a]$ is compact, and if $H(V)$ did lie in this set then c-continuity of H in V would imply continuity of H in V. Thus there is a sequence $(s_n, t_n) \in I \times I$ satisfying:

(i) (s_n, t_n) converges to (s_0, t_0);

(ii) $H(s_n, t_n) \notin S^1 \cup [0, a)$, $\forall n = 1, 2, \ldots$

Since $\{H(s_n, t_n)|\ n = 1, 2, \ldots\}$ is countable, $\exists b \in L$ so that

$$H(s_n, t_n) \in [a, b], \quad \forall n = 1, 2, \ldots$$

Consider $S^1 \cup [0, a) \cup (b, \Omega) = U$ say. Since U is open in X and $X - U = [a, b]$ is compact, c-continuity of H implies that $H^{-1}(U)$ is open in $I \times I$. But this contradicts our choice of the sequence (s_n, t_n) since $(s_0, t_0) \in H^{-1}(U)$ but $(s_n, t_n) \notin H^{-1}(U)$ for $n > 0$.

Thus if H is c-continuous, H is also continuous. ∎

Reference

[1] K. R. Gentry and H. B. Hoyle III, *c-continuous fundamental groups*, Fund. Math. 76 (1972), pp. 9–17.

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