Write \( F = x^{-1}(\Gamma) \), \( A' = A \cap F \) and let us consider the metric spaces \((A', \varrho)\). We shall show that this is an absolutely Borel space \(^*\). We adopt the notation of the proof of Lemma 3. Let \( F_\tau = \bigcup \{ G_\tau : \tau \in \mathcal{S} \} \). Since \( G_\tau \) is an \( F_\tau \)-set and \( \varrho(G_\tau, G_\tau) \geq 1/m \) for distinct \( \tau, \sigma \), we infer that \( F = X \setminus \bigcup F_\tau \) is a \( G_\Delta \)-set in \((X, \varrho)\). Thus \((F, \varrho)\) is an absolutely Borel space and so is \((A', \varrho)\), as \( A' \) is an \( F_\tau \)-set in \((F, \varrho)\). By Lemma 1 the space \((A', \varrho)\) is not \( \sigma \)-discrete and thus by a Theorem of A. H. Stone \([6]\), Theorem 1) it must contain a Cantor set. This gives the contradiction, because separable subspaces of \((A', \varrho)\) are countable (compare with \([6]\), Sec. 5).

Remark 4. Let \( E \) be the space considered in the Example (Sec. 1). One can prove (see R. Pol, Comment. Math. 22 (1977)) that the product \( E^\aleph_0 \) is perfectly normal, while \( E \) is not paracompact.

\(^*\) A metrizable space is absolutely Borel if it can be embedded as a Borel subspace in a completely metrizable space.

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A hereditarily normal strongly zero-dimensional space with a subspace of positive dimension and an \( N \)-compact space of positive dimension

by

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Abstract. In this paper we give a solution of an old Čech's problem on dimension by constructing a hereditarily normal strongly zero-dimensional space containing a subspace of positive dimension. We give also an example of an \( N \)-compact space of positive dimension.

The aim of this paper is to construct spaces with the properties mentioned in the title.

The problem of existence of a hereditarily normal space \( X \) containing a subspace with the covering dimension greater than the covering dimension of \( X \) is an old problem of Čech (see [2]; compare also [7] Appendix, [3], [11] Problem 11-14, [1] VII, Introduction). Recently, V. V. Filippov [6] showed that the existence of a Suslin Tree yields a space of this kind. Further examples, with many additional properties, were constructed by V. V. Fedorchuk [5]; he used, however, some additional set theoretic assumptions, too. The example we shall construct needs only the usual axioms for the set theory. It solves at the same time a problem on the local dimension raised by C. H. Dowker in [3].

The problem of existence of a closed subspace with the positive covering dimension in a product of countable discrete spaces appears in the natural way in the theory of \( N \)-compactness (see [12]). It was solved recently by S. Mrówka [10] (see also [13]). We give another example of this kind (it seems to us that it is simpler than the Mrówka's one).

1. Notation and terminology. Our terminology will follow [4]. We shall use the following notation: \( I \) denotes the closed real unit interval, \( Q \) stands for rationals of \( I, \mathbb{P} \) for irrationals of \( I \) and \( N \) for natural numbers. For an ordinal \( \alpha \) we shall denote by \( D(\alpha) \) the set of all ordinals less than \( \alpha \) with the discrete topology and by \( W(\alpha) \) the same set with the order topology. The word "dimension" will denote the covering dimension \( \dim \) (see [4], § 7.1); a space \( X \) with \( \dim X = 0 \) is called strongly zero-dimensional. We say that the local dimension of a space \( X \) is at most \( n \) (abbreviated \( \text{locdim} X \leq \alpha \)) if each point \( x \in X \) has an open neighbour-
hood \( U \) with \( \dim U < n \) (see [3] and [11] Definition 11-6). All spaces under discussion are assumed completely regular.

2. Auxiliary construction. The construction of the Broom due to Knaster and Kuratowski (see [5] and [4] P. 6.3.23) is a source of the following observation which play the key role in the sequel.

Let \( X \) be a topological space, \( A \) a subspace of \( X \) and let \( Q_0 = [0, 1] \) be a subset of \( Q \) such that the set \( Q_1 = Q \setminus Q_0 \) is dense in \( Q \). Let

\[
B(X, A) = (X \times Q_0) \cup (A \times P) \cup [(X \setminus A) \times Q_1].
\]

be the subspace of the Cartesian product \( X \times I \). For \( Y \subset X \) put

\[
C(Y) = (Y \times I) \cap B(X, A) = B(Y, A \cap Y).
\]

We have the following

**Lemma 1.** If \( A \) is not an \( F_\sigma \)-set in \( X \), then for arbitrary \( q, q' \in Q_0 \) the sets \( X \times \{q\} \) and \( X \times \{q'\} \) cannot be separated in \( B(X, A) \) by the empty set. In particular, \( \dim B(X, A) > 0 \).

**Proof.** Suppose that \( B(X, A) \) is the union of a disjoint open-and-closed subsets \( U \) and \( U' \) such that \( U \cup X \times \{q\} \) and \( U' \cup X \times \{q'\} \) is closed in \( X \times I \). Let \( F = U \cup U' \), where \( \bar{F} \) denotes the closure in \( X \times I \). We shall show that \( A = \bigcup \{ F(x) : x \in X \setminus A \} \) is closed in \( X \). Indeed, if \( x \in F(x) \) for some \( x \in X_1 \), then \( A \supseteq X \setminus F' \), hence \( x \in A \); for every \( x \in X \) there exists a \( t \in I \) such that \( (x, t) \in F \) and \( x \in A \), then \( t \in X_1 \) so that \( x \in F(t) \).

3. A hereditarily normal strongly zero-dimensional space with a subspace of positive dimension. C. H. Dowker [3] showed that the existence of such a space is equivalent to the existence of a hereditarily normal space \( L \) with \( \text{locdim} L = 0 < \text{dim} L \) (see [11] Remark 11-18); for the construction of \( L \) we shall need the following

**Lemma 2.** There exists a perfectly normal and locally second-countable space \( K \) with \( \omega_1 \) a subset of a locally countable subset \( A \) which is not an \( F_\sigma \)-set in \( K \).

We take the space \( X \) defined in Example of [14] as the space \( K \); we recall the construction below. Let \( B(n) = D(\omega_1)^n \) be the Baire space of weight \( \omega_1 \) (see [4] Example 4.2.12). For each \( n \in B(n) \) let \( \omega = \min \{ x : x < \omega \} \) for \( n \neq Y \) and let \( K \) be the graph \( \{(x, \omega(x)) : x \in B(n)\} \in B(n) \times W(\omega_1) \) of the function \( \omega \). The space \( K \) is perfectly normal (see [14] Proposition 1) and, since \( K \cap \left( B(n) \times W(\omega_1) \right) = K \cap \left( D(\omega_1)^n \times W(\omega_1) \right) \) for every \( \omega \leq \omega_1 \), \( K \) is locally second-countable and \( \text{locdim} K = 0 \). Finally, if we choose for each \( \xi < \omega_1 \), a point \( x_\xi \in D(\omega_1)^n(\xi) \) then the set \( A = \{(x_\xi, 0) : \xi < \omega_1 \} \) has the required property, by [14] Remark 3, Proposition 2.

**Example 1.** There exists a perfectly normal locally second-countable space \( L \) such that \( \text{locdim} L = 0 < \text{dim} L \).

Let us put \( \mathcal{Q}_0 = [0, 1] \) and let \( L = B(K, A) \), where \( K \) and \( A \) are as in Lemma 2. By Morita's theorem \( L \) is perfectly normal (see for example [4] P. 4.5.16) and it is locally second-countable. By Lemma 1 we have \( \text{dim} L > 0 \). It remains to show that \( \text{locdim} L = 0 \). Take an arbitrary point \( (x, t) \in L \), where \( x \in K, t \in I \). There exists an open-and-closed neighbourhood \( U \) of \( x \) such that \( \text{dim} U = 0 \) and \( |U \cap A| < \aleph_0 \) for each

\[
C(U) = (U \times Q_0) \cup ((U \cap A) \times (P \cup Q_0)) \cup (U \setminus A) \times Q_1
\]

is the countable union of its closed strongly zero-dimensional subset \( U \times \{t\} \) for \( t \in Q_0 \), \( (U \setminus A) \times \{t\} \) for \( t \in Q_1 \) and \( (\{y\} \times (P \cup Q_0)) \) for \( y \in U \cap A \). Hence by the Sum Theorem \( \text{dim} C(U) = 0 \). It follows that the point \( (x, t) \) has an open-and-closed strongly zero-dimensional neighborhood.

We shall use Dowker's construction (see [11] Theorem 11-17) to obtain the following

**Example 2.** There exists a hereditarily normal strongly zero-dimensional Lindelöf space containing a subspace of positive dimension.

Let \( L^* = L \cup \{p\} \) where \( L \) is the space from Example 1 and \( p \) is a point which does not belong to \( L \). The topology of \( L^* \) consists of all open subsets of \( L \) and the sets \( V \) such that \( \omega \in V \) and \( L \cup V \) is a second-countable closed subspace of \( L \). It is easy to see that \( L^* \) is Lindelöf. By the construction it follows that each separable subset of \( K \) is contained in an open-and-closed second-countable and strongly zero-dimensional subspace of \( K \); the same holds in \( L \). Thus the space \( L^* \) is hereditarily normal, because for any separated sets \( A, B \) either \( A \cup B \subseteq L \) or one of the sets is second-countable. Finally, it is not hard to verify that \( \text{dim} L^* = 0 < \text{dim} L \).

**Remark 1.** The space \( L \) we have constructed is collectionwise normal. Indeed, the space \( K \) is perfect and collectionwise normal (see [14] Remark 2) and thus the same is true for the product \( K \times I \) which contains \( L \) (see [4], P. 4.5.16, P. 5.5.19 and P. 5.5.1). Let us notice that \( L \) cannot be paracompact because in the class of paracompact spaces \( \text{locdim} = \text{dim} \) (see [11], Corollary 11-8).

**Remark 2.** As proved by C. H. Dowker [3], for the function \( \text{locdim} \) the Finite Sum Theorem holds in the class of normal spaces. It is easy to show that the space \( Z = (N \times U) \cup \{a\} \), where the sets of the form \( \{a\} \cup \{k \times L\} \) form a base of neighbourhoods at the point \( a \), is the union of countably many closed subsets with \( \text{locdim} = 0 \), whereas \( \text{locdim} Z > 0 \). Thus the Countable Sum Theorem fails for \( \text{locdim} \) in the class of perfectly normal spaces.
4. An $N$-compact space of positive dimension. A space $X$ is $N$-compact if it can be embedded as a closed subspace in a product of copies of $N$ (see [16]).

Let $S$ be a set of cardinality $\aleph_1$. For every $T\subset S$ by $p_T: N^S \to N^T$ we shall denote the projection; if $|T|\leq \aleph_0$ then the set $p_T^{-1}(x)$, where $x \in N^T$, will be called an $\aleph_0$-cube.

The following lemma was proved by the authors in [15] (Example 2).

**Lemma 3.** There exists a subset $E$ of $N^S$ which has the following properties:
(i) $E$ is locally an $F_\sigma$-set in $N^S$ (1),
(ii) $E$ is not an $F_\sigma$-set in $N^S$.

Notice that $N^S \times E$ is also the union of $\aleph_0$-cubes, because by (i) it is the union of $G_\delta$-sets. For every $G_\delta$-set in $N^S$ the union of $\aleph_0$-cubes.

**Example 3 (cf. [10]).** An $N$-compact space $M$ which is not strongly zero-dimensional.

Let $Q_0$ be a dense subset of $Q$ such that $Q_1 = Q \setminus Q_0$ is also dense in $Q$. Define $M = B(N^S, E)$, where $E$ is as in Lemma 3. By Lemma 1 and (ii) it follows that $\dim M > 0$. It remains to show that $M$ is $N$-compact.

First we shall prove that

$(1)$ $M$ is realcompact.

Indeed, the space $N^S \times I$ is realcompact and the complement $(N^S \times I) \setminus M = (E \times Q_0) \cup [(N^S \times E) \times P]$ is the union of $\aleph_0$-cubes and, as each $\aleph_0$-cube is a $G_\delta$-set in $N^S \times I$, (1) follows by Mrowka's theorem (compare with [4] P. 3.12.25).

Let $U$ be an open and closed subset of $N^S$. Then

$(2)$ $\dim C(U) = 0$ if and only if $U \cap E$ is an $F_\sigma$-set in $U$.

If $U \cap E$ is not an $F_\sigma$-set in $U$ then by Lemma 1 $\dim C(U) > 0$. Conversely, let $U \cap E$ be an $F_\sigma$-set in $N^S$. Since $U$ is open and closed it depends on countably many coordinates, i.e. $U = p_T(U \cap N^{S \setminus T})$ for some countable set $T, T \subseteq S$ (see [4], P. 2.7.12). Because $U \cap E$ is an $F_\sigma$-set which is the union of $\aleph_0$-cubes,

$$U \cap E = p_T(U \cap E) \times N^{S \setminus T},$$

where $T \subset S$ is countable,

by Theorem 2 of [15]. By Remark 2 of [15] there exists $T \subset S$, $T = T_1 \cup T_2$, $|T| \leq \aleph_0$ such that $p_T(U \cap E)$ is an $F_\sigma$-set in $N^T$. Thus we have

$$U = U' \times N^{S \setminus T},$$

where $U'$ is open in $N^T$,

and

$$U \cap E = E' \times N^{S \setminus T},$$

where $E' = p_T(U \cap E) \cap \bigcup_{i=1}^{m} F_i$,

$(1)$ This means that for each $x \in N^S$ there exists a neighbourhood $V$ of $x$ such that $V \cap E$ is an $F_\sigma$-set in $V$.

where $F_i$ are closed in $N^T$. It follows that

$$C(U) = (U \times Q_0) \cup [(U \cap E) \times P] \cup [(U \cap E) \times Q_1]$$

$$= (U' \times N^{S \setminus T} \times Q_0) \cup (E' \times N^{S \setminus T} \times P) \cup [(U' \times E') \times N^{S \setminus T} \times Q_1]$$

$$\subseteq \{(U' \times Q_0) \cup (E' \times P) \cup [(U' \times E') \times Q_1]\} \times N^{S \setminus T}.$$

The space $Z = (U' \times Q_0) \cup (E' \times P) \cup [(U' \times E') \times Q_1]$ is a metrizable separable space and it is the union of its closed zero-dimensional subsets $U' \times \{t\}$, for $t \in Q_0$, $(U' \times E') \times \{t\}$, for $t \in Q_1$, and $F_i \times (P \cup Q_0)$, for $t = 1, 2, ...$ Hence

$$\dim Z = 0$$

by the Sum Theorem. Thus $C(U)$ is the product of zero-dimensional second-countable spaces and hence $\dim C(U) = 0$ by Morita's theorem ([9], Theorem 3). The proof of (2) is completed.

Let $U$ be an open and closed subset of $M$. For $q \in Q_0$ put

$$U(q) = \{x \in N^S : (x, q) \in U\};$$

clearly $U(q)$ is open and closed in $N^S$.

We shall verify that

$(3)$ $(U(q), U(q')) \cap E$ is an $F_\sigma$-set in $U(q) \setminus U(q')$ for every $q, q' \in Q_0$.

Indeed, the set $V = U \cap C(U(q) \setminus U(q'))$ is open and closed in $C(U(q) \setminus U(q'))$ and $(U(q), U(q')) \times \{q\} \cap V \subseteq C(U(q) \setminus U(q')) \setminus U(q) \times U(q') \times \{q\}$. Hence from Lemma 1 it follows that $(U(q), U(q')) \cap E$ is an $F_\sigma$-set in $U(q) \setminus U(q')$.

For an open and closed set $U \subseteq M$ we define

$(4)$ $C(J_U) \subseteq U$.

and

$(5)$ $U \setminus C(J_U)$ is the countable union of strongly zero-dimensional open and closed subsets of $M$.

Consider an $(x, t) \in C(J_U)$. Then $x \in \cap_{q \in Q_0} U(q)$ and because $Q_0$ is dense in $Q$ and $\{q \}$ is dense in $Q$ (see [4] P. 2.7.12) and $U$ is closed, we have $U = C((x, t)) \cap \{q \}$, for $t \in T$. Thus (4) holds. To establish (5) let us assume that $(x, t) \in U \setminus C(J_U)$. Then $(x, t) \notin U(q)$ for some $q \in Q_0$. Since $U$ is open and $Q_0$ is dense in $Q$ there exists $q \in Q_0$ such that $(x, q) \in U$. Thus $(x, q) \notin U(q) \setminus U(q')$. We have obtained the equality

$$U \setminus C(J_U) = \bigcup_{q, q' \in Q_0} C(U(q) \setminus U(q')) \cap U$$

which proves (5) by (3) and (2).
We shall prove now that $M$ is $N$-compact. Since by (i) and (2) it follows that

\[ \text{locdim} M = 0, \]

it suffices only to verify that every open-and-closed ultrafilter in $M$ with the countable intersection property has nonempty intersection (see [16], p. 478). Let $\mathcal{U}$ be such an ultrafilter. We shall show that

(7) there exists $U \in \mathcal{U}$ with $\text{dim} U = 0$.

Suppose on the contrary that $\text{dim} U > 0$ for each $U \in \mathcal{U}$. Fix an arbitrary $a_0 \in Q_0$, and let

\[ \mathcal{V} = \{ U(a_0); U \in \mathcal{U} \}. \]

We shall prove that $\mathcal{V}$ is an open-and-closed ultrafilter in $\mathcal{N}^\mathbb{N}$ and has the countable intersection property. $\mathcal{V}$ is a filter because for $U_1, U_2 \in \mathcal{U}$ the intersection $U_1(a_0) \cap U_2(a_0) = (U_1 \cap U_2)(a_0)$ belongs to $\mathcal{V}$ and if an open-and-closed set $A \in \mathcal{N}^\mathbb{N}$ contains $U_1(a_0)$ then $A = (C(A) \cup U_1)(a_0)$ also belongs to $\mathcal{V}$ (because $\mathcal{U}$ is a filter). Now let $U$ be an open-and-closed subset of $\mathcal{N}^\mathbb{N}$. Then either $C(U) \in \mathcal{U}$ or $\mathcal{N} \setminus C(U) = C(\mathcal{N}^\mathbb{N} \setminus U) \in \mathcal{V}$; hence either $U \notin \mathcal{V}$ or $\mathcal{N}^\mathbb{N} \setminus U \notin \mathcal{V}$; thus $\mathcal{V}$ is an ultrafilter. Let $U_i \in \mathcal{U}$ for $i = 1, 2, \ldots$. We shall show that $\bigcap_{i=1}^{\infty} U_i(a_0) \neq \emptyset$. As shown in (5), we have $\bigcup_{i=1}^{\infty} (U_i \cap C(U_i)) = \bigcup_{i=1}^{\infty} V_i$, where the sets $V_j$ are strongly zero-dimensional and open-and-closed in $M$. Since $\mathcal{V}$ is an ultrafilter it follows from the negation of (7) that $\mathcal{N}^\mathbb{N} \setminus V_j \in \mathcal{U}$ for $j = 1, 2, \ldots$. By the countable intersection property of $\mathcal{U}$ there exists a point

\[ (x_0, t_0) \in \bigcap_{i=1}^{\infty} U_i \cap \bigcap_{j=1}^{\infty} (\mathcal{N}^\mathbb{N} \setminus V_j) = \bigcap_{i=1}^{\infty} U_i \setminus \bigcup_{j=1}^{\infty} V_j = \bigcap_{i=1}^{\infty} U_i \setminus \bigcup_{i=1}^{\infty} (U_i \cap C(U_i)) = \bigcap_{i=1}^{\infty} C(U_i). \]

We obtain

\[ x_0 \in \bigcap_{i=1}^{\infty} U_i \cap \bigcap_{i=1}^{\infty} U_i(a_0). \]

Now $\mathcal{V}$, being an open-and-closed ultrafilter in $\mathcal{N}^\mathbb{N}$ with the countable intersection property, has the nonempty intersection, and thus there exists an $x \in \bigcap \mathcal{V}$. By (6) there exists an open-and-closed strongly zero-dimensional neighbourhood $U$ of $x$. We have $U \in \mathcal{U}$ contrary to our assumption that $\mathcal{U}$ does not contain strongly zero-dimensional sets. This completes the proof of (7).

Let us take an open-and-closed set $U_0 \in \mathcal{U}$ with $\text{dim} U_0 = 0$. The set $U_0$ is realcompact by (1) and thus it is $N$-compact (see [16], p. 478). The family $\mathcal{W}$

\[ \{ V \in \mathcal{U}; U_0 \cap V \neq \emptyset \} \]

is an open-and-closed ultrafilter in $U_0$ with the countable intersection property and hence $\emptyset \notin \bigcap \mathcal{W} \subset \bigcap \mathcal{W}$. This completes the proof that $M$ is $N$-compact.

Remark 3. If we take in the above construction $K$ instead of $\mathcal{N}^\mathbb{N}$ and $A$ instead of $E$, where $K$ and $A$ are as in Lemma 2, then we obtain a space $M'$ which is perfectly normal, locally second-countable, $N$-compact, and satisfies $\text{dim} M' = \text{locdim} M' = 0$ (the space $M'$ is a slight modification of Example 1). The property of $N$-compactness of $M'$ is analogous to the proof in Example 3 and reduces to the proof of $N$-compactness of $K(F)$ (which follows by Mrówka's result [10] from the fact that $K$ can be mapped continuously in a one-to-one way into the metrizable strongly zero-dimensional space $B(n_1)$) and of realcompactness of $M'$ (which follows from the fact that $M'$ can be mapped in a one-to-one way into the space $B(n_1) \times I$ (see [4] Exercise 3.11.B)). The remaining properties of $M'$ can be proved in the same way as the properties of the space $L$ in Example 1.

We are grateful to Professor R. Engelking for valuable discussions about the subject of this paper.

Added in proof.

(a) In the paper A hereditarily normal strongly zero-dimensional space containing subspaces of arbitrarily large dimensional, Fund. Math. (to appear) the authors have developed essentially the idea described in Section 3.

(b) E. Pol, Bull. Acad. Polon. Sci. 24 (1976), pp. 749-752 gave under CH an example of a locally compact perfectly normal space $X_0$ with $\text{locdim} X_0 = 0$ and $\text{dim} X_0 > n$, where $n = 1, 2, \ldots$; some very strong examples of this kind, also under CH, were constructed recently by V. V. Fedorčuk, On the dimension of hereditarily normal spaces (to appear).

References


(1) In fact, one can prove that $K$ is strongly zero-dimensional.
Addition and correction to the paper
"On stability and products"
Fund. Math. 93 (1976), pp. 81-95

by

J. Wierzejewski (Wrocław and Nijmegen)

In the paper quoted in the title the second part of Corollary 5.5 was formulated wrongly. Namely it should have the following form:

The class of all \( \omega \)-stable theories for which \( \sigma_\varepsilon \) is finite is closed under finite products.

Now we shall show that \( \omega \)-stable cannot be omitted. The notation and terminology are taken from [1].

Namely, let \( \mathcal{B} = \langle Q, \leq, \times, O \rangle, W, C, D, R, \sim_\omega \rangle \), where

- \( Q \) is the set of rationals numbers,
- \( W \) is a unary relation and \( W(a) \) iff \( a \in Q \),
- \( C \) is a unary relation and \( C(a) \) iff \( a \in Q \),
- \( D \) is a ternary relation and \( D(a, b, c) \) iff \( W(a), W(b) \) and \( \exists q \in Q, c = (a, b, q) \),
- \( R \) is a ternary relation and \( R(a, b, c) \) iff \( D(a, b, c) \) and

\[ a \leq b \rightarrow \exists q \in Q, c = (a, b, q) \text{ and } q \text{ is a natural number,} \]
\[ a \geq b \rightarrow \exists q \in Q, c = (a, b, q) \text{ and } q \neq 0, \]

\( \sim_\omega \) are equivalence relations on \( Q \) with infinitely many classes and

\( \sim_{\omega + 1} \) divides every equivalence class of \( \sim_\omega \) into infinitely many equivalence classes of \( \sim_{\omega + 1} \). Moreover every equivalence class of \( \sim_\omega \) is a dense linear ordering without endpoints (with the ordering taken from \( Q \)).

Note that we can define a formula which linearly orders \( W \) into type \( \eta \).

Fact 1. For every \( \mathfrak{B} \vdash \text{Th}(\mathcal{B}) \) and every \( p \in \text{SA} \) either rank \( (p) = 1 \), or rank \( (p) = \infty \).

Indeed, every type contains one of the following sets of formulas:

1) \( \{ x_0 = a \} \) for some \( a \in A \),