A perfectly normal locally metrizable non-paracompact space

by

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Abstract. We construct an example of a perfectly normal locally second-countable and non-paracompact space by a modification of a metrizable space.

The aim of this paper is to describe a construction which by a modification of a metric space yields a locally metrizable, perfect, collectionwise normal and non-paracompact space containing a locally countable non-$F_{\sigma}$ set. An application in the dimension theory, given in [4], has been the motivation for such a construction.

1. Terminology and notation. We shall use the terminology of [3]. For an ordinal $\alpha$ we shall denote by $D(\alpha)$ the set of all ordinals less than $\alpha$ with the discrete topology and by $W(\alpha)$ the same set with the order topology. The symbol $\text{Lim}$ stands for limit countable ordinals. A set $X \subseteq W(\alpha)$ is called stationary if it intersects each closed, cofinal set in $W(\alpha)$; equivalently (cf. [3], Appendix 1.5), if for each function $\varphi: \Sigma \to W(\omega_1)$ with $\varphi(\alpha) < \alpha$ there exists $\xi < \omega_1$ such that $[\varphi^{-1}(\xi)] = \xi$.

If $M$ is a set and $\varrho$ a metric on $M$, then $w(M, \varrho)$ denotes the weight of the metric space $(M, \varrho)$ and $A^\varrho = \{ x \in M : \varrho(x, A) = 0 \}$ denotes the closure of $A \subseteq M$ with respect to $\varrho$. The set of natural numbers is denoted by $\mathbb{N}$, I denotes the unit real interval and $[M]$ stands for the cardinality of a set $M$.

2. The definition of $X$. Let $X$ be a set and $\varrho$ a metric on $X$ such that $w(X, \varrho) = \mathbb{N}$. Suppose that for $\xi < \omega_1$ we have given sets $X_\xi$ satisfying the following conditions (cf. [3], (3), (4)):

\begin{enumerate}
  \item $X_{\xi} \subseteq \ldots \subseteq X_1 \subseteq \ldots \subseteq X_0 = X$, $X_0 = X_\xi$, $w(X_\xi, \varrho) \leq \omega_0$,
  \item $X = \bigcup_{\xi \in \omega_1} X_\xi$ and, for $\xi \in \text{Lim}$, $X_\xi = \bigcup_{\eta < \xi} X_\eta$.
\end{enumerate}

We can obtain such sets by taking $X_\xi = \{ x_\xi ; a < \xi \}$ for a set $\{ x_\xi ; a < \omega_1 \}$ dense in the space $(X, \varrho)$.

Let us introduce a topology in the set $X$ taking as a base the sets $U \cap X_\xi$ where $U$ is open with respect to $\varrho$ and $\xi < \omega_1$. By open and closed sets in $X$ we shall understand sets which are open or closed with respect to that topology.

References

Let us put for \( x \in X \)
\[
(3) \quad x(\xi) = \min \{ \xi : x \in X_\xi \}.
\]

It is easy to see that \( X \) is first-countable and that
\[
(4) \quad (x_n \to x) = \{ q(x_n, \xi) \to 0 \} \quad \text{and} \quad x(x_n) \leq x(x) \quad \text{for almost all} \ n.
\]

The topology of \( X \) is, by (4) and (1), the weak topology introduced by the functions \( \{ x(\xi) \} \) and \( \{ x \to W(\omega) \} \); in other words the function \( x \to (x, x(\xi)) \) maps the space \( X \) homeomorphically onto the graph of the function \( x \) considered as a subspace of the product \( (X, q) \times W(\omega) \). It follows that
\[
(5) \quad X_\xi \cap (X_\xi, q) \times W(\xi+1)
\]
and hence an open and closed set \( X_\xi \) is metrizable and separable.

**Example.** Let \( X = D(\omega^2) \) and let \( q \) be the standard metric on \( X \), i.e. \( (X, q) = B(\omega_1) \) be the Baire space of weight \( \omega_1 \) (see [1], Example 4.2.2). The sets \( X_\xi \) = \( D(\xi) \) satisfy (1) and (2) and for \( x \in X \) we have \( x(x) = \min \{ a : a > x(\xi), \xi \in \omega_1 \} \). We shall consider \( X \) with the topology defined as above. For each \( \xi \in \omega_1 \) let us choose a point \( x_\xi \in X \) with \( x(x_\xi) = \xi \) and put \( E = \{ x_\xi : \xi \in \omega_1 \} \). The space \( E \) is homeomorphic to the graph of \( x \) restricted to the set \( E \), i.e.
\[
E \equiv \{(x_\xi, \xi) : \xi \in \omega_1 \} \times B(\omega_1) \times W(\omega_1).
\]
Notice that the topology of \( E \) is the supremum of the metric topology introduced by \( q \) and the order topology induced by the relation \( x_\xi \leq x_\eta \equiv (\xi \leq \eta) \) (compare with [1] Problem 3. F(c), or [3] Example 6.3).

3. **Auxiliary Lemmas.** The following lemma can be derived easily from the Theorem 1 of [5] (cf. also [5] Remark 5). We shall give however a simple proof of it for the sake of completeness.

**Lemma 1.** Let \( A \) be a subspace of \( X \) such that the set \( \kappa(A) \) is stationary. Then the space \( A \) is not discrete.

**Proof.** Write \( A = \kappa(A) \cap \text{Lim} \), choose for each \( \lambda \in A \) a point \( a_\lambda \in A \) with \( \kappa(a_\lambda) = \lambda \) and put \( A_\lambda = \{ a_\lambda : a \in A, a \leq \lambda \} \). First we shall prove that there exists \( \lambda \in A \) with \( q(a_\lambda, A_\lambda) > 0 \) for every \( \lambda \in A \), i.e. \( A = \cup A_\lambda \), where \( A_\lambda = \{ a \in A : q(a_\lambda, A_\lambda) > 0 \} \). There exists \( \eta \in \omega_1 \) such that \( A_\lambda \) is stationary.

Since, by (2), \( a_\lambda \in \cup A_\lambda \) we can choose for each \( \lambda \in A \) an ordinal \( \phi(\xi) < \lambda \) and a point \( b_\lambda \in X_{\phi(\xi)} \) such that \( q(a_\lambda, b_\lambda) < 1/3n \). There exists \( \xi < \eta \) with \( |\eta - \xi| < \eta \), because \( A \) is stationary. Thus \( b_\lambda : \lambda \in \omega_1 \} \subset X_\xi \) and, by (1), there exist \( \alpha, \lambda \in \phi(\xi) \) such that \( a_\lambda \leq \lambda \) and \( b(\alpha, \lambda) \leq 1/3n \). We have obtained \( q(a_\lambda, a_\lambda) < 1/3n \) which is impossible, as \( a_\lambda \in A_\lambda \) and \( \lambda \in A_\lambda \).

\[(1)\] The space \( (E, q) \) was investigated by A. H. Stone in [6] Section 5 as an example in Borel Theory.

It follows that for some \( \lambda \in A \) there exist \( \xi, \lambda < \lambda \) with \( q(a_\lambda, a_\lambda) \to 0 \) which gives \( a_\lambda \to a_\lambda \), by (4). The proof is completed.

Let us put for \( A \subset X \)
\[
R(A) = X \setminus A.
\]

In the sequel the key role will be played by the following

**Lemma 2.** For each \( A \subset X \) the set \( R(A) \) is not stationary.

**Proof.** Suppose to the contrary that the set \( \kappa(R(A)) = \Sigma \) is stationary. For each \( \lambda \in \Sigma \) let us choose
\[
(7) \quad x_\lambda \in R(A) \quad \text{with} \quad x(\xi) \leq \xi,
\]
and for \( m \in \omega \)
\[
(8) \quad a_\lambda^m \in A \quad \text{with} \quad q(a_\lambda^m, x_\lambda) < 1/m.
\]
Let us put for \( \xi \in \Sigma \)
\[
(9) \quad \varphi(\xi) = \sup \{ q(a_\lambda^m) : m \in \omega \}.
\]
We can easily define by the transfinite induction a closed, cofinal set \( \Gamma \subset W(\omega_1) \) such that
\[
(10) \quad \text{if} \quad \xi \in \Gamma \cap \Sigma \text{ and } \xi < \lambda \in \Gamma \quad \text{then} \quad \varphi(\xi) \leq \lambda.
\]
The set \( A = \Gamma \cap \Sigma \) is stationary and hence, by Lemma 1 and (4), there exist \( \lambda \in A \) and a sequence \( (\lambda_m) \subset A \) such that \( \lambda_m < \lambda \) and \( q(x_\lambda, x_\lambda) \to 0 \). We have, by (8),
\[
q(a_\lambda^m, x_\lambda) \in (q(a_\lambda, x_\lambda) + 1/m
\]
and, by (9) and (10),
\[
\varphi(a_\lambda^m) \leq \varphi(\lambda_m) = \varphi(\lambda) \leq \varphi(\xi).
\]
We have obtained, by (4), \( a_\lambda^m \to x_\lambda \) and hence the contradiction \( x_\lambda \in A \cap R(A) = \emptyset \).

**Lemma 3.** Let \( \Gamma \subset W(\omega_1) \) be a closed and cofinal set. Let us write \( F = \kappa^{-1}(\Gamma) \) and \( G = X \setminus F \). Then
\[
(11) \quad G \text{ has a base } \sigma \text{-discrete in } X;
\]
\[
(12) \quad F = F^{-1}(0) \text{ for a continuous function } f : X \to \Gamma;
\]
\[
(13) \quad \text{we can assign to each set } L \subset G \text{ an open set } G(L) \supset L \text{ in such a way that}
\]
\[
\begin{array}{c}
(1) \quad \text{if} \quad L' \subseteq L'' \quad \text{then} \quad G(L') \subset G(L''),
\end{array}
\]
\[
(11) \quad G(L) \cap F = F \cap F.
\]

**Proof.** Let \( \{ a_{s} : s \in S \} \) be the family of all order components of the set \( W(\omega_1) \setminus \Gamma \). Let us write \( G_s = \kappa^{-1}(\Sigma_s) \) and take for each \( s \in S \) the ordinal \( \mu_s \) such that \( \mu_s + 1 = \min(E_s) \) (we assume that 0 \( \in \Gamma \)). Let us put
\[
G_m = \{ x \in G : q(x, X_m) > 1/m \}.
\]
Since for different \( s, t \in A \) we have either \( G_s \subset X_t \) or \( G_t \subset X_s \), it follows that \( G_m \supset G_m \supset G_m \supset 1/m \). Thus each family \( \Sigma_m = \{ G_m : s \in S \} \) is discrete in \( X \). Since each
We shall prove that $X$ is perfect. From (14) we infer that

$$X \setminus A_0 = (X \setminus A_0) \cup (G \setminus A_0).$$

The first member of the union is an $F_\sigma$-set with respect to $g$ and thus it is an $F_\sigma$-set in $X'$; the second is an $F_\sigma$-set in $X$ by (11) and (12) of Lemma 3. Hence $X \setminus A_0$ is an $F_\sigma$-set in $X$.

**Proposition 2.** If the set $x(x)$ is stationary (this is satisfied in the case considered in Example) then the space $X$ is not paracompact.

**Proof.** Let us choose for each $\zeta \in x(x)$ a point $x_\zeta \in X$ with $x(x_\zeta) = \zeta$. The open set $X_\zeta$ contains only countably many points of the set $A = \{x_\zeta : \zeta \in x(x)\}$ and thus $A$ is locally countable in $X$. But, by Lemma 1, the space $A$ is not $\sigma$-discrete and thus the space $X$ cannot be paracompact.

**4. Remarks.** We shall establish some further properties of our construction.

**Remark 1.** By Theorem 1 of [5] the stationarity of $\pi(X)$ depends on the metric $\sigma$ only; namely, it is equivalent to the property that the metric space $(X, \sigma)$ cannot be expressed as the union of countably many locally separable subspaces. This is the case if $(X, \sigma)$ is a complete space each nonempty open subspace of which has the weight $n_1$ (see [7] Section 2).

**Remark 2. The space $X$ is collectionwise normal.**

We sketch the proof. First notice that the following strengthening of Lemma 2 holds.

**Lemma 2'.** Let $\mathcal{F}$ be a discrete family of closed sets in $X$. Then the union $\bigcup \{x(R(A)) : A \in \mathcal{F} \}$ is not stationary.

Let us put $\Sigma_2 = x(R(A))$ and let $x_\Sigma \in R(A)$ satisfies $x(x_\Sigma) = \min$. Using reasonings analogous to those in the proof of Lemma 2 we can prove that the set $\{x(x_\Sigma) : A \in \mathcal{F} \}$ is not stationary. Since, by Lemma 2, each set $\Sigma_2$ is stationary we conclude by Fodor's theorem ([2] Hilfssatz) that the union $\bigcup \{\Sigma_2 : A \in \mathcal{F} \}$ is not stationary.

Our remark can be derived now from Lemma 2' in the same way as Proposition 1 from Lemma 2 (we must use in addition the property (13), (i)).

**Remark 3.** Let $g$ be a complete metric on a set $X$ and assume that each nonempty open set in $(X, g)$ has the weight $n_1$ (cf. Remark 1). Let us choose for each $\xi \in x(x)$ a point $x_\xi \in X$ with $x(x_\xi) = \xi$ and put $A = \{x_\xi : \xi \in x(x)\}$. Then Proposition 2 can be strengthened in the following way.

**Proposition 2'.** The set $A$ is not an $F_\sigma$-set in $X$ (being locally countable in $X$).

Suppose the contrary. Then $A = \bigcup A_m$, where $A_m$ are closed subsets of $X$.

By Lemma 2 we can find a closed, cofinal set $G = \bigcup G_m$ such that

$$\Gamma \cap \{x(R(A_m)) : m \in N \} = \emptyset.$$
Write $F = \pi^{-1}(\Gamma)$, $A' = A \cap F$ and let us consider the metric spaces $(A', \varrho)$. We shall show that this is an absolutely Borel space (5). We adopt the notation of the proof of Lemma 3. Let $F_n = \bigcup \{ G_s : s \in S \}$. Since $G_n$ is an $F_\sigma$-set and $\varrho(G_m, G_n) \geq 1/m$ for distinct $s, t$, we infer that $F = X \setminus \bigcup F_n$ is a $G_\delta$-set in $(X, \varrho)$. Thus $(F, \varrho)$ is an absolutely Borel space and so is $(A', \varrho)$, as $A'$ is an $F_\sigma$-set in $(F, \varrho)$. By Lemma 1 the space $(A', \varrho)$ is not $\sigma$-discrete and thus by a Theorem of A. H. Stone ([6], Theorem 1) it must contain a Cantor set. This gives the contradiction, because separable subspaces of $(A', \varrho)$ are countable (compare with [6], Sec. 5).

Remark 4. Let $E$ be the space considered in the Example (Sec. 1). One can prove (see R. Pol, Comment. Math. 22 (1977)) that the product $E^{\aleph_0}$ is perfectly normal, while $E$ is not paracompact.

(*) A metrizable space is absolutely Borel if it can be embedded as a Borel subspace in a completely metrizable space.

References


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A hereditarily normal strongly zero-dimensional space
with a subspace of positive dimension and
an $N$-compact space of positive dimension

by

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Abstract. In this paper we give a solution of an old Čech's problem on dimension by constructing a hereditarily normal strongly zero-dimensional space containing a subspace of positive dimension. We give also an example of an $N$-compact space of positive dimension.

The aim of this paper is to construct spaces with the properties mentioned in the title.

The problem of existence of a hereditarily normal space $X$ containing a subspace with the covering dimension greater than the covering dimension of $X$ is an old problem of Čech (see [2]); compare also [7] Appendix, [3], [11] Problem 11-14, (1 VII, Introduction). Recently, V. V. Filippov [6] showed that the existence of a Souslin Tree yields a space of this kind. Further examples, with many additional properties, were constructed by V. V. Fedorčuk [5]; he used, however, some additional set theoretic assumptions, too. The example we shall construct needs only the usual axioms for the set theory. It solves at the same time a problem on the local dimension raised by C. H. Dowker in [3].

The problem of existence of a closed subspace with the positive covering dimension in a product of countable discrete spaces appears in the natural way in the theory of $N$-compactness (see [12]). It was solved recently by S. Mrówka [10] (see also [13]). We give another example of this kind (it seems to us that it is simpler than the Mrówka's one).

1. Notation and terminology. Our terminology will follow [4]. We shall use the following notation: $I$ denotes the closed real unit interval, $Q$ stands for rationals of $I$, $P$ — for irrationals of $I$ and $N$ — for natural numbers. For an ordinal $\alpha$ we shall denote by $D(\alpha)$ the set of all ordinals less than $\alpha$ with the discrete topology and by $W(\alpha)$ the same set with the order topology. The word "dimension" will denote the covering dimension $\dim$ (see [4], § 7.1); a space $X$ with $\dim X = 0$ is called strongly zero-dimensional. We say that the local dimension of a space $X$ is at most $n$ (abbreviated $\text{ldim} X \leq \alpha$) if each point $x \in X$ has an open neighbour-