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Commutative rings in which every proper ideal is maximal

by

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Abstract. We will give the full description of commutative rings in which every proper principal ideal is a prime ideal.

Introduction. Perticani studied in [2] the class of commutative rings with identity in which every proper ideal is maximal. He gave a full description of such a ring R only in the case when R has at least two different proper ideals. In the case where R has only one proper ideal he reduced the problem of characterizing such rings to the one of the computation of cohomology groups. In this paper we will give a full description in both cases. The first case is a trivial conclusion of the Chinese Remainder Theorem and the second will follow very easily from the Cohen Structure Theorem of complete local rings.

All throughout R denotes a commutative ring with identity. We have the same notation as in [3]. The following lemma shows that three classes of rings with pathological properties are only one class and we do not use it in the following.

PROPOSITION 1. *Let R be a ring. Then the following are equivalent:*

1. *every proper ideal is maximal,*
2. *every proper ideal is a primeideal,*
3. *every proper principal ideal is a primeideal.*

Proof. $1\to 2\to 3$ is trivial. To see that $3\to 1$ let A be a proper ideal of R and $a\in A$, $a\neq 0$. Suppose $bc\in A$. If $bc\neq 0$, then $b\in (bc)\subseteq A$ or $c\in (bc)\subseteq A$. If $bc=0$, then $b\in (a)\subseteq A$ or $c\in (a)\subseteq A$. It follows that R/A is an integraldomain. Clearly R/A is a regular ring. Therefore A is a maximal ideal. Q.E.D.

Call a ring R a *max-ring* if every proper ideal is maximal.

LEMMA 2 (see Theorem 1.1 and Theorem 1.4 of [2]). *Suppose R is a max-ring and R contains at least two different proper ideals then R is isomorphic to a product of two fields.*

Proof. Let A_1, A_2 be proper ideals of R and $A_1\neq A_2$. It follows immediately that $A_1\cap A_2=(0)$. Since A_1, A_2 are comaximal it follows from the Chinese

Remainder Theorem that R is of the form $R \cong R_1/A_1 \times R_2/A_2$, hence a product of two fields. Q.E.D.

Suppose R is a max-ring. Because of Lemma 2 we can assume w.l.o.g. that R contains exactly one proper ideal. Thus R is an artinian local ring. Let A be the only proper ideal of R. Obviously A is a principal ideal.

LEMMA 3 (see Proposition 2.1 of [2]). *Suppose R is a max-ring with only one proper ideal, say $A = (a)$, $a \neq 0$. Then $A^2 = (0)$.*

Proof. Since R is artinian and local, a is nilpotent, say $a^n = 0$. It is obvious that $n = 2$, since otherwise there would exist $b \in R$ such that $a = b^{n-1}a^n = 0$. Q.E.D.

DEFINITION. Let p be a prime or zero. We will call a ring a *coefficient ring* if it is a noetherian local ring L whose maximal ideal A is equal pL and whose characteristic is a power p^n of p. Let R be an arbitrary local ring. We will call a subring L of R a *coefficient ring* for R if L is a coefficient ring and $\bar{L} = \bar{R}$, where $-: R \to R/M$ is the canonical mapping and M is the only maximal ideal of R.

Facts (see [1]),

1. Every local ring R with nilpotent maximal ideal contains a coefficient ring for itself.

2. Every field F of characteristic p ($p = 0$ or p a prime) is the residuefield of some coefficient ring L_n of characteristic p^n for each $n \geqslant 1$. Any two such coefficients rings of equal characteristic are isomorphic over F by a unique isomorphism.

3. Every coefficient ring L of characteristic p ($p = 0$ or a prime) is a field. If $\operatorname{char} L = p^n$, $p>0$ and $n \geqslant 2$ then L is of the form $L = V/p^nV$ where V is a discrete and complete valuationring of characteristic zero and residuefield of characteristic $p>0$, $p \notin \Pi_V^2$.

We are now able to give very easily the complete description of rings with only one proper ideal.

THEOREM 4. *Suppose R is a commutative ring with identity. Then every proper principal ideal of R is a primeideal if and only if R is of the form:* (i) $R \cong K_1 \oplus K_2$ *where K_1, K_2 are fields, or* (ii) $R \cong K[x]/(x^2)$ *with K a field, or $R \cong L$ where L is a coefficient ring of characteristic p^2 and $p>0$ (so that $L \cong V/p^2V$ where V is a discrete valuationring of characteristic* 0 *and residuefield of characteristic $p>0$ and p generates M_V the only maximal ideal of V).*

Proof. "$\leftarrow$" obviously.

"$\rightarrow$" Because of Lemma 2 we can assume that R has exactly one proper ideal. Let $a \in R$, $a \neq 0$ and $A = (a)$ be the only proper ideal of R and let $\bar{R}$ be the residuefield of R. From Fact 1 we can conclude that there exists a subring L of R which is a coefficient ring for R. Clearly $\operatorname{char} R = \operatorname{char} \bar{R}$ or otherwise $\operatorname{char} \bar{R} = p>0$ and $\operatorname{char} R = p^n$ for some $n \geqslant 2$.

Case A. $\operatorname{char} R = \operatorname{char} \bar{R}$. Then L is a field and since $\bar{L} = \bar{R}$ it follows that $R = L[a] \cong L[x]/(x^2)$.

Case B. $\operatorname{char} R = p^n$, $p>0$, $n \geqslant 2$. Then p is a nonunit. Therefore $p^2 = 0$ and $(p) = A$ is the only proper ideal of R. Let $b \in L$ such that $\bar{b} = \bar{a}$. Then $a = b + cp$ for some $c \in R$. Again let $k \in L$ such that $\bar{k} = \bar{c}$. It follows that $a = b + kp \in L$. Hence $R = L$. This proves the theorem.

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