Certain continua in S^n with homeomorphic complements have the same shape

by

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Abstract. In this paper we prove that if X_1 and X_2 are globally 1-alg continua in S^n ($n\geqslant 6$) such that X_i has the shape of codimension $\geqslant 3$, closed, $0<(2m_i-n+1)$-connected, m-dimensional topological manifold, $i=1,2$, then $S^n-X_1\approx S^n-X_2$ if and only if $\mathrm{Sh}(X_1)=\mathrm{Sh}(X_2)$.

1. Introduction. An interesting problem is to classify compacta in a manifold M such that the following statement holds: $\mathrm{Sh}(X)=\mathrm{Sh}(Y)$ if and only if $M-X\approx M-Y$, where X, Y are compacta in M.

This question has been answered affirmatively in some cases:

(1) Z-sets in the Hilbert cube by Chapman [2].

(2) Compacta in trivial range in R^n satisfying the small loops condition by Hollingsworth and Rushing [9] (this result generalizes [3] and [7]).

(3) Codimension 3 continua in R^n satisfying the small loops condition and having the shape of a finite complex in trivial range by Theorem 2.4 of [5] and a remark in [11].

(4) Globally 1-alg continua in S^n having the shape of finite complex K in trivial range with either (i) $\pi_1(K)=0$ or (ii) $\pi_1(K)$ abelian and $\pi_2(K)=0$ [11].

(5) Globally 1-alg continua in S^n having the shape of either a topological group in trivial range or a S^k-like continua, for $k\neq n-2$ by Venema [19]. (This result generalizes [14] and [6].)

In this note, we will give an affirmative answer for the class of globally 1-alg continua in S^n ($n\geqslant 6$) having the shape of a codimension 3, closed, $0<(2m-n+1)$-connected topological manifold M^m.

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R. J. Daverman for a suggestion which eliminate the h-cobordism theorem from my original proof of Lemma 1.

2. Notation and definitions. Throughout this note, we use the following notations

$\overset{(PL)}{\approx}$	(PL-) homeomorphic or isomorphic,
$\simeq$	homotopy equivalence or homotopic,
$\sim$	homologous,
∂V, $\operatorname{Int} V$	boundary, interior of a manifold V,
i or $A \subset B$	inclusion map,
f_*	induced maps on homology groups,
H_*	singular homology, Z-coefficients.

For basic shape theory results, we refer to [1] and [12]. For convenience, in this paper we use both shape theories [1] and [12] as is justified in [13].

A *continuum* is a compact, connected space.

A continuum X in S^n is said to be *globally* 1-*alg* in S^n if for every neighborhood U of X in S^n, there is a neighborhood V of X ($V \subset U$) such that if $f\colon S^1 \to V-X$, $f \sim 0$ in $V-X$, then $f \simeq 0$ in $U-X$.

For definitions of the *end* of a manifold, *stable end*, etc. ..., we refer to [15].

For definitions of regular neighborhood, PL-embedding, PL-homeomorphism, etc. ..., we refer to Hudson [8].

A *closed manifold* is a compact manifold without boundary.

An embedding $f\colon M^m \to \operatorname{Int} Q^n$ is said to be *locally flat* if for every $x \in f(M)$, there is a neighborhood U of x in Q such that $(U, U \cap f(M)) \approx (R^n, R^m)$.

Let K be a subset of a manifold M, we say that K has a PL-*radial neighborhood* in M if there is a closed PL-manifold neighborhood W of K in M such that $W-K \overset{PL}{\approx} \partial W \times [0, 1)$.

We will use h-cobordism theorem ([17], p. 59) in PL or TOP-version in appropriate situations.

3. Main results and details of the proof.

LEMMA 1. *Let K_i be a continuum in S^n having a radial neighborhood N_i in S^n $(i = 1, 2)$. Let $\varphi\colon S^n-K_1 \to S^n-K_2$ be a homeomorphism. Then, there is a homotopy equivalence $f\colon \partial N_1 \to \partial N_2$ such that for $[\alpha] \in H_q(\partial N_1)$, $\alpha \sim 0$ in S^n-K_1 if and only if $f \circ \alpha \sim 0$ in S^n-K_2.*

Proof. We may assume that $\varphi(N_1-K_1) \subset N_2-K_2$. There exist homeomorphisms

$$\theta_i\colon \partial N_i \times [0, 1) \to N_i-K_i, \quad i = 1, 2$$

with

(1) $$\theta_2(\partial N_2 \times [\tfrac{1}{2}, 1)) \subset \varphi(N_1-K_1),$$

(2) $$\varphi\theta_1(\partial N_1 \times [\tfrac{3}{4}, 1)) \subset \theta_2(\partial N_2 \times [\tfrac{1}{2}, 1)).$$

Let $r_i\colon N_i-K_i \to \partial N_i$ be the retraction where $r_i\theta_i(x, t) = x$ and

$$r_2'\colon \theta_2(\partial N_2 \times [\tfrac{1}{2}, 1)) \to \partial N_2' \ (\equiv \theta_2(\partial N_2 \times \{\tfrac{1}{2}\}))$$

with $r_2'\theta_2(x, t) = \theta_2(x, \frac{1}{2})$, for every $x \in \partial N_2$, $\frac{1}{2} \leqslant t < 1$.

Let $\Psi\colon \partial N_2' \to \partial N_2$ be the trivial homeomorphism, $\Psi\theta_2(x, \frac{1}{2}) = x$, for every $x \in \partial N_2$, then $r_2 = \Psi r_2'$.

Let $g_i\colon \partial N_i \to \theta_i(\partial N_i \times \{\frac{3}{4}\})$, and $g_2'\colon \partial N_2' \to \theta_2(\partial N_2 \times \{\frac{3}{4}\})$ be the obvious map, $g_2'\theta_2(x, \frac{1}{2}) = \theta_2(x, \frac{3}{4})$ for every $x \in \partial N_2$, then $g_2 = g_2'\Psi^{-1}$. Define

$$\begin{aligned} f'&\colon \partial N_1 \to \partial N_2' && \text{as } r_2'\varphi g_1, \\ f&\colon \partial N_1 \to \partial N_2 && \text{as } r_2\varphi g_1 = \Psi f', \\ \tilde{f}&\colon \partial N_2' \to \partial N_1 && \text{as } r_1\varphi^{-1}g_2', \\ \tilde{f}'&\colon \partial N_2 \to \partial N_1 && \text{as } r_1\varphi^{-1}g_2 = \tilde{f}\Psi^{-1}. \end{aligned}$$

Clearly,

(i) $g_2'r_2' \simeq 1_{\theta_2(\partial N_2 \times [\frac{1}{2}, 1))}$ in $\theta_2(\partial N_2 \times [\frac{1}{2}, 1))$, and

(ii) $g_1 r_1 \simeq 1_{N_1-K_1}$ in N_1-K_1.

It follows that

$$\begin{aligned} \tilde{f}f' &= r_1\varphi^{-1}g_2'r_2'\varphi g_1 \simeq r_1\varphi^{-1}\varphi g_1 \quad \text{(by (1) and (i))} \\ &= r_1 g_1 = 1_{\partial N_1}, \end{aligned}$$

and

$$\begin{aligned} f\tilde{f}' &= r_2\varphi g_1 r_1\varphi^{-1}g_2 \simeq r_2\varphi\varphi^{-1}g_2 \quad \text{(by (ii))} \\ &= r_2 g_2 = 1_{\partial N_2}. \end{aligned}$$

Moreover, we have $f\tilde{f}' = \Psi f'\tilde{f}\Psi^{-1}$. Hence,

$$f'\tilde{f} = \Psi^{-1}f\tilde{f}'\Psi \simeq \Psi^{-1}1_{\partial N_2}\Psi = 1_{\partial N_2'}.$$

Therefore, f' and $\tilde{f}$ are homotopy equivalences. Furthermore, since g_1, g_2', r_2', r_1 can obviously be extended to homotopy equivalences $\bar{g}_1, \bar{g}_2', \bar{r}_2', \bar{r}_1$ of S^n-K_1 and S^n-K_2 with

(a) $\bar{g}_1\bar{r}_1 \simeq 1$, $\bar{r}_1\bar{g}_1 \simeq 1$,

(b) $\bar{g}_2'\bar{r}_2' \simeq 1$, $r_2'g_2' \simeq 1$,

then, so can $\tilde{f}$ and f'.

Thus, the other parts of the conclusion follow. ■

Remark. In particular, Lemma 1 is true if K_1, K_2 are finite subcomplexes of S^n.

OBSERVATION. Let K be a finite subcomplex of $S^n \subset S^{n+1}$. Then, K has a regular neighborhood of the form $N = W \times [-1, 1]$, where W is a regular neighborhood of K in S^n. In this case, let $\pi\colon N \to K$ be a deformation retraction induced by the collapse $N \searrow K$, and let $\nu\colon K \to \partial N$ defined by $\nu(x) = (x, 1)$, for every $x \in K$.

Consider the following diagram

$$\begin{array}{ccc} \partial N & \xrightarrow{i} & N \\ & \overset{\bar\pi}{\searrow} \underset{\nu}{\nwarrow} & \pi \downarrow \uparrow k \\ & & K \end{array}$$

where $\bar\pi = \pi|\partial N$.

It is easy to prove that

(i) $\bar\pi\nu \simeq 1_K$,

(ii) $i \simeq k\bar\pi$,

(iii) $i(\nu\bar\pi) \simeq 1_N$.

(This observation is also true if K is a globally 1-alg, simply-connected CANR in S^n $(n \geqslant 6)$ having the homotopy type of a finite complex of dimension $\leqslant n-3$.)

LEMMA 2. *Let K be a finite subcomplex of R^n. Let $N = W\times[-1, 1]$, π, ν be as above. Then for every $q \geqslant 1$, given $[\alpha] \in H_q(\partial N)$, $\alpha \sim 0$ in $R^{n+1}-K$ if and only if $[\alpha] \in \nu_* H_q(K)$.*

Proof. The homology sequence of the pair $(N, \partial N)$ can be decomposed into split short exact sequences

$$0 \to H_{q+1}(N, \partial N) \to H_q(\partial N) \xrightarrow{i_*} H_q(N) \to 0$$

for every $q \geqslant 1$, since $i_*(\nu\bar\pi)_* = 1_*$. Hence,

$$\begin{aligned} H_q(\partial N) &= \nu_*\bar\pi_* H_q(N) \oplus \operatorname{Ker} i_* \\ &= \nu_* H_q(K) \oplus \operatorname{Ker}\bar\pi_* \quad (\pi_*, k_* \text{ are isomorphisms}). \end{aligned}$$

Now, the following commutative diagram

$$\begin{array}{ccccccc} 0 \to H_{q+1}(R^{n+1}, R^{n+1}-\operatorname{Int}N) & \xrightarrow{\approx} & H_q(R^{n+1}-\operatorname{Int}N) \to 0 \\ \uparrow \approx & & \uparrow j_* \\ 0 \longrightarrow H_{q+1}(N, \partial N) & \xrightarrow{\partial} & H_q(\partial N) \longrightarrow \ldots \end{array}$$

where $j\colon \partial N \hookrightarrow R^{n+1}-\operatorname{Int}N$, proves that $j_*|\operatorname{Im}\partial$ is an isomorphism from $\operatorname{Im}\partial = \operatorname{Ker} i_*$ onto $H_q(R^{n+1}-\operatorname{Int}N) = H_q(R^{n+1}-K)$. Hence, given $[\alpha] \in H_q(\partial N)$, then $j_*[\alpha] = 0$ if and only if $[\alpha] \in \nu_* H_q(K)$, since $\nu(K)$ is contractible in $R^n\times\{1\} \subset R^{n+1}-K$. ■

THEOREM 1. *Let K_1, K_2 be simply-connected subcomplexes of S^n. Then $S^{n+1}-K_1 \approx S^{n+1}-K_2$ implies $K_1 \approx K_2$.*

Proof. Let N_i be a regular neighborhood of K_i in S^{n+1}, $\bar\pi_i\colon \partial N_i \to K_i$ and $\nu_i\colon K_i \to \partial N_i$ as in Lemma 2. Let $f\colon \partial N_1 \to \partial N_2$ be a homotopy equivalence as in Lemma 1.

Define $h = \bar\pi_2 f \nu_1\colon K_1 \to K_2$.

Since K_1, K_2 are simply-connected finite complexes, it suffices to show that $h_*\colon H_q(K_1) \to H_q(K_2)$ is an isomorphism for every $2 \leqslant q \leqslant n-2$ ([18], Theorem 7.6.25 and Corollary 7.6.24).

Consider the following commutative diagram $(2 \leqslant q \leqslant n-2)$.

$$\begin{array}{ccc} (\nu_1)_* H_q(K_1) \oplus \operatorname{Ker}(\bar\pi_1)_* = H_q(\partial N_1) & \xrightarrow{f_*} & H_q(\partial N_2) = (\nu_2)_* H_q(K_2) \oplus \operatorname{Ker}(\bar\pi_2)_* \\ \uparrow (\nu_1)_* & & (\nu_2)_* \uparrow \downarrow (\bar\pi_2)_* \\ H_q(K_1) & \xrightarrow{h_*} & H_q(K_2)\,. \end{array}$$

Since $(\bar\pi_2)_*|(\nu_2)_* H(K_2)$ is an isomorphism from $(\nu_2)_* H_q(K_2)$ onto $H_q(K_2)$, the proof of Theorem 1 will be complete if we can show that $f_*|(\nu_1)_* H_q(K_1)$ is an isomorphism from $(\nu_1)_* H_q(K_1)$ onto $(\nu_2)_* H_q(K_2)$.

However, this property can be proved from the property of f given in Lemma 1 and the property of the subgroup $(\nu_i)_* H_q(K_i)$ of $H_q(\partial N_i)$, $i = 1, 2$ given in Lemma 2, for $2 \leqslant q \leqslant n-2$. ■

LEMMA 3. *Let X be a globally 1-alg continuum in S^n, $n \geqslant 6$, having the shape of a simply-connected finite complex K, with $\dim K \leqslant n-3$. Then there is a finite subcomplex K' of S^n, $\dim K' \leqslant \dim K$, such that $S^{n+1}-X \approx S^{n+1}-K'$ and $\operatorname{Sh}(X) = \operatorname{Sh}(K')$.*

Proof. By Theorem 3 [11], X has a PL-radial neighborhood W in S^n such that $W \simeq K$. Let $\eta\colon K \to \operatorname{Int} W$ be a homotopy equivalence. By Stallings' theorem ([8], Theorem 12.1), we have a subcomplex K' of $\operatorname{Int} W$ with $\dim K' \leqslant \dim K$ and $K' \hookrightarrow \operatorname{Int} W$ is a homotopy equivalence. Similar to the proof of Corollary 3 in [11], we can prove that W is a regular neighborhood of K'.

It is clear that $N = W\times[-1, 1]$ is a regular neighborhood of K' in S^{n+1}; particularly, $N-K' \overset{\mathrm{PL}}{\approx} \partial N\times[0, 1)$.

Therefore, the lemma will follow if we can prove that N is also a PL-radial neighborhood of X in S^{n+1}. To prove this, it suffices to prove the following statement:

"Let $N_1 = W_1\times[-\frac{1}{2}, \frac{1}{2}]$, where $W_1 = \partial W\times[\frac{1}{2}, 1) \cup X$, then $\overline{N-N_1} \overset{\mathrm{PL}}{\approx} \partial N\times[0, 1]$".

Let K'' be a finite subcomplex of W_1 contained in $\operatorname{Int} W_1$, with $K'' \simeq K$ and $\dim K'' \leqslant \dim K$ such that W_1 is a regular neighborhood of K'' in S^n as above. It is clear that N and N_1 are regular neighborhoods of K'' in S^{n+1}. The statement follows by the uniqueness of regular neighborhoods ([16], Corollary 2.16.2). ■

THEOREM 1'. *Let X and Y be globally 1-alg continua in S^n $(n \geqslant 6)$ having the shape of simply connected finite complexes K_1, K_2, with $\dim K_i \leqslant n-3$, $i = 1, 2$. Then, $S^{n+1}-X \approx S^{n+1}-Y$ implies $\operatorname{Sh}(X) = \operatorname{Sh}(Y)$.*

Proof. By Lemma 3, we have $S^{n+1}-X \approx S^{n+1}-K_1'$ and $S^{n+1}-Y \approx S^{n+1}-K_2'$, where K_1', K_2' are subcomplexes of S^n with $\dim K_i' \leqslant \dim K_i$ and $K_i' \simeq K_i$, $i = 1, 2$.

Hence, $S^{n+1}-X \approx S^{n+1}-Y$ implies that $S^{n+1}-K_1' \approx S^{n+1}-K_2'$.

The result follows from Theorem 1. ■

Now we start to prove the main result.

Consider a simply-connected topological manifold M. Since M is a CANR, M is dominated by a finite simplicial complex. Therefore, by Theorem A [20], M satisfies the condition F_n for every n. On the other hand, M^m also satisfies the condition D_m; hence, by Theorem F [20], $M \simeq K$, where K is a finite CW-complex with $\dim K = \max\{3, m\}$.

In the case $n \geqslant 6$ and $m \leqslant n-3$, then $\dim K \leqslant n-3$. Combining this observation with Theorem 3 of [11], we have the following lemma.

LEMMA 4. *Let M be a codimension 3, closed, simply connected locally flat topological submanifold of S^n $(n \geqslant 6)$, then M has a PL-radial neighborhood W in S^n with $W \simeq M$.* ■

COROLLARY. *Let X and Y be globally 1-alg continua in S^n $(n \geqslant 6)$ having the shape of a closed, $0<(2m-n+1)$-connected topological manifold M^m $(m \leqslant n-3)$. Then $S^n - X \approx S^n - Y$.*

Proof. By Weller's embedding tneorem [21], we may assume that M is a locally flat topological submanifold of S^n. Hence, it will suffice to show that $S^n - X \approx S^n - M$.

By Theorem 3 [11] and the observation above, X has a PL-radial neighborhood W, which has the homotopy type of M.

Let $f\colon M \to \operatorname{Int} W$ be a homotopy equivalence. By Weller's embedding theorem [21], we can assume that f is a locally flat embedding of M into $\operatorname{Int} W \subset S^n$. Now, by topological unknotting theorem [21], it suffices to show that $S^n - X \approx S^n - f(M)$.

Let V be a PL-radial neighborhood of $f(M)$ in S^n such that $V \subset \operatorname{Int} W$ and $f(M) \hookrightarrow V$ is a homotopy equivalence. Let $H = W - \operatorname{Int} V$.

CLAIM. *H is a PL h-cobordism.*

It is clear that H is a PL-manifold with $\partial H = \partial W \cup \partial V$ and $\pi_1(\partial W) = \pi_1(\partial V) = 0$. Moreover, since $f(M) \hookrightarrow V$ and $f(M) \hookrightarrow W$ are homotopy equivalences, $V \hookrightarrow W$ is a homotopy equivalence. Therefore, $H_*(H, \partial V) = 0$ by the excision theorem. Hence, $\pi_*(H, \partial V) = 0$ since $\pi_1(H) = 0$ and [18], Theorem 7.5.4. Theorem 3.2 of [4] shows that H strong deformation retracts onto ∂V.

On the other hand, it is clear that $\partial V \hookrightarrow H - \partial W$ is a homotopy equivalence and that the PL-manifold $H - \partial W$ has a unique tame end ε (∂W is locally flat) with $\pi_1(\varepsilon) = 0$. Thus, we can apply Theorem 1.6 of [16] to conclude that

$$H - \partial W \overset{\mathrm{PL}}{\approx} \partial V \times [0, 1)\,.$$

Now employing this fact and a collar of ∂W in H, it follows that H strong deformation retracts onto ∂W.

Therefore, W is a PL-radial neighborhood of $f(M)$ in S^n by the product structure of H, and the statement $S^n - X \approx S^n - f(M)$ is proved. ■

THEOREM 2. *Let X_1 and X_2 be globally 1-alg continua in S^n $(n \geqslant 6)$ having the shape of codimension 3, closed, $0<(2m_i-n+1)$-connected topological manifolds $M_i^{m_i}$, $i = 1, 2$ (respectively). Then, $S^n - X_1 \approx S^n - X_2$ if and only if $\operatorname{Sh}(X_1) = \operatorname{Sh}(X_2)$.*

Proof. (i) The "if part" is the previous corollary.

(ii) The proof of the "only if part", by Theorem 1, will complete if we can show that there exist finite complexes M_1', M_2' in S^{n-1}, with $\dim M_i' \leqslant n-3$, such that $M_i \simeq M_i'$ $(i = 1, 2)$ and $S^n - M_1' \approx S^n - M_2'$.

Hence, it suffices to show that $S^n - X_i \approx S^n - M_i'$ with such M_i''s.

For the case $n \geqslant 6$ and $m = n-3$, every $0<(2m-n+1)$-connected, closed manifold has the homotopy type of the m-sphere, by the Poincaré duality theorem and the Whitehead theorem. Hence $S^n - X \approx S^n - S^{n-3}$ by the previous corollary.

The case $n = 6$ and $m \leqslant n-4$, is trivial.

For the case $n \geqslant 7$, $m \leqslant n-4$, we may assume that M_i is a locally flat submanifold of S^{n-1} $(n-1 \geqslant 6)$ by Weller's embedding theorem [21]. Finally, by the "if part" and Lemma 3, $S^n - X_i \approx S^n - M_i \approx S^n - M_i'$, as desired. ■

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Commutative rings in which every proper ideal is maximal

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Abstract. We will give the full description of commutative rings in which every proper principal ideal is a prime ideal.

Introduction. Perticani studied in [2] the class of commutative rings with identity in which every proper ideal is maximal. He gave a full description of such a ring R only in the case when R has at least two different proper ideals. In the case where R has only one proper ideal he reduced the problem of characterizing such rings to the one of the computation of cohomology groups. In this paper we will give a full description in both cases. The first case is a trivial conclusion of the Chinese Remainder Theorem and the second will follow very easily from the Cohen Structure Theorem of complete local rings.

All throughout R denotes a commutative ring with identity. We have the same notation as in [3]. The following lemma shows that three classes of rings with pathological properties are only one class and we do not use it in the following.

PROPOSITION 1. *Let R be a ring. Then the following are equivalent:*

1. *every proper ideal is maximal,*
2. *every proper ideal is a primeideal,*
3. *every proper principal ideal is a primeideal.*

Proof. $1\to 2\to 3$ is trivial. To see that $3\to 1$ let A be a proper ideal of R and $a\in A$, $a\neq 0$. Suppose $bc\in A$. If $bc\neq 0$, then $b\in (bc)\subseteq A$ or $c\in (bc)\subseteq A$. If $bc=0$, then $b\in (a)\subseteq A$ or $c\in (a)\subseteq A$. It follows that R/A is an integraldomain. Clearly R/A is a regular ring. Therefore A is a maximal ideal. Q.E.D.

Call a ring R a *max-ring* if every proper ideal is maximal.

LEMMA 2 (see Theorem 1.1 and Theorem 1.4 of [2]). *Suppose R is a max-ring and R contains at least two different proper ideals then R is isomorphic to a product of two fields.*

Proof. Let A_1, A_2 be proper ideals of R and $A_1\neq A_2$. It follows immediately that $A_1\cap A_2=(0)$. Since A_1, A_2 are comaximal it follows from the Chinese