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On the classification of locally compact separable metric spaces

by

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Abstract. Let M be a locally compact separable metric space. Let P, C, and N be the space of irrationals, the Cantor set, and the positive integers, respectively. For any space X let $H(X)$ be the homeomorphism group. The existence of standard maps of P onto M and $C\times N$ onto M is established. This provides a classification of the locally compact separable metric spaces by considering certain subgroups of $H(P)$ and $H(C\times N)$.

1. Introduction. A result due to A. R. Vobach [3] completely classifies the compact metric spaces in terms of certain subgroups of the Cantor set. In view of the techniques used by Vobach, it is reasonable to wonder if such results are possible for other classes of spaces with the Cantor set replaced by some other universal space. In this note, we show that this is the case for locally compact separable metric spaces and the irrationals. The author is indebted to Professor Vobach for several valuable suggestions.

2. Preliminaries. Let N be the positive integers with the discrete topology, C any Cantor set, C' any Cantor set with a single point removed, and P the space of irrationals on the real line with the subspace topology. For any space X, let $H(X)$ denote the full homeomorphism group. Finally, map and $\simeq$ will mean continuous function and homeomorphic, respectively.

LEMMA 2.1. *If $A_i = N$ for each i, then $\Pi\{A_i\colon i\in N\}$, $C\times P$ and P are homeomorphic.*

Proof. To see that $\Pi\{A_i\colon i\in N\}$ and P are homeomorphic we refer the reader to [2], p. 25, Example 2.

Let $B_i = \{0,2\}$. Then $\Pi\{B_i\colon i\in N\}\simeq C$ and $B_i\times N\simeq N$. Therefore

$$C\times P\simeq \Pi\{B_i\colon i\in N\}\times \Pi\{A_i\colon i\in N\}\simeq \Pi\{B_i\times A_i\colon i\in N\}\simeq \Pi\{A_i\colon i\in N\}\simeq P.$$

LEMMA 2.2. *Let X, Y and Z be spaces $F\colon X\to Y$ an identification, and $G\colon X\to Z$ continuous. If GF^{-1} is single-valued, then GF^{-1} is continuous.*

* This material will appear in the author's doctoral dissertation, under the direction of Paul F. Duvall, Jr. at Oklahoma State University.

Proof. See [1], p. 123, Theorem 3.2.

DEFINITION 2.3 (A. R. Vobach, [3]). Let X and Y be spaces and f a map of X onto Y. Define $G(f, X, Y) = \{h \in H(X): fh = f\}$. If there is no danger of confusion we will simply write $G(f, Y)$. $G(f, Y)$ is a subgroup of $H(X)$.

DEFINITION 2.4. Let X and Y be spaces and f a map of X onto Y. f will be called a *standard map* if f is an identification and $a \neq b$, $f(a) = f(b)$ implies there are sequences $\{x_n\}$ in X, $\{h_n\}$ in $G(f, X, Y)$ such that $x_n \to a$ and $h_n(x_n) \to b$.

REMARKS. In [3], Vobach does not require a standard map to be an identification. However, Vobach's only concern is the case when $X = C$ and Y is compact and metrizable. Obviously, any map is closed and therefore an identification. We note, however, the hypothesis that f is an identification is necessary in the general case. This will become evident later on.

LEMMA 2.5. *C' and $C \times N$ are homeomorphic.*

Proof. We assume without loss of generality that C is the standard "middle-thirds" Cantor set and that $C' = C - \{1\}$. Let $\{C_j\}$ be a pairwise disjoint collection of Cantor sets such that $C' = \bigcup \{C_j: j \in N\}$. For each n in N let $h_n: C \times \{n\} \to C_n$ be a homeomorphism. Define h from $C \times N$ to C' by $h(c, n) = h_n(c)$. If $h(c, n) = h(d, m)$, then clearly $m = n$ and therefore $c = d$. Let U be open in C'. Then $U = \bigcup \{C_j \cap U: j \in N\}$. We have

$$h^{-1}(U) = \bigcup \{h^{-1}(U \cap C_j): j \in N\} = \bigcup \{h_j^{-1}(U \cap C_j): j \in N\}$$

which is open. Let V be open in $C \times N$. Then $V = \bigcup \{V_n \times \{k_n\}: n \in N\}$ and

$$h(V) = \bigcup \{h(V_n \times \{k_n\}: n \in N\} = \bigcup \{h_{k_n}(V_n): n \in N\}$$

which is open in C, since each C_n is both open and closed in C, and hence open in C'. Therefore h is a homeomorphism.

LEMMA 2.6 (A. R. Vobach, [3]). *Let M be a compact metric space and C a Cantor set. Then there is a standard map of C onto M.*

Proof. See [3].

LEMMA 2.7. *Let X, Y and Z be spaces. If $h: X \to Y$ is a homeomorphism and $f: Y \to Z$ is standard, then $fh: X \to Z$ is standard.*

Proof. Clearly fh is an identification since f is an identification. If $f(h(a)) = f(h(b))$, then there are sequences $\{y_n\}$ in Y and $\{h_n\}$ in $G(f, Y, Z)$ such that $y_n \to h(a)$ and $h_n(y_n) \to h(b)$. Define $x_n = h^{-1}(y_n)$ and $k_n = h^{-1}h_n h$. Then $x_n \to a$ and $h^{-1}(h_n(h(x_n))) = h^{-1}(h_n(y_n)) \to b$. $h^{-1}h_n h$ is in $G(fh, X, Z)$ as required.

LEMMA 2.8. *Let M be a locally compact separable metric space. Then there is a standard map of C' onto M.*

Proof. Let M^* be the one-point compactification of M and q the ideal point. Let $\{T_j^1\}_{j=1}^{n(1)}, \{T_j^2\}_{j=1}^{n(2)}, \ldots,$ be a sequence of finite closed covers of M^* satisfying

1. mesh$\{T_j^k\}_{j=1}^{n(k)} \leqslant 1/k$,

2. $T_j^k \cap T_i^j \neq \emptyset$ is the union of two or more elements of the $(k+1)$-st cover, and
3. for each k, q belongs to exactly one element of $\{T_j^k\}_{j=1}^{n(k)}$.

For notational convenience we list part of the construction which appears in [3].

Divide the closed unit interval $[0, 1]$ into $2n(1)$ equal subintervals and label every other one, including end-points, as $E_1^1, E_2^1, \ldots, E_{n(1)}^1$. Suppose we have $E_{i(1),\ldots,i(k)}^k$, where the sequence $i(1), \ldots, i(k)$ is such that $T_{i(1)}^1 \supset \ldots \supset T_{i(k)}^k$. We divide it into $2m(i(k))$ equal subintervals, where $m(i(k))$ is the number of elements of $\{T_i^{k+1}\}_{i=1}^{n(k+1)}$ which are contained in $T_{i(k)}^k$. Denote every second one of these intervals by $E_{i(1),\ldots,i(k),j(1)}^{k+1}$, $E_{i(1),\ldots,i(k),j(2)}^{k+1}, \ldots, E_{i(1),\ldots,i(k),j(m(i(k)))}^{k+1}$, where the $j(r)$'s, $r = 1, \ldots, m(i(k))$, are the subscripts of the T_i^{k+1} that are contained in $T_{i(k)}^k$.

Set $S_k = \bigcup E_{i(1),\ldots,i(k)}^k$ for all sequences $i(1), \ldots, i(k)$ for which $T_{i(1)}^1 \supset T_{i(2)}^1 \supset \ldots \supset T_{i(k)}^k$. Set $C = \bigcup \{S_k: k \in N\}$. C is a Cantor set.

Now, we define $F_1(s) = T_{i(1)}^k$ for s in $C \cap E_{i(1)}^1$. Similarly, $F_2(s) = T_{i(2)}^2$ for s in $C \cap E_{i(1),i(2)}^2$ where $T_{i(1)}^1 \supset T_{i(2)}^2$. In general, set $F_n(s) = T_{i(n)}^n$ where s is in $C \cap \cap E_{i(1),\ldots,i(n)}^n$ and $T_{i(1)}^1 \supset \ldots \supset T_{i(n)}^n$. Define $p(s) = \bigcap \{F_n(s): n \in N\}$. The remainder of the discussion in [3] is devoted to showing that p is a standard map. Given that p is standard we assert that $p^{-1}(q)$ is a singleton and that $p' \equiv p|_{C-p^{-1}(q)}$ is a standard map of $C' = C - p^{-1}(q)$ onto M. Suppose $p(a) = p(b) = q$. Then there are sequences $i(1), i(2), \ldots$ and $j(1), j(2), \ldots$ such that

$$a = \bigcap \{E_{i(1),i(2),\ldots,i(k)}^k: k \in N\}, \quad b = \bigcap \{E_{j(1),j(2),\ldots,j(k)}^k: k \in N\},$$

and

$$q = \bigcap \{T_{i(k)}^k: k \in N\} = \bigcap \{T_{j(k)}^k: k \in N\}.$$

Therefore, by 3, $i(k) = j(k)$ for all k and hence $a = b$. Therefore $p^{-1}(q)$ is a single point. Certainly $p': C' \to M$ is continuous and onto. We now show that p' is standard. For each h in $G(p, C, M^*)$ let $h' \equiv h|_{C'}$. $h' \in G(p', C', M)$ since $h(p^{-1}(q)) = p^{-1}(q)$. If $p'(a) = p'(b)$, then $p(a) = p(b)$ and so there are sequences $\{x_n\}$ in C and $\{h_n\}$ in $G(p, C, M^*)$ such that $x_n \to a$ and $h_n(x_n) \to b$. Select a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to a and is contained in C'. Let h'_{n_k} be defined as on the previous page. Then $h'_{n_k}(x_{n_k}) \to b$ and $h'_{n_k} \in G(p', C', M)$ as required.

It remains to show that p' is an identification. Let $U \subset M$ and suppose $p'^{-1}(U)$ is open in C'. Then $p'^{-1}(U)$ is open in C since C' is open in C. But $p'^{-1}(U) = p^{-1}(U)$ and p is an identification. Thus U is open in M^* and $q \notin U$. Therefore U is open in M and p' is an identification.

Remark. Note that Lemmas 2.5 and 2.7 guarantee the existence of a standard map of $N \times C$ onto any locally compact separable metric space.

LEMMA 2.9. *Let X be a homogeneous space and f a standard map of Y onto Z. Then $f \circ \Pi: X \times Y \to Z$ is standard where Π is the projection map on the Y-coordinate.*

Proof. Suppose $f(\Pi(x, y)) = f(\Pi(a, b))$. Then $f(y) = f(b)$ and so there are x_n in Y and h_n in $G(f, Y, Z)$ such that $x_n \to y$ and $h_n(x_n) \to b$. Let $g \in H(X)$ such that $g(x) = a$. Then $H_n \equiv (g, h_n) \in G(f \circ \Pi, X \times Y, Z)$, $(x, x_n) \to (x, y)$ and $H_n(x, x_n) \to (a, b)$. $f \circ \Pi$ is an identification since f is an identification and Π is an open map (by virtue of the fact that it is a projection map). Thus $f \circ \Pi$ is a standard map.

LEMMA 2.10. *Let M be a locally compact separable metric space. Then there is a standard map of P onto M.*

Proof. By Lemmas 2.8 and 2.7 there is a standard map of $N \times C$ onto M. By Lemmas 2.1 and 2.9 there is a standard map of P onto M.

LEMMA 2.11. *Let X, Y and Z be topological spaces and f a standard map of X onto Y. Then Y and Z are homeomorphic if and only if there is a standard map f of X onto Z satisfying $G(f, Y) = G(g, Z)$.*

Proof. Suppose h: $Y \to Z$ is a homeomorphism. Then hf: $X \to Z$ is a standard map and $G(f, Y) = G(hf, Z)$.

On the other hand, let us assume there is a standard map g of X onto Z such that $G(f, Y) = G(g, Z)$. Define h: $Y \to Z$ by $h(y) = g(f^{-1}(y))$. By Lemma 2.2 it suffices to show that h is single-valued. Let $f(a) = f(b)$. Then there exists x_n in X and h_n in $G(f, Y)$ such that $x_n \to a$ and $h_n(x_n) \to b$. Therefore, $g(a) = \lim g(x_n) = \lim g(h_n(x_n)) = g(b)$. Thus, h is single-valued and hence continuous. Similarly, h^{-1} is continuous. It follows that h is a homeomorphism.

3. Main results. The real labor has already been done. We will now state the main theorems and indicate their proofs.

THEOREM 3.1. *Let X and Y be locally compact separable metric spaces and f: $N \times C \to X$ a standard map. Then X and Y are homeomorphic if and only if there exists a standard map g: $N \times C \to Y$ satisfying $G(f, X) = G(g, Y)$.*

Proof. By Lemma 2.8 we know that our hypothesis concerning the existence of the standard map f is non-vacuous. The theorem follows from Lemma 2.11.

THEOREM 3.2. *Let X and Y be locally compact separable metric spaces and f a standard map of P onto X. Then X and Y are homeomorphic if and only if there is a standard map g of P onto Y satisfying $G(f, X) = G(g, Y)$.*

Proof. By Lemma 2.10 we know that our hypothesis concerning the existence of the standard map f is non-vacuous. The theorem follows from Lemma 2.11.

COROLLARY 3.3. *Let f and g be standard maps of P $\{N \times C\}$ onto locally compact separable metric spaces X and Y. Then X and Y are homeomorphic if there is a homeomorphism h in $H(P)\{H(N \times C)\}$ such that*

$$G(f, P, X) = hG(g, P, Y)h^{-1}\{G(f, N \times C, X) = hG(g, N \times C, Y)h^{-1}\}.$$

Proof. The proof of a similar corollary in [3] is sufficient in view of Lemmas 2.8 and 2.10.

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