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Some uniformization results

by

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Abstract. Some results on the uniformization of Borel sets are proved in this paper.

1. Introduction. Let X, Y be Polish spaces and $B \subseteq X\times Y$. We say C *uniformizes* B if $C\subseteq B$ and for all $x\in \mathrm{pr}_X B$, $C\cap B^x$ is a singleton where $B^x = \{y\colon (x,y)\in B\}$ and $\mathrm{pr}_X B$ is the projection of B to X. In general, a Borel set B does not have a Borel uniformization ([2], [6]). However, in some cases, such a uniformization exists, for example, if B^x is σ-compact for each x [1] or if $\mu(B^x)>0$ for each x where μ is a probability measure on the Borel σ-algebra of Y [3].

The chief aim of this paper is to prove the following:

THEOREM 1. *Let X, Y be Polish spaces and $B\subseteq X\times Y$ be Borel. B has a Borel uniformization if any one of the following is true:*

(1) *for all $x\in \mathrm{pr}_X B$, B^x contains an isolated point,*

(2) *for all $x\in \mathrm{pr}_X B$, B^x contains a point which is not its point of condensation,*

(3) *for all $x\in \mathrm{pr}_X B$, B^x is not meager.*

The paper is organized in the following way. Section 2 is devoted to preliminaries. In Section 3, a proof of Theorem 1 is given. In Section 4, a related result is proved.

2. Preliminaries. A set is called *meager* if it is a countable union of nowhere dense sets. A comeager set is one whose complement is meager. Let X, Y be Polish spaces, $B\subseteq X\times Y$ and $U\subseteq Y$. Following Vaught, we put $B_U^* = \{x\colon B^x\cap U$ is comeager in $U\}$. It is known that if B is Borel and U open, then B_U^* is Borel [7]. For any set A, let $\delta(A)$ denote the diameter of A.

If f is a function, put $Z_f = \{y\colon f^{-1}(y)$ is a singleton$\}$, $I_f = \{y\colon f^{-1}(y)$ contains an isolated point$\}$, $D_f = \{y\colon f^{-1}(y)$ is countable and non-empty$\}$, $C_f = \{y\colon f^{-1}(y)$ contains a point which is not its condensation point$\}$. It is known that if f is a Borel measurable function defined on a Borel subset of a Polish space into a separable metric space, then Z_f, I_f, D_f, C_f are coanalytic [4].

3. Proof of main theorem.

Proof of (1). Let $\{V_n\}$ be a countable open base for Y. For any n, define f_n on $B\cap(X\times V_n)$ by $f_n(x,y) = x$. Let $Z_n = \{x\colon B^x\cap V_n$ is a singleton$\}$. Then $Z_n = Z_{f_n}$

is coanalytic and $\mathrm{pr}_X B = \bigcup_n Z_n$ is Borel. Let $B_n \subseteq Z_n$ be disjoint Borel sets such that $\mathrm{pr}_X B = \bigcup_n B_n$. Let $C = \bigcup_n ((B_n \times V_n) \cap B)$. Then C is a Borel uniformization of B.

Proof of (2). Let $\{V_n\}$ and $\{f_n\}$ be defined as above and, for any n, let $Z_n = \{x\colon B^x \cap V_n$ is countable and non-empty$\}$. Then $Z_n = D_{f_n}$ is coanalytic and $\mathrm{pr}_X B = \bigcup_n Z_n$ is Borel. Let B_n be as before. Let $D = \bigcup_n ((B_n \times V_n) \cap B)$. Then $D \subseteq B$ is a Borel set such that for $x \in \mathrm{pr}_X B$, D^x is non-empty and countable. D can therefore be uniformized by a Borel set C, which also uniformizes B.

Remark. Proofs of (1) and (2) illustrate the use of the reduction principle in proving selection theorems. For a detailed exposition see [5].

Proof of (3). This follows from:

THEOREM 2. *If X, Y are Polish spaces and $B \subseteq X \times Y$ is a Borel set such that for all $x \in \mathrm{pr}_X B$, B^x is comeager, then B has a Borel uniformization.*

Assuming Theorem 2, we prove (3) as follows: Suppose $B \subseteq X \times Y$ is a Borel set such that for all $x \in \mathrm{pr}_X B$, B^x is not meager. Let $\{V_n\}$ be a countable open base for Y. Put $D_n = B^*_{V_n} - \bigcup_{m<n} B^*_{V_m}$ for all n. By Theorem 2, the Borel subset $B \cap (D_n \times V_n)$ of $X \times V_n$ can be uniformized by a Borel set C_n. As $\mathrm{pr}_X B = \bigcup_n B^*_{V_n} = \bigcup_n D_n$ [7], $C = \bigcup_n C_n$ uniformizes B. This concludes the proof of Theorem 1.

(1) and (2) yield the following:

COROLLARY. *Let X be absolutely Borel and Y a separable metric space. Let $f\colon X \to Y$ be Borel measurable. If $Y = I_f$ or C_f, then f admits a Borel selector, i.e., there is a Borel subset B of X such that f restricted to B is one-to-one and $f(B) = f(X) = Y$.*

Remark. (3) is a category analogue of the result of Blackwell and Ryll-Nardzewski [3] referred to in the introduction.

Proof of Theorem 2.

LEMMA. *Let X, Y be Polish spaces and $B \subseteq X \times Y$ be Borel. Given any open subset U of Y, there is a sequence $\{Z_k\}$ of subsets of $X \times Y$ satisfying the following:*

(a) Each Z_k is a Borel subset of $X \times U$.

(b) $\bigcap_k Z_k \subseteq B$.

(c) Given any non-empty open $W \subseteq U$, any k and any $\varepsilon > 0$, there is a Borel set $F \subseteq Z_k \cap (X \times W)$ such that for all x, F^x is closed, $\delta(F^x) < \varepsilon$ and if $x \in B^*_U$, then F^x is not meager.

Proof. Let $\underline{\underline{M}} = \{B \subseteq X \times Y\colon B$ is Borel and satisfies the above$\}$. We show that $\underline{\underline{M}}$ contains all Borel sets.

Step 1. $\underline{\underline{M}}$ contains closed sets.

Let $\{W_m\}$ and $\{V_n\}$ be countable open bases for X and Y respectively. Let $B \subseteq X \times Y$ be closed. Then there are open sets $U_k \subseteq X \times Y$, $k = 1, 2, \ldots$ such that $B = \bigcap_k U_k$. Let $U \subseteq Y$ be open. Put $Z_k = U_k \cap (X \times U)$ for all k. Clearly, $\{Z_k\}$ satisfies (a) and (b). To see that it also satisfies (c), fix any k, any $\varepsilon > 0$ and any non-empty open subset W of U. We shall construct F so that (c) is satisfied.

$Z_k \cap (X \times W) = U_k \cap (X \times W)$ is open and hence a countable union of sets of the form $W_m \times V_n$. Let $L = \{m\colon W_m$ appears in this union$\}$. Corresponding to each $m \in L$, choose exactly one n such that $W_m \times V_n$ appears in this union and let $V_m \neq \emptyset$ satisfy $V_m \subseteq \overline{V}_m \subseteq V_n$ and $\delta(V_m) < \varepsilon$. Let

$$F = \bigcup_{m \in L} \Big(W_m - \bigcup_{\substack{n<m\\ n \in L}} W_n\Big) \times \overline{V}_m .$$

Step 2. $\underline{\underline{M}}$ is closed under countable intersections.

Let $B_n \in \underline{\underline{M}}$ for $n = 1, 2, \ldots$ and let $U \subseteq Y$ be open. For each n, let the sequence of sets Z_{nk}, $k = 1, 2, \ldots$ satisfy (a), (b), (c) if B is replaced by B_n and Z_k by Z_{nk}. Rearrange the double sequence $\{Z_{nk}\}$ in the form of a simple sequence $\{Z_k\}$. Then $\{Z_k\}$ satisfies (a), (b), (c) if $B = \bigcap_n B_n$.

Step 3. $\underline{\underline{M}}$ is closed under countable unions.

Let $B_n \in \underline{\underline{M}}$ for $n = 1, 2, \ldots$ and let $U \subseteq Y$ be open. We construct a sequence $\{Z_k\}$ satisfying (a), (b), (c) if $B = \bigcup_n B_n$.

Let $\{V_m\}$ be a countable open base for U. For each m, n, let Z_{nmk}, $k = 1, 2, \ldots$ satisfy (a), (b), (c) with B replaced by B_n, U by V_m and Z_k by Z_{nmk}. For all n, m, k put

$$D_{nm} = B^*_{nV_m} - \bigcup_{j<n} B^*_{jV_m}, \qquad E_m = \bigcup_n D_{nm} = \bigcup_n B^*_{nV_m},$$

$$Z_{mk} = \bigcup_n (Z_{nmk} \cap (D_{nm} \times Y)), \qquad Z_k = \bigcup_m \Big(Z_{mk} - \bigcup_{i<m} (E_i \times V_i)\Big).$$

Clearly, Z_k is a Borel subset of $X \times U$ for each k.

$$\bigcap_k Z_k = \bigcap_k \bigcup_m \Big(Z_{mk} - \bigcup_{i<m} (E_i \times V_i)\Big) = \bigcup_m \bigcap_k \Big(Z_{mk} - \bigcup_{i<m} (E_i \times V_i)\Big)$$

since $Z_{mk} \subseteq E_m \times V_m$ for all k and m.

For any m,

$$\bigcap_k \Big(Z_{mk} - \bigcup_{i<m} (E_i \times V_i)\Big) = \bigcap_k \bigcup_n \Big(Z_{nmk} \cap (D_{nm} \times Y) - \bigcup_{i<m} (E_i \times V_i)\Big)$$
$$= \bigcup_n \bigcap_k \Big(Z_{nmk} \cap (D_{nm} \times Y) - \bigcup_{i<m} (E_i \times V_i)\Big)$$

since D_{nm}, $n = 1, 2, \ldots$ is a disjoint sequence.

Thus

$$\bigcap_k Z_k \subseteq \bigcup_m \bigcup_n \bigcap_k Z_{nmk} \subseteq \bigcup_n B_n .$$

Fix any k, any $\varepsilon > 0$ and any non-empty open subset W of U. For all m, n, put

$$H_m = E_m - \bigcup_{\substack{i<m\\ W \cap V_i \neq \emptyset}} E_i \quad \text{and} \quad G_{nm} = H_m \cap D_{nm} .$$

For all m such that $W \cap V_m \neq \emptyset$ and all n, choose a Borel set $F_{nm} \subseteq Z_{nmk} \cap \cap (X \times (W \cap V_m))$ such that for all x, F_{nm}^x is closed, $\delta(F_{nm}^x) < \varepsilon$ and if $x \in B_{nV_m}^*$, then F_{nm}^x is not meager. Let

$$F = \bigcup_n \bigcup_{\substack{m \\ W \cap V_m \neq \emptyset}} ((G_{nm} \times Y) \cap F_{nm}).$$

F is clearly a Borel subset of $X \times W$. To see that $F \subseteq Z_k$, let $(x, y) \in F$. There exist unique m and n such that $W \cap V_m \neq \emptyset$ and $(x, y) \in (G_{nm} \times Y) \cap F_{nm}$. As $F_{nm} \subseteq Z_{nmk}$ and $G_{nm} \subseteq D_{nm}$, $(x, y) \in Z_{nmk} \cap (D_{nm} \times Y) \subseteq Z_{mk}$. Let $i < m$. If $W \cap V_i \neq \emptyset$, $x \notin E_i$ and if $W \cap V_i = \emptyset$, $y \notin V_i$ since $(x, y) \in F_{nm} \subseteq X \times W$ implies $y \in W$. Thus $(x, y) \notin \bigcup_{i<m} (E_i \times V_i)$. Hence $(x, y) \in Z_k$.

Clearly F^x is closed and $\delta(F^x) < \varepsilon$ for all x. Let $(\bigcup_n B_n)^x \cap U$ be comeager in U. We show that there is some m, n such that $W \cap V_m \neq \emptyset$ and $x \in G_{nm}$. Then $F^x = F_{nm}^x$ and $x \in B_{nV_m}^*$. Hence F^x is not meager.

It is enough to show that there is an m such that $W \cap V_m \neq \emptyset$ and $x \in E_m$. Clearly, $(\bigcup_n B_n)^x \cap U$ is comeager in U implies $(\bigcup_n B_n)^x \cap W$ is comeager in W so that there is some n for which $B_n^x \cap W$ is not meager. Hence there is a $V_m \subseteq W$ such that $x \in B_{nV_m}^* \subseteq E_m$.

Proof of the theorem. Let $\{V_n\}$ be a countable base for Y. Let $\{Z_k\}$ satisfy (a), (b), (c) of the lemma if U is replaced by Y. Let $C_1 \subseteq Z_1$ be a Borel set such that for all x, C_1^x is closed, $\delta(C_1^x) < 1$ and for $x \in \mathrm{pr}_X B$, C_1^x is not meager.

For all n, put $H_{1n} = C_{1V_n}^* - \bigcup_{m<n} C_{1V_m}^*$ and choose a Borel set $F_{2n} \subseteq Z_2 \cap \cap (X \times V_n)$ such that for all x, F_{2n}^x is closed, $\delta(F_{2n}^x) < \frac{1}{2}$ and $x \in \mathrm{pr}_X B$ implies F_{2n}^x is not meager. Let $C_2 = \bigcup_n ((H_{1n} \times Y) \cap F_{2n})$. Then C_2 is a Borel subset of Z_2. Also $C_2 \subseteq C_1$, for if $x \in H_{1n}$, $x \in C_{1V_n}^*$ and hence $V_n \subseteq C_1^x$ as C_1^x is closed; thus $(H_{1n} \times Y) \cap \cap F_{2n} \subseteq H_{1n} \times V_n \subseteq C_1$ for all n. Clearly, for all x, C_2^x is closed and $\delta(C_2^x) < \frac{1}{2}$. Let $x \in \mathrm{pr}_X B$. Then C_1^x is not meager and therefore $x \in \bigcup_n C_{1V_n}^* = \bigcup_n H_{1n}$. Hence $C_2^x = F_{2n}^x$ for some n and therefore is non-meager. By induction, we define a descending sequence $\{C_k\}$ of Borel sets such that for all k, $C_k \subseteq Z_k$ and for all x, C_k^x is closed, $\delta(C_k^x) < 1/k$ and $x \in \mathrm{pr}_X B$ implies C_k^x is non-meager. $C = \bigcap_k C_k$ uniformizes B.

4. A related result.

THEOREM 3. *Let X, Y be Polish spaces and $B \subseteq X \times Y$ be a Borel set such that for all $x \in \mathrm{pr}_X B$, B^x is comeager. Then there exist a sequence $\{Z_k\}$ of Borel sets in $X \times Y$ such that $\bigcap_k Z_k \subseteq B$ and for all k and x, Z_k^x is open in Y and $x \in \mathrm{pr}_X B$ implies Z_k^x is dense (and hence comeager) in Y.*

Remark. Theorem 2 follows from Theorem 3 and the reduction principle. Theorem 3 is a particular case of:

THEOREM 4. *Let X, Y be Polish spaces and $B \subseteq X \times Y$ be a Borel set. Given any open $U \subseteq Y$, there is a sequence $\{Z_k\}$ of sets such that*

(a) *Z_k is a Borel subset of $X \times U$ for each k.*

(b) *$\bigcap_k Z_k \subseteq B$.*

(c) *For all k and x, Z_k^x is an open subset of Y and if $x \in B_U^*$, then Z_k^x is comeager in U.*

Proof. Let $\underline{M} = \{B \subseteq X \times Y$: B is Borel and satisfies the above$\}$. We show that $\underline{M}$ contains all Borel subsets of $X \times Y$.

Step 1. Clearly $\underline{M}$ contains all G_δ sets and hence all closed sets.

Step 2. It is easy to see that $\underline{M}$ is closed under countable intersections.

Step 3. $\underline{M}$ is closed under countable unions.

Let $B_n \in \underline{M}$ for $n = 1, 2, \ldots$ and let $U \subseteq Y$ be open. Let $\{V_m\}$ be a countable open base for U. For any fixed m, n let Z_{nmk}, $k = 1, 2, \ldots$ satisfy (a), (b), (c) if B is replaced by B_n, U by V_m and Z_k by Z_{nmk}. For all m and k, define E_m and Z_{mk} as in the lemma of Section 3 and let $Z_k = \bigcup_m (Z_{mk} - \bigcup_{i<m} (E_i \times \overline{V}_i))$. It is easy to see that $\{Z_k\}$ satisfies (a) and (b) if $B = \bigcup_n B_n$ and that for all k and x, Z_k^x is open. Let $x \in (\bigcup_n B_n)_U^*$. To show that Z_k^x is comeager in U, it is enough to show that it is dense in U. We prove this in two steps.

Step 1. $V_x = \bigcup_{\substack{m \\ x \in E_m}} V_m$ is dense in U.

Step 2. $\overline{Z_k^x} \supseteq V_x$.

Proof of Step 1. As $x \in (\bigcup_n B_n)_U^*$, $x \in (\bigcup_n B_n)_{V_m}^*$ for all m. Thus given m, there is some n such that $B_n^x \cap V_m$ is not meager and hence there is some $V_s \subseteq V_m$ such that $x \in B_{nV_s}^* \subseteq E_s$. Now, $V_s \subseteq V_x \cap V_m$ so that $V_x \cap V_m \neq \emptyset$.

Proof of Step 2.

$$Z_k^x = \bigcup_m (Z_{mk}^x - \bigcup_{\substack{i<m \\ x \in E_i}} \overline{V}_i) = \bigcup_{\substack{m \\ x \in E_m}} (Z_{mk}^x - \bigcup_{\substack{i<m \\ x \in E_i}} \overline{V}_i).$$

If $x \in E_m$, there is an n for which $x \in D_{nm}$ so that $Z_{mk}^x = Z_{nmk}^x$. Hence $\overline{Z_{mk}^x} \supseteq \overline{V}_m$ and therefore $\overline{Z_{mk}^x} \supseteq \overline{V}_m$. Hence

$$\overline{(Z_{mk}^x - \bigcup_{\substack{i<m \\ x \in E_i}} \overline{V}_i)} \supseteq \overline{V}_m - \bigcup_{\substack{i<m \\ x \in E_i}} \overline{V}_i.$$

Thus

$$\overline{Z_k^x} \supseteq \bigcup_{\substack{m \\ x \in E_m}} \overline{V}_m \supseteq V_x.$$

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On the classification of locally compact separable metric spaces

by

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Abstract. Let M be a locally compact separable metric space. Let P, C, and N be the space of irrationals, the Cantor set, and the positive integers, respectively. For any space X let $H(X)$ be the homeomorphism group. The existence of standard maps of P onto M and $C\times N$ onto M is established. This provides a classification of the locally compact separable metric spaces by considering certain subgroups of $H(P)$ and $H(C\times N)$.

1. Introduction. A result due to A. R. Vobach [3] completely classifies the compact metric spaces in terms of certain subgroups of the Cantor set. In view of the techniques used by Vobach, it is reasonable to wonder if such results are possible for other classes of spaces with the Cantor set replaced by some other universal space. In this note, we show that this is the case for locally compact separable metric spaces and the irrationals. The author is indebted to Professor Vobach for several valuable suggestions.

2. Preliminaries. Let N be the positive integers with the discrete topology, C any Cantor set, C' any Cantor set with a single point removed, and P the space of irrationals on the real line with the subspace topology. For any space X, let $H(X)$ denote the full homeomorphism group. Finally, map and $\simeq$ will mean continuous function and homeomorphic, respectively.

LEMMA 2.1. *If $A_i = N$ for each i, then $\Pi\{A_i\colon i\in N\}$, $C\times P$ and P are homeomorphic.*

Proof. To see that $\Pi\{A_i\colon i\in N\}$ and P are homeomorphic we refer the reader to [2], p. 25, Example 2.

Let $B_i = \{0, 2\}$. Then $\Pi\{B_i\colon i\in N\}\simeq C$ and $B_i\times N\simeq N$. Therefore

$$C\times P\simeq \Pi\{B_i\colon i\in N\}\times \Pi\{A_i\colon i\in N\}\simeq \Pi\{B_i\times A_i\colon i\in N\}\simeq \Pi\{A_i\colon i\in N\}\simeq P.$$

LEMMA 2.2. *Let X, Y and Z be spaces $F\colon X\to Y$ an identification, and $G\colon X\to Z$ continuous. If GF^{-1} is single-valued, then GF^{-1} is continuous.*

* This material will appear in the author's doctoral dissertation, under the direction of Paul F. Duvall, Jr. at Oklahoma State University.