It is an interesting question whether the following generalization of the Bruckner–Świątkowski theorem is true:

If a function f is a Baire class 1 function with the Darboux property, T satisfies Khintchine's condition n.e., f'_T exists n.e. and $f'_T \geq 0$ a.e. in (a, b), then f is non-decreasing and continuous in (a, b).

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Decomposition spaces and shape in the sense of Fox

by

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Abstract. It is proved in the paper that if X, Y are finite dimensional metrizable spaces, $f\colon X \to Y$ is a closed continuous map such that $f^{-1}(y)$ is approximatively k-connected for $y \in Y$ and $k = 0, 1, \ldots, \dim Y$, then $\mathrm{Sh}(X) \geq \mathrm{Sh}(Y)$ (in the sense of Fox [5]). By applying the theorem it is shown that for every finite dimensional locally compact metric space X there exists a Δ-space Y such that $\dim X = \dim Y$, $\mathrm{Sh}_W(X) = \mathrm{Sh}_W(Y)$ and $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

§ 1. Introduction. In [5] Fox introduced the notion of shape for metric spaces and proved that for compacta this notion coincides with the notion of shape in the sense of Borsuk [4]. In the previous paper [9] we proved that a certain decomposition map induces a weak shape equivalence. The purpose of this paper is to prove that a similar theorem holds for shape in the sense of Fox. Let X be a finite dimensional metric spaces and let $\mathscr{D}$ be an upper semicontinuous decomposition of X each element of which is a closed set being approximatively k-connected for $k = 0, 1, \ldots, \max(\dim X, \dim Y)$. Then we shall show that the equality $\mathrm{Sh}(X) = \mathrm{Sh}(X_{\mathscr{D}})$ holds, where $X_{\mathscr{D}}$ is the decomposition space of X by $\mathscr{D}$ and $\mathrm{Sh}(X)$ is the shape of X in the sense of Fox. As an application of this theorem we can obtain a generalization of Ball's theorem [1]. Finally, we shall prove that for every finite dimensional and locally compact metric space X there is a Δ-space Y such that $\dim X = \dim Y$, $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ and $\mathrm{Sh}_W(X) = \mathrm{Sh}_W(Y)$, where $\mathrm{Sh}_W(X)$ is the weak shape of X defined by Borsuk [3].

Throughout this paper all of spaces are metrizable and maps are continuous. By an AR-space and an ANR-space we mean always those for metric spaces and by dimension we mean the covering dimension.

§ 2. The shape in the sense of Fox. We first recall the basic notions introduced by Fox [5]. Let X and Y be metric spaces and let M and N be AR-spaces containing X and Y as closed sets respectively. By $U(X, M)$ we mean the inverse system consisting of open neighborhoods U of X in M and all inclusion maps $u\colon U' \to U$, $U' \subset U$. Similarly, by $V(Y, N)$ denote the inverse system of open neighborhoods of U in N. A *mutation* $f\colon U(X, M) \to V(Y, N)$ from $U(X, M)$ to $V(Y, N)$ is defined as a collection of maps $f\colon U \to V$, $U \in U(X, M)$, $V \in V(Y, N)$, such that

(2.1) if $f \in \boldsymbol{f}$ and $u\colon U' \to U$ and $v\colon V \to V'$ are inclusions of U' and V into U and V', $U, U' \in \boldsymbol{U}(X, M)$ and $V, V' \in \boldsymbol{V}(Y, N)$, then $vfu \in \boldsymbol{f}$;

(2.2) every neighborhood $V \in \boldsymbol{V}(Y, N)$ is the range of a map $f \in \boldsymbol{f}$;

(2.3) if $f_1, f_2 \in \boldsymbol{f}$ and $f_1, f_2\colon U \to V$, then there is a $U' \subset U$, $U' \in \boldsymbol{U}(X, M)$, such that $f_1 u \simeq f_2 u$, where $u\colon U' \to U$ is the inclusion map.

Consider two mutations $\boldsymbol{f}\colon \boldsymbol{U}(X, M) \to \boldsymbol{V}(Y, N)$ and $\boldsymbol{g}\colon \boldsymbol{V}(Y, N) \to \boldsymbol{W}(Z, P)$. The *composition* $\boldsymbol{gf}\colon \boldsymbol{U}(X, M) \to \boldsymbol{W}(Z, P)$ of $\boldsymbol{f}$ and $\boldsymbol{g}$ is given by all the compositions $gf\colon U \to M$ which are defined. Two mutations $\boldsymbol{f}, \boldsymbol{g}\colon \boldsymbol{U}(X, M) \to \boldsymbol{V}(Y, N)$ are *homotopic*, $\boldsymbol{f} \simeq \boldsymbol{g}$, if

(2.4) for any maps $f, g\colon U \to V$ from $\boldsymbol{f}$ and $\boldsymbol{g}$ respectively there is a $U' \in \boldsymbol{U}(X, M)$, $U' \subset U$, such that $fu \simeq gu$, where $u\colon U' \to U$ is the inclusion map.

Two metrizable spaces X and Y are said to be of the same *shape* in the sense of Fox (notation: $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$) if there exist two mutations $\boldsymbol{f}\colon \boldsymbol{U}(X, M) \to \boldsymbol{V}(Y, N)$ and $\boldsymbol{g}\colon \boldsymbol{V}(Y, N) \to \boldsymbol{U}(X, M)$ such that

(2.5) $\boldsymbol{fg} \simeq \boldsymbol{u}$ and $\boldsymbol{gf} \simeq \boldsymbol{v}$, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are mutations consisting of all inclusions in $\boldsymbol{U}(X, M)$ and $\boldsymbol{V}(Y, N)$ respectively.

If the mutations $\boldsymbol{f}$ and $\boldsymbol{g}$ satisfy the first of conditions (2.4), then we say that the shape of X *dominates* the shape of Y and we write $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$.

Let k be a non negative integer. According to Borsuk ([2], p. 266) a metric space X is said to be *approximatively* k*-connected* if there is an AR-space M containing X as a closed set and satisfying the condition: For every neighborhood V of X in M there is a neighborhood U of X such that every map of a k-sphere S^k into U is null-homotopic in V. By the same way as in the proof of [2], Theorem (2.1) we know

(2.6) if $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$ and X is approximatively k-connected then Y is approximatively k-connected.

§ 3. Main theorem and its applications.

THEOREM 1. *Let X and Y be finite dimensional metric spaces and let $f\colon X \to Y$ be a closed map from X onto Y. If $\dim Y \leqslant n$ and for each $y \in Y$ $f^{-1}(y)$ is approximatively k-connected, $k = 0, 1, \ldots, n$ then $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$. In addition, if $\dim X \leqslant n$, then $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.*

The proof of the theorem is given by a similar process to the proof of [9], Theorem 2. We first state lemmas used in the proof of the theorem.

Let X be a metric space and let $\mathscr{D}$ be an upper semicontinuous decomposition of X consisting of closed sets. Denote by f the decomposition map of X onto the decomposition space Y for $\mathscr{D}$. Let M be a metric space containing X such that each element of $\mathscr{D}$ is closed in M. A collection $\mathscr{U}$ of open sets in M is said to be a *cover* of $\mathscr{D}$ if $X \subset \bigcup \{U\colon U \in \mathscr{U}\}$ and for each $U \in \mathscr{U}$ $U \cap X$ is non empty and saturated, i.e. $U \cap X = f^{-1}f(U \cap X)$. The following lemma has been proved in [9] in case each element of D is compact.

LEMMA 1. *Every cover of D has a star refinement.*

Proof. Let $\mathscr{U}$ be a cover of $\mathscr{D}$. Put $M' = \bigcup \{U\colon U \in \mathscr{U}\}$. Consider the decomposition $\mathscr{D}'$ of M' consisting of each element of $\mathscr{D}$ and each point in $M' - X$. Let f be the decomposition map of M' onto the decomposition space Z of M' by $\mathscr{D}'$. Since $\mathscr{D}'$ is upper semicontinuous, f is a closed map and hence Z is paracompact by the theorem of Michael [15]. Since each set $U \cap X$, $U \in \mathscr{U}$, is saturated, $f(\mathscr{U})$ is an open cover of Z. Take an open cover $\mathscr{W}$ of Z such that $\mathscr{W} \overset{*}{>} f(\mathscr{U})$. (For two collections $\mathscr{A}$ and $\mathscr{B}$, by $\mathscr{A} > \mathscr{B}$ (resp. $\mathscr{A} \overset{*}{>} \mathscr{B}$) we mean $\mathscr{A}$ is a refinement (resp. star refinement) of $\mathscr{B}$.) Put $\mathscr{V} = f^{-1}\mathscr{W}$. Then it is obvious that a cover $\mathscr{V}$ of $\mathscr{D}$ is a star refinement of $\mathscr{U}$.

A cover $\mathscr{U}$ of $\mathscr{D}$ is said to be an n*-refinement* of a cover $\mathscr{V}$ of $\mathscr{D}$ if there is a sequence $\mathscr{V}_0, \mathscr{V}'_1, \mathscr{V}_1, \ldots, \mathscr{V}_n, \mathscr{V}'_{n+1}, \mathscr{V}_{n+1}$ of covers of $\mathscr{D}$ such that $\mathscr{V}_0 = \mathscr{U}$, $\mathscr{V}_{n+1} = \mathscr{V}$, $\mathscr{V}_i \overset{*}{>} \mathscr{V}'_{i+1}$ and $\mathscr{V}'_{i+1} > \mathscr{V}_{i+1}$, $i = 0, 1, \ldots, n$, and for each $V \in \mathscr{V}_i$, $i = 1, \ldots, n+1$, there is a $V' \in \mathscr{V}'_i$ such that every map $g\colon S^j \to V'$, $j = 0, 1, \ldots, n$, is null-homotopic in V. When $\mathscr{U}$ is an n-refinement of $\mathscr{V}$, we write $\mathscr{U} \overset{n}{\geqslant} \mathscr{V}$.

The following lemma is a consequence of Lemma 1.

LEMMA 2. *Let M be an* AR*-space and let X be a closed subset of M. Let $\mathscr{D}$ be an upper semicontinuous decomposition of X each element of which is approximatively k-connected for $k = 0, 1, \ldots, n$. Then every cover of $\mathscr{D}$ has an n-refinement.*

As an immediate consequence of the definition of an n-refinement we have

LEMMA 3. *Let $\mathscr{U}$ and $\mathscr{V}$ be covers of $\mathscr{D}$ such that $\mathscr{U} \overset{n}{\geqslant} \mathscr{V}$.*

(3.1) *Let K be an $(n+1)$-dimensional simplicial complex and K^0 the set of its vertices. If $f\colon K^0 \to M$ is a map such that for each closed simplex σ of K there is a $U \in \mathscr{U}$ containing $f(\sigma \cap K^0)$, then f has an extension $g\colon K \to M$ such that for a closed simplex σ of K there is a $V \in \mathscr{V}$ containing $g(\sigma)$.*

(3.2) *Let K be an n-dimensional simplicial complex. If f and g are maps of K into M such that for each closed simplex σ of K there is a $U \in \mathscr{U}$ containing $f(\sigma) \cup g(\sigma)$ then there is a homotopy $H\colon K \times I \to M$ connecting f and g such that for each closed simplex σ $H(\sigma \times I)$ is contained in some $V \in \mathscr{V}$.*

Remark. In Lemma 3, let $\{\mathscr{V}_i, \mathscr{V}'_i\}$ be a sequence of covers of $\mathscr{D}$ in the definition of n-refinements such that $\mathscr{U} = \mathscr{V}_0$ and $\mathscr{V} = \mathscr{V}_{n+1}$. If σ is an i-simplex of K, then we can construct a map g and a homotopy H such that $g(\sigma)$ is in some element of $\mathscr{V}'_i$ and $H(\sigma \times I)$ is in some element of $\mathscr{V}'_{i+1}$ for each $i = 1, 2, \ldots, n$. In particular, if σ is an n-simplex, then we can assume that $g(\sigma)$ is in some element of $\mathscr{V}'_n$.

Proof of Theorem 1. Let M and N be AR-spaces containing X and Y as closed sets respectively. By $\boldsymbol{U}(X, M)$ denote the inverse system of neighborhoods U of X in M and all inclusion maps $u\colon U'\to U$, $U'\subset U$. Similarly by $\boldsymbol{V}(Y, N)$ denote the inverse system of neighborhoods V of Y in N. Let $\tilde{f}\colon M\to N$ be an extension of $f\colon X\to Y$. Then f generates a mutation $\boldsymbol{f}\colon \boldsymbol{U}(X, M)\to \boldsymbol{V}(Y, N)$. To prove the theorem we have to construct a mutation $\boldsymbol{g}\colon \boldsymbol{V}(Y, N)\to \boldsymbol{U}(X, M)$ such that $\boldsymbol{fg}\simeq\boldsymbol{v}$, where $\boldsymbol{v}$ is the mutation consisting of all inclusions in $\boldsymbol{V}(Y, N)$. Let $U\in \boldsymbol{U}(X, M)$. We first construct a map g whose range is U (cf. (2.2)). Throughout the proof of the theorem we shall keep the notations used in the construction of g. Consider $\{U\}$ as a cover of the decomposition $\mathscr{D} = \{f^{-1}(y)\colon y\in Y\}$ of X. Take covers $\mathscr{U}'$ and $\mathscr{U}$ of $\mathscr{D}$ such that

(3.3) $$\mathscr{U} \overset{n}{\geqslant} \mathscr{U}' \overset{n}{\geqslant} \{U\}.$$

Since $f\colon X\to Y$ is closed, there is a locally finite open cover $\mathscr{V}'$ of Y such that order $\mathscr{V}'\leqslant n+1$ and $f^{-1}\mathscr{V}'\overset{*}{>}\mathscr{U}$. Let $\mathscr{V}$ be a locally finite collection consisting of open sets of N such that $\mathscr{V}\cap Y = \mathscr{V}'$ and $\mathscr{V}$ and $\mathscr{V}'$ are similar. Put $V = \bigcup\{W\colon W\in\mathscr{V}\}$. By K denote the nerve of $\mathscr{V}$ and let $\varphi\colon V\to K$ be a canonical map. Let us define a map $g'\colon K\to U$ as follows. For a vertex w of K, let W be the element of $\mathscr{V}$ corresponding to w. Choose a point $x_W\in f^{-1}(W\cap Y)$ of X and put $g''(w) = x_W$ for each vertex w of K. For every closed simplex σ, $g''(\sigma\cap K^0)$ is contained in some element of $\mathscr{U}$, where K^0 is the 0-skeleton of K. Hence, by Lemma 3 (3.1), g'' is extended to a map $g'\colon K\to U$ such that for each closed simplex σ of K $g'(\sigma)$ is in some element of $\mathscr{U}'$. Define $g\colon V\to U$ by $g = g'\varphi$. Let $\boldsymbol{g}$ be the collection of all maps $g\colon V\to U$ and $ugv\colon V'\to U'$, where $U\subset U'$, $U, U'\in\boldsymbol{U}(X, M)$ and $V'\subset V$, $V', V\in\boldsymbol{V}(Y, N)$, u and v are the inclusion maps of U, V' into U', V respectively and g is a map which is constructed by the above-mentioned process. It is obvious that $\boldsymbol{g}$ satisfies conditions (2.1) and (2.2). Let $g_1, g_2\in\boldsymbol{g}$. Let $U_i, \mathscr{U}_i, \mathscr{U}_i', \mathscr{V}_i, V_i, K_i, g_i', \varphi_i$, $i = 1, 2$, be neighborhoods in $\boldsymbol{U}(X, M)$, covers of $\mathscr{D}$, ..., canonical maps which are used for the constructions of g_1 and g_2. Take covers $\mathscr{U}_3$, $\mathscr{U}_3'$ of $\mathscr{D}$ such that $\mathscr{U}_3 \overset{n}{\geqslant} \mathscr{U}_3' > \mathscr{U}_1\wedge\mathscr{U}_2$. Let $\mathscr{V}_3$ be a locally finite collection of open sets of N satisfying the conditions:

(3.4) $$\text{order } \mathscr{V}_3\leqslant n+1 \quad\text{and}\quad V_3 = \bigcup\{V\colon V\in\mathscr{V}_3\}\supset Y,$$

(3.5) $\mathscr{V}_3\cap Y$ and $\mathscr{V}_3$ are similar, that is, for every finite subcollection V_i, $i = 1, \dots, k$, of $\mathscr{V}_3$ $\bigcap_{i=1}^{k} V_i\neq\emptyset$ if and only if $\bigcap_{i=1}^{k} V_i\cap Y\neq\emptyset$,

(3.6) $\mathscr{V}_3>\mathscr{V}_1\wedge\mathscr{V}_2$ and $f^{-1}(\mathscr{V}_3\cap Y)\overset{*}{>}\mathscr{U}_3$, where for collections $\mathscr{V}$, $\mathscr{V}_1$ and $\mathscr{V}_2$ $\mathscr{V}\cap Y = \{V\cap Y\colon V\in\mathscr{V}\}$ and $\mathscr{V}_1\wedge\mathscr{V}_2 = \{V\cap V'\colon V\in\mathscr{V}_1$ and $V'\in\mathscr{V}_2\}$.

Such an open collection $\mathscr{V}_3$ is constructed as follows. Put $\mathscr{W} = \{V\cap V'\cap f(U)\cap Y\colon V\in\mathscr{V}_1, V'\in\mathscr{V}_2, U\in\mathscr{U}_3\}$. Since $\mathscr{W}$ is an open cover of Y and $\dim Y\leqslant n$, there is a locally finite open cover $\mathscr{V}'$ of Y such that $\mathscr{V}'\overset{*}{>}\mathscr{W}$ and order of $\mathscr{V}'\leqslant n+1$. Since N is a metric space, there is an open collection $\mathscr{V}_3$ of N such that $\mathscr{V}_3\cap Y = \mathscr{V}'$ and $\mathscr{V}_3\cap Y$ and $\mathscr{V}_3$ are similar. We can assume that $\mathscr{V}_3>\mathscr{V}_1\wedge\mathscr{V}_2$. Since $f^{-1}f(U\cap X) = U\cap X$ for each $U\in\mathscr{U}_3$, we have

$$f^{-1}(\mathscr{V}_3\cap Y) = \tilde{f}^{-1}(\mathscr{V}_3\cap Y)\cap X\overset{*}{>}\mathscr{U}_3.$$

Thus the collection $\mathscr{V}_3$ satisfies (3.4), (3.5) and (3.6). Let K_3 be the nerve of $\mathscr{V}_3$ and $\varphi_3\colon V_3\to K_3$ a canonical map. As in the construction of a map g we can find a map $g_3'\colon K_3\to U_1\cap U_2$ such that for each closed simplex σ of K_3 $g_3'(\sigma)$ is in some element of $\mathscr{U}_3$. Put $g_3 = g_3'\varphi_3$. Then $g_3\colon V_3\to U_1\cap U_2$ is in $\boldsymbol{g}$. Let $u_i\colon U_1\cap U_2\to U_i$ and $v_i\colon V_3\to V_i$, $i = 1, 2$, be the inclusion maps. We shall show that $u_ig_3\simeq g_iv_i$ for $i = 1, 2$. This shows that $\boldsymbol{g}$ satisfies the condition (2.3) and as a consequence $\boldsymbol{g}$ is a mutation. Let π be a simplicial projection of K_3 into K_1 (cf. (3.6)). Since $\varphi_1v_1\simeq\pi\varphi_3$, $g_1v_1 = g_1'\varphi_1v_1\simeq g_1'\pi\varphi_3$. Therefore, to prove $u_1g_3\simeq g_1v_1$ it is enough to show that $u_1g_3'\simeq g_1'\pi\colon K_3\to U_1$. By the definitions of g_1' and g_3', we can know that for each closed simplex σ of K_3 there is an element U' of $\mathscr{U}_1'$ containing $g_3'(\sigma)\cup g_1'\pi(\sigma)$ (cf. Lemma 3 (3.1) and Remark). Hence, by Lemma 3 (3.2), $u_1g_3'\simeq g_1'\pi$ in U_1. Thus we know $u_1g_3\simeq g_1v_1$. Similarly $u_2g_3\simeq g_2v_2$. Therefore $\boldsymbol{g}$ is a mutation.

Next, we shall prove that $\boldsymbol{fg}\simeq\boldsymbol{v}$. To do it, let $W\in\boldsymbol{V}(Y, N)$. Since N is an AR-space, there is an open collection $\mathscr{W}'$ of N such that $V' = \bigcup\{W'\colon W'\in\mathscr{W}'\}\subset W$, $V'\in\boldsymbol{V}(Y, N)$, for each $W'\in\mathscr{W}'$ $W'\cap Y$ is non empty and if h, h' are maps of a space Z into V' such that h and h' are $\mathscr{W}'$-close then $h\simeq h'$ in W. Here, if for each $z\in Z$ there is a $W_z'\in\mathscr{W}'$ containing both $h(z)$ and $h'(z)$, then we say that h and h' are *$\mathscr{W}'$-close*. Put $U = \tilde{f}^{-1}(V')$. Then $U\in\boldsymbol{U}(X, M)$. For the element U, find a neighborhood $V\in\boldsymbol{V}(Y, N)$ and a map $g\colon V\to U$ by the above-mentioned process. Let g be constructed by choosing covers $\mathscr{U}, \mathscr{U}'$ of $\mathscr{D}$ and an open collection $\mathscr{V}$ in N such that $\mathscr{U}\overset{n}{\geqslant}\mathscr{U}'\overset{*}{>}\tilde{f}^{-1}\mathscr{W}'$, order $\mathscr{V}\leqslant n+1$ and $f^{-1}(\mathscr{V}\cap Y)\overset{*}{>}\mathscr{U}$. Let K be the nerve of $\mathscr{V}$ and let $g'\colon K\to U$ be the map such that $g = g'\varphi$, where φ is a canonical map of V into K. From the definition of g and the choice of $\mathscr{U}'$, we can know that the map $\tilde{f}g$ and the inclusion map v of V into W are $\mathscr{W}'$-close. Therefore $\tilde{f}g\simeq v$ in W. This shows that $\boldsymbol{fg}\simeq\boldsymbol{v}$.

Finally, suppose that $\dim X\leqslant n$. We have to prove $\boldsymbol{gf}\simeq\boldsymbol{u}$. Let $U\in\boldsymbol{U}(X, M)$ and $g\in\boldsymbol{g}$ be a map constructed for U by the process in above. Let $\mathscr{U}$ and $\mathscr{V}$ be a cover of $\mathscr{D}$ and an open collection in N used for the construction of g. Since M is an AR-space, we can know that there are open collections $\mathscr{W}'$ and $\mathscr{W}$ in M satisfying the following conditions:

(3.7) $$X\subset W' = \bigcup\{W''\colon W''\in\mathscr{W}'\}\quad\text{and}\quad \mathscr{W}'>\mathscr{U};$$

(3.8) if h and h' are maps of a space into W' such that h and h' are $\mathscr{W}'$-close then $h\simeq h'$ in U;

(3.9) $$X\subset W = \bigcup\{W'': W''\in\mathscr{W}\}\subset W';$$

(3.10) $$\mathscr{W}>\mathscr{W}'\wedge f^{-1}\mathscr{V} \quad \text{and order } \mathscr{W}\leqslant n+1;$$

(3.11) if L is the nerve of $\mathscr{W}$ and ψ: $W\to L$ is a canonical map, then there is a map h: $L\to W'$ such that for each closed simplex σ of L there is an element W'' of $\mathscr{W}'$ such that $\psi^{-1}(\sigma)\cup h(\sigma)\subset W''$;

(3.12) let π: $L\to K$ be a simplicial projection (cf. (3.10)), where K is the nerve of $\mathscr{V}$, then the maps $g'\pi$ and $u''h$ of L into U are $\mathscr{U}'$-close, where u'' is the inclusion map: $W'\subset U$ and $\mathscr{U}'$ is a cover of $\mathscr{D}$ used for the construction of g (see (3.3)).

Consider the map $f' = \tilde{f}|W$: $W\to V$. We shall show that $gf'\simeq u$: $W\to U$, where u is the inclusion map. Since $\pi\psi$ and $\varphi f'$ are contiguous, $\pi\psi\simeq\varphi f'$: $W\to K$, where φ is a canonical map of V into K. Let σ be a closed simplex of L. By (3.12) there is an element U' of $\mathscr{U}'$ such that $g'\pi(\sigma)\cup h(\sigma)\subset U'$ and hence $g'\pi\simeq h$ in U by Lemma 3. From (3.8) and (3.11) $u\simeq u''h\psi$. Therefore $u\simeq u''h\psi\simeq g'\pi\psi\simeq g'\varphi f' = gf'$. Thus we know $\boldsymbol{gf\simeq u}$. This completes the proof of Theorem 1.

Example 1. Consider the following sets in the plane E^2:

$$A = \{(0,y)\colon -1\leqslant y\leqslant 1\}, \quad B = \{(x,y)\colon y = \sin\pi/x,\ 0<x\leqslant 1\},$$

$$Y = \{(x,0)\colon 0\leqslant x\leqslant 1\}.$$

Put $T = A\cup B$ and $X = T-\{(0,-1)\}$. Define f: $T\to Y$ by $f(x,y) = x$ for $(x,y)\in T$, and put $g = f|X$: $X\to Y$. Then, for each $y\in Y$, $f^{-1}(y)$ and $g^{-1}(y)$ consist of a segment, a half line or one point and hence these sets are k-approximatively connected for every k. However $\mathrm{Sh}(T) = \mathrm{Sh}(Y) = \mathrm{Sh}(1) \neq \mathrm{Sh}(X)$. Here $\mathrm{Sh}(1)$ means a trivial shape. Because, note that for every non-zero abelian group G $\check{H}^1(X\colon G)$ has an infinite number of elements, where $\check{H}^*$ means the Čech cohomology group. Obviously the Čech cohomology group is an invariant for shape in the sense of Fox. (This follows from Mardešić's characterization ([12] and [13]) of shape in the sense of Fox and the fact that the Čech cohomology group $\check{H}^n(X\colon G)$ is isomorphic to the group of the homotopy classes of maps of X into Eilenberg–MacLane space $K(G,n)$ (cf. [7]); to obtain a direct proof is easy.) This example shows that the closedness of f in Theorem 1 can not be removed.

Example 2. Consider the set $A\cup B$ in Example 1. Let F be a continuum obtained from $A\cup B$ by connecting two points $(0,1)$ and $(1,0)$ by an arc whose interior does not intersect $A\cup B$. (F is a Warsaw circle.) Let S be a circle. By f': $F\to S$ denote a map such that $f(A)$ is a point a of S and $f'|F-A$: $F-A\to S-\{a\}$ is a homeomorphism. Let X be a topological sum of a countable infinite number of copies of F. Similarly, let Y be a countable infinite number of copies of S. Let f: $X\to Y$ be a map which is defined by f' on each copy of F. Then f is a perfect map and for each $y\in Y$ $f^{-1}(y)$ is one point or an arc. However, as shown by Godlewski and Nowak [6], $\mathrm{Sh}_S(X)\neq\mathrm{Sh}_S(Y)$. Here $\mathrm{Sh}_S(X)$ means the strong shape of X defined by Borsuk [3]. This shows that Theorem 1 does not hold for strong shape.

Next, as an application of Theorem 1, we shall generalize Ball's theorem [1]. Let X and Y be finite dimensional and locally compact metric spaces. By $\mathscr{C}_X$ and $\mathscr{C}_Y$ denote the decompositions of X and Y consisting of all components respectively. Let $\square X$ and $\square Y$ be the decomposition spaces of X and Y by $\mathscr{C}_X$ and $\mathscr{C}_Y$ and let p: $X\to\square X$ and q: $Y\to\square Y$ be the decomposition maps. Suppose that each element of $\mathscr{C}_X$ and $\mathscr{C}_Y$ is compact. The following theorem generalizes a theorem of Ball ([1] Theorem 2.4).

Theorem 2. *Under the hypothesis mentioned above, suppose that* $\mathrm{Sh}(X)\leqslant\mathrm{Sh}(Y)$ *(resp.* $\mathrm{Sh}_W(X)\leqslant\mathrm{Sh}_W(Y)$*). Then there is a homeomorphism into* Λ: $\square X\to\square Y$ *such that for each locally compact set* F *of* $\square X$

(3.13) $$\mathrm{Sh}\big(p^{-1}(F)\big)\leqslant\mathrm{Sh}\big(q^{-1}(\Lambda(F))\big) \quad (\textit{resp. } \mathrm{Sh}_W\big(p^{-1}(F)\big)\leqslant\mathrm{Sh}_W\big(q^{-1}(\Lambda(F))\big)).$$

Moreover, if $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ *(resp.* $\mathrm{Sh}_W(X) = \mathrm{Sh}_W(Y)$*), then there is a homeomorphism* Λ: $\square X\to\square Y$ *for which the equality holds in* (3.13).

Proof. Since X and Y are locally compact and each component in $\mathscr{C}_X$ and $\mathscr{C}_Y$ is compact, the maps p and q are perfect and $\dim\square X = \dim\square Y = 0$. Let M and M' be AR-spaces containing X and $\square X$ as closed sets respectively, and let $\tilde{p}$: $M\to M'$ be an extension of p. Since each element of $\mathscr{C}_X$ is connected, it is approximatively 0-connected. Since $\dim\square X = 0$, by Theorem 1, we know that there is a mutation $\boldsymbol{h}$: $\boldsymbol{U}(\square X, M')\to\boldsymbol{U}(X,M)$ such that $\boldsymbol{ph\simeq u'}$, where $\boldsymbol{U}(\square X, M')$ and $\boldsymbol{U}(X,M)$ are the inverse systems consisting of neighborhoods of $\square X$ and X in M' and M respectively, $\boldsymbol{u'}$ is the mutation consisting of the inclusion maps in $\boldsymbol{U}(\square X, M')$ and $\boldsymbol{p}$ is the mutation generated by $\tilde{p}$. Similarly, if N and N' are AR-spaces containing Y and $\square Y$ as closed sets and $\tilde{q}$: $N\to N'$ is an extension of q: $Y\to\square Y$, then there is a mutation $\boldsymbol{k}$: $\boldsymbol{V}(\square Y, N')\to\boldsymbol{V}(Y,N)$ such that $\boldsymbol{qk\simeq v'}$, where $\boldsymbol{V}(\square Y, N')$ and $\boldsymbol{V}(Y,N)$ are the inverse systems of neighborhoods of $\square Y$ and Y in N' and N respectively, $\boldsymbol{v'}$ is the mutation consisting of the inclusions in $\boldsymbol{V}(\square Y, N')$ and $\boldsymbol{q}$ is the mutation generated by $\tilde{q}$. Since $\mathrm{Sh}(X)\leqslant\mathrm{Sh}(Y)$, there are mutations $\boldsymbol{f}$: $\boldsymbol{U}(X,M)\to\boldsymbol{V}(Y,N)$ and $\boldsymbol{g}$: $\boldsymbol{V}(Y,N)\to\boldsymbol{U}(X,M)$ such that $\boldsymbol{gf\simeq u}$, where $\boldsymbol{u}$ is the mutation consisting of the inclusions in $\boldsymbol{U}(X,M)$. Consider the mutation $\boldsymbol{qfh}$: $\boldsymbol{U}(\square X, M')\to\boldsymbol{V}(\square Y, N')$. Since $\dim\square Y = 0$, by [11], Lemma, there is a unique map Λ: $\square X\to\square Y$ which generates the mutation qfh. Similarly we know that there is a unique map Λ': $\square Y\to\square X$ which generates the mutation pgk. First, let us prove

(3.14) $\Lambda'\Lambda = 1_{\square X}$, where $1_{\square X}$ is the identity map of $\square X$.

Suppose that $\Lambda'\Lambda(a)\neq a$ for $a\in\square X$. Since $\dim\square X = 0$, there are open sets V'

and V'' of M' such that $a \in V'$, $\Lambda'\Lambda(a) \in V''$, $V' \cap V'' = \emptyset$ and $V' \cup V'' \in U(\Box X, M')$. Put $U' = \tilde{p}^{-1}(V')$ and $U'' = \tilde{p}^{-1}(V'')$. Then $U = U' \cup U'' \in U(X, M)$. Since $\boldsymbol{gf} \simeq \boldsymbol{u}$, there exist $W \in U(X, M)$, $f \in \boldsymbol{f}$ and $g \in \boldsymbol{g}$ such that $W \subset U$ and $gf \simeq u$: $W \to U$, where u is the inclusion. Therefore we have $gf(p^{-1}(a)) \subset gf(p^{-1}(V') \cap W) \subset U'$. This relation contradicts that $\Lambda'\Lambda(a) \in V''$. Thus (3.14) holds. We know that Λ is a homeomorphism into. Next, we shall show that

(3.15) if H is a closed subset of $\Box X$ then $\mathrm{Sh}(p^{-1}(H)) \leqslant \mathrm{Sh}(q^{-1}(\Lambda(H)))$.

Consider the inverse systems $U(p^{-1}(H), M)$ and $V(q^{-1}(\Lambda(H)), N)$ consisting of neighborhoods of $p^{-1}(H)$ and $q^{-1}(\Lambda(H))$ in M and N respectively. To prove (3.15), it is enough to show that $\boldsymbol{f}$ is a mutation of

$$U(p^{-1}(H), M) \quad \text{to} \quad V(q^{-1}(\Lambda(H)), N)$$

and $\boldsymbol{g}$ is a mutation of

$$V(q^{-1}(\Lambda(H)), N) \quad \text{to} \quad U(p^{-1}(H), M).$$

It is easy to see that this fact is shown by (3.14) and the following assertion.

(3.16) For every open neighborhood W of $q^{-1}(\Lambda(H))$ in N there exist $V \in V(Y, N)$, open sets V' and V'' of N such that $q^{-1}(\Lambda(H)) \subset V' \subset W$, $V' \cap V'' = \emptyset$, $V = V' \cup V''$, and $V' \cap Y$ and $V'' \cap Y$ are saturated, i.e. $q^{-1}q(V' \cap Y) = V' \cap Y$ and $q^{-1}q(V'' \cap Y) = V'' \cap Y$.

Let us prove (3.16). Since q is a closed map, for an open neighborhood W of $q^{-1}(\Lambda(H))$ in N $q(Y-W)$ and $\Lambda(H)$ are disjoint closed sets in $\Box Y$. Since $\dim \Box Y = 0$, it follows that there are open sets W' and W'' in N' such that $\Lambda(H) \subset W'$, $q(Y-W) \subset W''$, $W' \cap W'' = \emptyset$ and $W' \cup W'' \in V(\Box Y, N')$. Put $V' = q^{-1}(W') \cap \cap W$ and $V'' = q^{-1}(W'')$. Then V', V'' and $V = V' \cup V''$ satisfy the conditions of (3.16). Thus (3.16) and hence (3.15) was proved. As shown in the proof of [1], Lemma 2.3, the theorem follows from (3.15), ([6], Theorem 4.2) and the fact that every locally compact 0-dimensional metric space is a union of a discrete family of compact sets. The assertion for weak shape is proved by making use of [9], Theorems 1 and 2 in place of [12], Theorem 1 and Theorem 1 in the above proof. This completes the proof.

The following corollary concerns a problem raised by Ball ([1], P 888).

COROLLARY. *Under the same hypothesis as in Theorem 2, suppose that each element of $\mathscr{C}_X$ is of trivial shape. Then there is a homeomorphism into Λ: $X \to Y$ such that for every set K of $\Box X$*

(3.17) $\mathrm{Sh}(p^{-1}(K)) \leqslant \mathrm{Sh}(q^{-1}(\Lambda(K)))$ (*resp.* $\mathrm{Sh}_W(p^{-1}(K)) \leqslant \mathrm{Sh}_W(q^{-1}(\Lambda(K)))$).

Moreover, if $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ (*resp.* $\mathrm{Sh}_W(X) = \mathrm{Sh}_W(Y)$), *then there is a homeomorphism Λ: $X \to Y$ and the equality holds in* (3.17).

Proof. By Theorems 1 and 2 we know that $\mathrm{Sh}(p^{-1}(K)) = \mathrm{Sh}(K) = \mathrm{Sh}(\Lambda(K)) \leqslant \mathrm{Sh}(q^{-1}(\Lambda(K)))$. This shows the first part of the corollary. If $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$, then Λ is a homeomorphism and $\mathrm{Sh}(p^{-1}(a)) = \mathrm{Sh}(q^{-1}(\Lambda(a)))$ for each $a \in \Box X$. Thus each element of $\mathscr{C}_Y$ is of trivial shape. By Theorem 1 $\mathrm{Sh}(\Lambda(K)) = \mathrm{Sh}(q^{-1}(\Lambda(K)))$. This completes the proof.

The following definition was given for compact spaces in [10].

DEFINITION. A metric space X is said to be a *Δ-space* if there is an inverse sequence $\{K_n, \pi_{n,n+1}\}$ consisting of simplicial complexes K_n with metric topology and simplicial maps $\pi_{n,n+1}$: $K_{n+1} \to K_n$ such that $\underleftarrow{\lim}\{K_n\} = X$ (cf. [11]).

THEOREM 3. *Let X be a finite dimensional and locally compact metric space. Then there exists a Δ-space Y such that $\dim X = \dim Y$, $\mathrm{Sh}_W(X) = \mathrm{Sh}_W(Y)$ and $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.*

Proof. Let $\{\mathscr{U}_n\}$ be a sequence of locally finite open covers of X such that each element of $\mathscr{U}_n$ has a compact closure, order $\mathscr{U}_n \leqslant \dim X + 1$; $\overline{\mathscr{U}}_{n+1} > \mathscr{U}_n$ for $n = 1, 2, \ldots$ and $\mathrm{mesh}\,\mathscr{U}_n \to 0$ $(n \to \infty)$, where $\overline{\mathscr{U}}_{n+1} = \{\overline{U}: U \in \mathscr{U}_{n+1}\}$. Let K_n be the nerve of $\mathscr{U}_n$ with metric topology. For each n, let $\pi_{n,n+1}$ be a simplicial projection of K_{n+1} into K_n defined by mapping a vertex v of K_{n+1} corresponding to $V \in \mathscr{U}_{n+1}$ to a vertex w of K_n corresponding to $W \in \mathscr{U}_n$ such that $\overline{V} \subset W$. Consider the inverse sequence $\{K_n, \pi_{n,n+1}\}$ and put $Y = \underleftarrow{\lim}\{K_n\}$. Then Y is a Δ-space and $\dim X = \dim Y$. As shown by Kaul [8], there is a perfect map f from Y onto X such that for each $x \in X$ $f^{-1}(x)$ is the inverse limit of an inverse sequence consisting of closed simplexes and hence $f^{-1}(x)$ is of trivial shape. By Theorem 1 and [9], Theorem 2 we know $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ and $\mathrm{Sh}_W(X) = \mathrm{Sh}_W(Y)$. This completes the proof.

The assertion in Theorem 3 for weak shape was proved in [11], Theorem 2 by a different way. As shown there, we can not remove the local compactness of X in Theorem 3. To see it, let X be a set consisting of all rational numbers in a line. If Y is a 0-dimensional space, and $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ or $\mathrm{Sh}_W(X) = \mathrm{Sh}_W(Y)$, then X and Y are homeomorphic by [12], Theorem 1 and [9], Theorem 1 and hence Y is not completely metrizable. Since every finite dimensional Δ-space is completely metrizable, Y is not a Δ-space.

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Some uniformization results

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Abstract. Some results on the uniformization of Borel sets are proved in this paper.

1. Introduction. Let X, Y be Polish spaces and $B \subseteq X \times Y$. We say C *uniformizes* B if $C \subseteq B$ and for all $x \in \mathrm{pr}_X B$, $C \cap B^x$ is a singleton where $B^x = \{y\colon (x, y) \in B\}$ and $\mathrm{pr}_X B$ is the projection of B to X. In general, a Borel set B does not have a Borel uniformization ([2], [6]). However, in some cases, such a uniformization exists, for example, if B^x is σ-compact for each x [1] or if $\mu(B^x) > 0$ for each x where μ is a probability measure on the Borel σ-algebra of Y [3].

The chief aim of this paper is to prove the following:

THEOREM 1. *Let X, Y be Polish spaces and $B \subseteq X \times Y$ be Borel. B has a Borel uniformization if any one of the following is true*:

(1) *for all $x \in \mathrm{pr}_X B$, B^x contains an isolated point,*

(2) *for all $x \in \mathrm{pr}_X B$, B^x contains a point which is not its point of condensation,*

(3) *for all $x \in \mathrm{pr}_X B$, B^x is not meager.*

The paper is organized in the following way. Section 2 is devoted to preliminaries. In Section 3, a proof of Theorem 1 is given. In Section 4, a related result is proved.

2. Preliminaries. A set is called *meager* if it is a countable union of nowhere dense sets. A comeager set is one whose complement is meager. Let X, Y be Polish spaces, $B \subseteq X \times Y$ and $U \subseteq Y$. Following Vaught, we put $B_U^* = \{x\colon B^x \cap U$ is comeager in $U\}$. It is known that if B is Borel and U open, then B_U^* is Borel [7]. For any set A, let $\delta(A)$ denote the diameter of A.

If f is a function, put $Z_f = \{y\colon f^{-1}(y)$ is a singleton$\}$, $I_f = \{y\colon f^{-1}(y)$ contains an isolated point$\}$, $D_f = \{y\colon f^{-1}(y)$ is countable and non-empty$\}$, $C_f = \{y\colon f^{-1}(y)$ contains a point which is not its condensation point$\}$. It is known that if f is a Borel measurable function defined on a Borel subset of a Polish space into a separable metric space, then Z_f, I_f, D_f, C_f are coanalytic [4].

3. Proof of main theorem.

Proof of (1). Let $\{V_n\}$ be a countable open base for Y. For any n, define f_n on $B \cap (X \times V_n)$ by $f_n(x, y) = x$. Let $Z_n = \{x\colon B^x \cap V_n$ is a singleton$\}$. Then $Z_n = Z_{f_n}$