It is an interesting question whether the following generalization of the Bruckner–Świątkowski theorem is true:

If a function \( f \) is a Baire class 1 function with the Darboux property, \( T \) satisfies Khintchine’s condition n.e., \( f' \) exists n.e. and \( f' \geq 0 \) a.e. in \((a, b)\), then \( f \) is non-decreasing and continuous in \((a, b)\).

References


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Decomposition spaces and shape in the sense of Fox

by

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Abstract. It is proved in the paper that if \( X, Y \) are finite dimensional metrizable spaces, \( f: X \to Y \) is a closed continuous map such that \( f' \) is approximatively \( k \)-connected for \( y \in Y \) and \( k = 0, 1, \ldots, \dim Y \), then \( Sh(X) \geq Sh(Y) \) (in the sense of Fox [5]). By applying the theorem it is shown that for every finite dimensional locally compact metric space \( X \) there exists a \( D \)-space \( Y \) such that \( \dim X = \dim Y \), \( Sh_w(X) = Sh_w(Y) \) and \( Sh(X) = Sh(Y) \).

§ 1. Introduction. In [5] Fox introduced the notion of shape for metric spaces and proved that for compacta this notion coincides with the notion of shape in the sense of Borsuk [4]. In the previous paper [9] we proved that a certain decomposition map induces a weak shape equivalence. The purpose of this paper is to prove that a similar theorem holds for shape in the sense of Fox. Let \( X \) be a finite dimensional metric spaces and let \( \mathcal{D} \) be an upper semicontinuous decomposition of \( X \) such that \( k \) each element of which is a closed set being approximatively \( k \)-connected for \( k = 0, 1, \ldots, \max(\dim X, \dim Y) \). Then we shall show that the equality \( Sh(X) = Sh(X') \) holds, where \( X' \) is the decomposition space of \( X \) by \( \mathcal{D} \) and \( Sh(X) \) is the shape of \( X \) in the sense of Fox. As an application of this theorem we can obtain a generalization of Ball’s theorem [1]. Finally, we shall prove that for every finite dimensional and locally compact metric space \( X \) there is a \( D \)-space \( Y \) such that \( \dim X = \dim Y \), \( Sh(X) = Sh(Y) \) and \( Sh_w(X) = Sh_w(Y) \), where \( Sh_w(X) \) is the weak shape of \( X \) defined by Borsuk [3]. Throughout this paper all of spaces are metrizable and maps are continuous.

By an AR-space and an ANR-space we mean always those for metric spaces and by dimension we mean the covering dimension.

§ 2. The shape in the sense of Fox. We first recall the basic notions introduced by Fox [5]. Let \( X \) and \( Y \) be metric spaces and let \( M \) and \( N \) be AR-spaces containing \( X \) and \( Y \) as closed sets respectively. By \( U(X, M) \) we mean the inverse system consisting of open neighborhoods \( U \) of \( X \) in \( M \) and all inclusion maps \( u: U \to U' \), \( U' \subset U \). Similarly, by \( V(Y, N) \) denote the inverse system of open neighborhoods of \( U \) in \( N \). A mutation \( f: U(X, M) \to V(Y, N) \) from \( U(X, M) \) to \( V(Y, N) \) is defined as a collection of maps \( f: U \to V, U \in U(X, M), V \in V(Y, N) \), such that
(2.1) if \( f \in \mathcal{F} \) and \( u: U' \to U \) and \( v: V' \to V \) are inclusions of \( U' \) and \( V' \) into \( U \) and \( V \), \( U' \in \mathcal{U}(X, M) \) and \( V' \in \mathcal{V}(Y, N) \), then \( v \circ f \).

(2.2) every neighborhood \( V \in \mathcal{V}(Y, N) \) is the range of a map \( f \).

(2.3) if \( f_{1}, f_{2} \in \mathcal{F} \) and \( f_{1} \circ f_{2} : U \to V \), then there is a \( U' \in \mathcal{U}(X, M) \), such that \( f_{1} \circ f_{2} \) is \( U' \in \mathcal{U} \) is the inclusion map.

Consider two mutations \( f: U(X, M) \to V(Y, N) \) and \( g: V(Y, N) \to W(Z, P) \). The composition \( gf: U(X, M) \to W(Z, P) \) of \( f \) and \( g \) is given by all the compositions \( gf : U \to M \) which are defined. Two mutations \( f, g: U(X, M) \to V(Y, N) \) are homotopic, \( f \approx g \), if

(2.4) for any maps \( f, g: U \to V \) from \( f \) and \( g \) respectively, there is a \( U' \in U(X, M), U' \in \mathcal{U} \), such that \( f_{*}=g_{*} \), where \( u: U' \to U \) is the inclusion map.

Two metrizable spaces \( X \) and \( Y \) are said to be of the same shape in the sense of Fox (notation: \( Sh(X) \approx Sh(Y) \)) if there exist two mutations \( f: U(X, M) \to V(Y, N) \) and \( g: V(Y, N) \to U(X, M) \) such that

(2.5) \( fg \approx u \) and \( gf \approx v \), where \( u \) and \( v \) are mutations consisting of all inclusions in \( U(X, M) \) and \( V(Y, N) \) respectively.

If the mutations \( f \) and \( g \) satisfy the first of (2.4), then we say that the shape of \( X \) dominates the shape of \( Y \) and we write \( Sh(X) \approx Sh(Y) \).

Let \( k \) be a non-negative integer. According to Borsuk (22), p. 266) a metric space \( X \) is said to be approximately \( k \)-connected if there is an \( AR \)-space \( M \) containing \( X \) as a closed set and satisfying the condition: For every neighborhood \( V \) of \( M \) there is a neighborhood \( U \) of \( X \) such that every map of a \( k \)-sphere \( S^{k} \) into \( U \) is null-homotopic in \( V \). By the same as in the proof of (2.1), Theorem (2.1) we know

(2.6) if \( Sh(X) \approx Sh(Y) \) and \( X \) is approximately \( k \)-connected then \( Y \) is approximately \( k \)-connected.

§ 3. Main theorem and its applications.

Theorem 1. Let \( X \) and \( Y \) be finite dimensional metric spaces and let \( f: X \to Y \) be a closed map from \( X \) onto \( Y \). If \( dim \ Y \leq \alpha \) and for each \( y \in Y \) \( f^{-1}(y) \) is approximately \( k \)-connected, \( k = 0, 1, \ldots, n \) then \( Sh(X) \approx Sh(Y) \). In addition, if \( dim \ Y \leq n \), then \( Sh(X) \approx Sh(Y) \).

The proof of the theorem is given by a similar process to the proof of [9], Theorem 2. We first state lemmas used in the proof of the theorem.

Let \( X \) be a metric space and let \( \mathcal{D} \) be an upper semicontinuous decomposition of \( X \) consisting of closed sets. Denote by \( f \) the decomposition map of \( X \) onto the decomposition space \( Y \) for \( \mathcal{D} \). Let \( M \) be a metric space containing \( X \) such that each element of \( \mathcal{D} \) is closed in \( M \). A collection \( \mathcal{U} \) of open sets in \( M \) is said to be a cover of \( \mathcal{D} \) if \( X = \bigcup \{ U: U \in \mathcal{U} \} \) and for each \( U \in \mathcal{U} \) \( U \cap X \) is non-empty and saturated, i.e. \( U \cap X = f^{-1}(U \cap X) \). The following lemma has been proved in [9] in case each element of \( D \) is compact.

Lemma 1. Every cover of \( D \) has a star refinement.

Proof. Let \( \mathcal{U} \) be a cover of \( D \). Put \( M' = \bigcup \{ U: U \not\in \mathcal{U} \} \). Consider the decomposition \( \mathcal{D}' \) of \( M' \) consisting of each element of \( D \) and each point in \( M' \). Let \( f \) be the decomposition map of \( M' \) onto the decomposition space \( Z \) of \( M' \) by \( \mathcal{D}' \). Since \( \mathcal{D}' \) is upper semicontinuous, \( f \) is a closed map and hence \( Z \) is paracompact by the theorem of Michael [15]. Since each set \( U \in X, U \in \mathcal{U} \), is saturated, \( f(U) \) is an open cover of \( Z \). Take an open cover \( \mathcal{W} \) of \( Z \) such that \( \mathcal{W} > f(U) \). (For two collections \( \mathcal{A} \) and \( \mathcal{B} \), by \( \mathcal{A} > \mathcal{B} \) (resp. \( \mathcal{A} \geq \mathcal{B} \)) we mean \( \mathcal{A} \) is a refinement (resp. star refinement) of \( \mathcal{B} \).) Put \( \mathcal{W}' = f^{-1}(\mathcal{W}) \). Then it is obvious that a cover \( \mathcal{W}' \) of \( D \) is a star refinement of \( \mathcal{W} \).

A cover \( \mathcal{U} \) of \( D \) is said to be an \( n \)-refinement of a cover \( \mathcal{W} \) of \( D \) if there is a sequence \( \mathcal{W}_0, \mathcal{W}_1, \ldots, \mathcal{W}_n \) of covers of \( D \) such that \( \mathcal{W}_0 = D \), \( \mathcal{W}_{i} = \mathcal{W}_{i+1} \) and \( \mathcal{W}_{i+1} > \mathcal{W}_{i+1} \), \( i = 0, 1, \ldots, n \), and for each \( \mathcal{W}_j \), \( i = 0, 1, \ldots, n \), there is a \( \mathcal{V}_j \in \mathcal{V}_j \) such that every map \( \mathcal{G}: \mathcal{S}_j \to \mathcal{V}_j \), \( j = 0, 1, \ldots, n \), is null-homotopic in \( V \). When \( \mathcal{U} \) is an \( n \)-refinement of \( \mathcal{W} \), we write \( \mathcal{U} \geq \mathcal{W} \).

The following lemma is a consequence of Lemma 1.

Lemma 2. Let \( M \) be an AR-space and let \( X \) be a closed subset of \( M \). Let \( \mathcal{D} \) be an upper semicontinuous decomposition of \( X \) each element of which is approximately \( k \)-connected for \( k = 0, 1, \ldots, n \). Then every cover of \( D \) has an \( n \)-refinement.

As an immediate consequence of the definition of an \( n \)-refinement we have

Lemma 3. Let \( \mathcal{U} \) and \( \mathcal{V} \) be covers of \( D \) such that \( \mathcal{U} \geq \mathcal{V} \).

(3.1) Let \( K \) be an \((n+1)\)-dimensional simplicial complex and \( K^0 \) the set of its vertices. If \( f: K^0 \to M \) is a map such that for each closed simplex \( \sigma \) of \( K \) there is a \( U \in \mathcal{U} \) containing \( f(\sigma \cap K^0) \), then \( f \) has an extension \( g: K \to M \) such that for a closed simplex \( \sigma \) of \( K \) there is a \( V \in \mathcal{V} \) containing \( g(\sigma) \).

(3.2) Let \( K \) be an \( n \)-dimensional simplicial complex. If \( f \) and \( g \) are maps of \( K \) into \( M \) such that for each closed simplex \( \sigma \) of \( K \) there is a \( U \in \mathcal{U} \) containing \( f(\sigma) \cup g(\sigma) \) then there is a homotopy \( H: K \times I \to M \) connecting \( f \) and \( g \) such that for each closed simplex \( \sigma \) of \( K \times I \) is contained in some \( V \in \mathcal{V} \).

Remark. In Lemma 3, let \( (\mathcal{V}_0, \mathcal{V}_i) \) be a sequence of covers of \( D \) in the definition of \( n \)-refinements such that \( \mathcal{V}_0 \geq \mathcal{V}_n \) and \( \mathcal{V}_n = \mathcal{V}_{n+1} \). If \( \sigma \) is an \( i \)-simplex of \( K \), then we can construct a map \( g \) and a homotopy \( H \) such that \( g(\sigma) \) is in some element of \( \mathcal{V}_i \) and \( H(\sigma \times I) \) is in some element of \( \mathcal{V}_{i+1} \) for each \( i = 1, 2, \ldots, n \). In particular, if \( \sigma \) is an \( n \)-simplex, then we can assume that \( g(\sigma) \) is in some element of \( \mathcal{V}_n \).
Proof of Theorem 1. Let \( M \) and \( N \) be AR-spaces containing \( X \) and \( Y \) as closed sets respectively. By \( U(X, M) \) denote the inverse system of neighborhoods \( U \) of \( X \) in \( M \) and all inclusion maps \( u: U \to U', U \subset U \). Similarly by \( V(Y, N) \) denote the inverse system of neighborhoods \( V \) of \( Y \) in \( N \). In \( f: M \to N \) be an extension of \( f: X \to Y \). Then \( f \) generates a mutation \( f: U(X, M) \to V(Y, N) \). To prove the theorem we have to construct a mutation \( g: V(Y, N) \to U(X, M) \) such that \( fg = \text{id} \). Consider \( U \) as a cover of the decomposition \( \mathcal{D} = \{ f^{-1}(j): j \in J \} \) of \( X \). Take covers of \( \mathcal{W} \) and \( \mathcal{V} \) of \( \mathcal{D} \) such that
\[
\mathcal{W} \supseteq \mathcal{W}' \supseteq \{ U \}.
\]
Since \( f: X \to Y \) is closed, there is a locally finite open cover \( \mathcal{W}' \) of \( Y \) such that order \( \mathcal{W}' \leq \mathcal{W} \) and \( f^{-1}(\mathcal{W}') \supseteq \mathcal{W} \). Let \( \mathcal{V} \) be a locally finite collection consisting of open sets of \( N \) such that \( \mathcal{V} \cap Y = \mathcal{V}' \) and \( \mathcal{V}' \) and \( \mathcal{W}' \) are similar. Put \( V = \bigcup \{ W: W \in \mathcal{V}' \} \). By \( K \) denote the nerve of \( \mathcal{V}' \) and let \( V: K \to \mathcal{D} \) be a canonical map. Let us define a map \( g': K \to \mathcal{D} \) as follows. For each \( \mathcal{V}' \in \mathcal{V} \), the element of \( \mathcal{V} \) corresponding to \( \mathcal{V}' \) is a point \( x_{\mathcal{V}} \in f^{-1}(W \cap Y) \) of \( X \) and put \( g' = g(x_{\mathcal{V}}) \). For every closed simplex \( \sigma \) of \( X \), \( g'(\sigma) \cap X \) is contained in some element of \( \mathcal{W} \) where \( X^0 \) is the 0-skeleton of \( \mathcal{M} \). Hence, by Lemma 3 (3.1), \( g'' \) is extended to a map \( g': K \to \mathcal{D} \) such that for each closed simplex \( \sigma \) of \( K \) \( g''(\sigma) \) is in some element of \( \mathcal{W} \). Define \( g: V: K \to \mathcal{D} \) by \( g = g' \). Let \( g \) be the collection of all maps \( g: V: U \to \mathcal{D} \), \( U \in \mathcal{U} \). Put \( U \cap \mathcal{V} \cap Y = \mathcal{V}_{\mathcal{V}} \cap Y \), where \( \mathcal{U} \cap \mathcal{V} \cap Y = \mathcal{D} \), \( U \cap \mathcal{V} \cap Y \), \( U \cap \mathcal{V} \cap Y \), \( U \cap \mathcal{V} \cap Y \), and \( U \cap \mathcal{V} \cap Y \) are the inclusion maps of \( U \), \( \mathcal{V} \cap \mathcal{V} \) respectively and \( g \) is a map which is constructed by the above-mentioned process. It is obvious that \( g \) satisfies conditions (2.1) and (2.2). Let \( g_1, g_2 \in g \). Let \( U, \mathcal{V}, \mathcal{V}', \mathcal{V}, \mathcal{V}_1, \mathcal{V}_2 \in \mathcal{D} \), \( g_1, g_2 \in g \). Put \( U, \mathcal{V}, \mathcal{V}', \mathcal{V}, \mathcal{V}_1, \mathcal{V}_2 \in \mathcal{D} \), \( g_1, g_2 \in g \). Therefore, to prove \( u_1 g_2 = g_1 \) \( u_1 g_2 = g_1 \). It is enough to show that \( u_1 g_2 = g_1 \). Let \( K_{g_1} \subset U \cap \mathcal{V} \cap Y \). Then \( K_{g_2} \subset U \cap \mathcal{V} \cap Y \). Let \( K \) be the nerve of \( \mathcal{V} \) and \( g: K \to \mathcal{D} \) be a map such that \( g = g \). Then \( K_{g_2} \subset U \cap \mathcal{V} \cap Y \). Therefore \( g \) is a mutation.

Next, we shall prove that \( fg = \text{id} \). To do it, let \( W \in \mathcal{W} \). Since \( N \) is an AR-space, there is an open collection \( \mathcal{W}' \) of \( N \) such that \( \mathcal{W}' = \bigcup \{ W': W' \subset \mathcal{W} \} \subset \mathcal{W} \), \( \mathcal{W}' \subset V(Y, N) \), for each \( W' \in \mathcal{W}' \). \( \mathcal{W}' \subset Y \) is non empty and if \( h, h' \) are maps of a space \( Z \) into \( V \) such that \( h \) and \( h' \) are \( \mathcal{W}' \)-close then \( h \) and \( h' \) in \( W \). Here, for each \( z \in Z \) there is a \( W_z \subset \mathcal{W}' \) containing both \( h(z) \) and \( h'(z) \), then \( \mathcal{W}' \subset \mathcal{W}' \)-close. Put \( U = f^{-1}(\mathcal{W}') \). Then \( U \subset U(X, M) \). For the element \( U \), there is a neighborhood \( V \in V(Y, N) \) and a map \( g: V: U \to \mathcal{D} \) by the above-mentioned process. Let \( g \) be constructed by choosing covers \( \mathcal{W}, \mathcal{W}' \) of \( \mathcal{D} \) and an open collection \( \mathcal{V} \) of \( N \) such that \( \mathcal{W} \supseteq \mathcal{W}' \supseteq \mathcal{W}' \supseteq \mathcal{W} \), \( \mathcal{W}' \subset \mathcal{W}' \supseteq \mathcal{W}' \supseteq \mathcal{W} \). Put \( g = g \). Let \( K \) be the nerve of \( \mathcal{V} \) and \( g: K \to \mathcal{D} \) be the map such that \( g = g \). \( K \subset \mathcal{D} \) is a canonical map of \( V \subset \mathcal{W} \). From the definition of \( g \) and the choice of \( \mathcal{W} \), we can know that the map \( fg \) and the inclusion map of \( V \subset \mathcal{W} \) are \( \mathcal{W}' \)-close. Therefore \( fg = \text{id} \). This shows that \( fg = \text{id} \).

Finally, suppose that \( \dim X < \infty \). We have to prove \( fg = \text{id} \). Let \( U \subset U(X, M) \) be a cover of \( \mathcal{D} \) and an open collection in \( N \) used for the construction of \( g \). Since \( M \) is an AR-space, we can know that there are open collections \( \mathcal{W} \) and \( \mathcal{W}' \) in \( M \) satisfying the following conditions:
\[
\mathcal{W} \subset \mathcal{W}' = \bigcup \{ W': W' \subset \mathcal{W} \} \quad \text{and} \quad \mathcal{W}' \supseteq \mathcal{W}'.
\]
if $h$ and $h'$ are maps of a space into $W'$ such that $h$ and $h'$ are $W'$-close then $h \approx h'$ in $U$;

\[ X \subset W = \bigcup \{ W'': W'' \cap W' \subset W'' \}; \]

\[ \bigwedge' > W' \setminus f^{-1}W' \text{ and order } \bigwedge' < n + 1; \]

if $L$ is the nerve of $W'$ and $\psi: W \to L$ is a canonical map, then there is a map $h: L \to W'$ such that for each closed simplex $\sigma$ of $L$ there is an element $W''$ of $W'$ such that $\psi^{-1}(\sigma) \cup h(\sigma) = W''$;

let $\pi: L \to K$ be a simplicial projection (cf. (3.10)), where $K$ is the nerve of $\psi'$, then the maps $g' \pi$ and $u'': L \to U$ are $\bigwedge'$-close, where $u''$ is the inclusion map: $W'' \subset U$ and $\bigwedge'$ is a cover of $U$ used for the construction of $g$ (see (3.3)).

Consider the map $f' = \tilde{f}|W': W \to V$, We shall show that $g f'' = u: W \to U$, where $u$ is the inclusion map. Since $\nu'$ and $\phi f''$ are $\nu'$-close $\nu'' \approx \phi f''$: $W \to K$, where $\phi$ is a canonical map of $V$ into $K$. Let $\nu$ be a closed simplex of $L$. By (3.12) there is an element $U''$ of $W'$ such that $g'\nu(\sigma) \cup h(\sigma) = U''$ and hence $g(\nu) = h(\nu)$ in $U$ by Lemma 3.

Consider the following sets in the plane $E^2$:

\[ A = \{ (0, y) : 0 \leq y \leq 1 \}, \quad B = \{ (x, y) : y = \sin \pi x, 0 < x \leq 1 \}, \]

\[ Y = \{ (x, 0) : 0 \leq x \leq 1 \}. \]

Put $T = A \cup B$ and $X = T \setminus \{ (0, -1) \}$. Define $f: T \to Y$ by $f(x, y) = x$ for $(x, y) \in T$, and put $g = f|Y: X \to Y$. Then, for each $y \in X$, $f^{-1}(y)$ and $g^{-1}(y)$ consist of a segment, a half line or one point and hence these sets are $k$-approximatively connected for $k$. However, $\text{Sh}(T) = \text{Sh}(Y) = \text{Sh}(T)$ and $\text{Sh}(X)$. Here $\text{Sh}(T)$ means a trivial shape. Because, note that for every non-zero abelian group $G$ $\tilde{H}^1(\text{CG})$ is an infinite number of elements, where $\tilde{H}^1$ means the Čech cohomology group. Obviously the Čech cohomology group is an invariant for shape in the sense of Fox. (This follows from Mardšeš's characterization (12) and (13) of shape in the sense of Fox and the fact that the Čech cohomology group $\tilde{H}^1(X; G)$ is isomorphic to the group of the homotopy classes of maps of $X$ into Eilenberg-MacLane space $K(G, n)$ (cf. [7]); to obtain a direct proof is easy.) This example shows that the closedness of $f$ in Theorem 1 can not be removed.

Consider the set $A \cup B$ in Example 1. Let $F$ be a continuum obtained from $A \cup B$ by connecting two points $(0, 1)$ and $(1, 0)$ by an arc whose interior does not intersect $A \cup B$. (If $F$ is a Warsaw circle.) Let $S$ be a circle. By $f': F \to S$ denote a map such that $f'(a)$ is a point of $S$ and $f'F = A: F: A \to S - \{ a \}$ is a homeomorphism. Let $X'$ be a topological sum of a countable infinite number of copies of $F$. Similarly, let $Y$ be a countable infinite number of copies of $S$. Let $f: X \to Y$ be a map which is defined by $f'$ on each copy of $F$. Then $f$ is a perfect map, and for each $y \in Y$ $f^{-1}(y)$ is one point or an arc. However, as shown by Godlewski and Nowak [6], $\text{Sh}(X) \neq \text{Sh}(Y)$. Here $\text{Sh}(X)$ means the strong shape of $X'$ defined by Borsuk [3]. This shows that Theorem 1 does not hold for strong shape.

Next, as an application of Theorem 1, we shall generalize Ball's theorem [1]. Let $X$ and $Y$ be finite dimensional and locally compact metric spaces. By $\Phi_\pi$ and $\Phi_T$ denote the decompositions of $X$ and $Y$ consisting of all components respectively. Let $\square X$ and $\square Y$ be the decomposition spaces of $X$ and $Y$ by $\Phi_\pi$ and $\Phi_T$ and let $p: X \to \square X$ and $q: Y \to \square Y$ be the decomposition maps. Suppose that each element of $\Phi_\pi$ and $\Phi_T$ is compact. The following theorem generalizes a theorem of Ball ([1] Theorem 2.4).

**Theorem 2.** Under the hypothesis mentioned above, suppose that $\text{Sh}(X) \leq \text{Sh}(Y)$ (resp. $\text{Sh}(X) \leq \text{Sh}(Y)$). Then there is a homeomorphism into $A$: $\square X \to \square Y$ such that for each locally compact set $F$ of $\square X$

\[ \text{Sh}(p^{-1}(F)) \leq \text{Sh}(q^{-1}(A(F))) \text{ (resp. } \text{Sh}(p^{-1}(F)) \leq \text{Sh}(q^{-1}(A(F)))) \]

Moreover, if $\text{Sh}(X) = \text{Sh}(Y)$ (resp. $\text{Sh}(X) = \text{Sh}(Y)$), then there is a homeomorphism $A$: $\square X \to \square Y$ for which the equality holds in (13).

**Proof.** Since $X$ and $Y$ are locally compact and each component in $\Phi_\pi$ and $\Phi_T$ is compact, the maps $p$ and $q$ are perfect and $\dim \square X = \dim \square Y = 0$. Let $M_0$ and $M$ be AR-spaces containing $X$ and $Y$ as closed sets respectively, and let $\beta: M_0 \to M$ be an extension of $p$. Since each element of $\Phi_\pi$ is connected, it is approximatively 0-connected. Since $\dim \square X = 0$, by Theorem 1, we know that there is a mutation $h: (\square X, M_0) \to (\square X, M)$ such that $\text{Sh}(X) \leq \text{Sh}(Y)$, and $\text{Sh}(X, M)$ are the inverse systems consisting of neighborhoods of $\square X$ and $\square Y$, respectively, $\nu'$ is the mutation consisting of the inclusion maps in $\square X, M_0$ and $p$ is the mutation generated by $\beta$. Similarly, if $N_0$ and $N$ are AR-spaces containing $Y$ and $\square Y$ as closed sets respectively, and $\beta': M_0' \to M'$ is an extension of $q$. Since $\text{Sh}(X) \leq \text{Sh}(Y)$, and $\text{Sh}(X, M)$ are the inverse systems consisting of neighborhoods of $\square Y$ and $\square Y$ in $\square Y$ and $\square Y$ respectively, $\nu'$ is the mutation consisting of the inclusions in $\square Y, N_0$ and $q$ is the mutation generated by $\beta$. Since $\text{Sh}(X) \leq \text{Sh}(Y)$, there are mutations $f: X(M_0) \to V(Y, N)$ and $g: V(Y, N) \to U(X, M)$ such that $\nu' = \alpha$, where $\nu$ is the mutation consisting of the inclusions in $U(X, M)$ and $\nu'$ is the mutation generated by $\beta$. Since $\text{Sh}(X) \leq \text{Sh}(Y)$, there is a unique map $A': \square Y \to \square X$ which generates the mutation $\nu'$. Similarity we know that there is a unique map $A': \square Y \to \square X$ which generates the mutation $\nu'_0$.

First, let us prove $A': \square Y \to \square X$ which generates the mutation $\nu'_0$.

\[ A' = 1_\square X, \quad \text{where } 1_\square X \text{ is the identity map of } \square X. \]

Suppose that $A'(a) \neq a$ for $a \in \square X$. Since $\dim \square X = 0$, there are open sets $V'$
and \(V''\) of \(M'\) such that \(a \in V', A'A(a) \in V''\), \(V' \cap V'' = \emptyset\) and \(V' \cup V'' \in U(\Box X, M')\). Put \(U' = \beta^{-1}(V')\) and \(U'' = \beta^{-1}(V'')\). Then \(U = U' \cup U'' \in U(X, M)\). Since \(g|_X = \emptyset\), there exist \(W \in U(X, M)\) and \(g \in g\) such that \(W \subseteq U\) and \(g|_W = W \cap U\). Where \(U\) is the inclusion. Therefore we have \(g|_W \subseteq g(W) \subseteq U\). This relation contradicts that \(A'(A(a)) \in V''\). Thus (3.14) holds. We know that \(A\) is a homeomorphism into. Next, we shall show that

(3.15) if \(H\) is a closed subset of \(\Box X\) then \(Sh(p^{-1}(H)) \subseteq Sh[q^{-1}(A(K))].\)

Consider the inverse systems \(U(p^{-1}(H), M)\) and \(V[q^{-1}(A(K)), N]\) consisting of neighborhoods of \(p^{-1}(H)\) and \(q^{-1}(A(K))\) in \(M\) and \(N\) respectively. To prove (3.15), it is enough to show that \(f\) is a mutation of

\[ U(p^{-1}(H), M) \rightarrow V[q^{-1}(A(K)), N] \]

and \(g\) is a mutation of

\[ V[q^{-1}(A(K)), N] \rightarrow U(p^{-1}(H), M) \]

It is easy to see that this fact is shown by (3.14) and the following assertion.

(3.16) For every open neighborhood \(W\) of \(q^{-1}(A(K))\) in \(N\) there exist \(V \in \varGamma(V, N)\), open sets \(V'\) and \(V''\) of \(N\) such that \(q^{-1}(A(K)) \subseteq V' \subseteq W\), \(V' \cap V'' = \emptyset\), \(V = V' \cup V''\), and \(V' \cap Y\) and \(V'' \cap Y\) are saturated, i.e. \(q^{-1}(q(V' \cap Y)) = V' \cap Y\) and \(q^{-1}(q(V'' \cap Y)) = V'' \cap Y\).

Let us prove (3.16). Since \(g\) is a closed map, for an open neighborhood \(W\) of \(q^{-1}(A(K))\) in \(N\) \(\varGamma(V, Y = W)\) and \(A(K)\) are disjoint closed sets in \(\Box Y\). Since \(\varGamma(Y = W) = \emptyset\), it follows that there are open sets \(W'\) and \(W''\) of \(\varGamma(Y = W)\) such that \(\varGamma(A(K) = W') \subseteq W', q(Y = W) = W'' \cap W' = \emptyset\) and \(W' \cup W'' = W\). Put \(V' = q^{-1}(W')\) and \(V'' = q^{-1}(W'')\). Then \(V' \cap Y = V'' \cap Y = \emptyset\), and \(V' = V' \cup V''\) satisfies the conditions of (3.16). Thus (3.16) and hence (3.15) was proved. As shown in the proof of [1], Lemma 2.3, the theorem follows from (3.16), (6), Theorem 4.2 and the fact that every locally compact 0-dimensional metric space is a union of a discrete family of compact sets. The assertion for weak shape is proved by making use of [9], Theorems 1 and 2 in place of [12], Theorem 1 and Theorem 1 in the above proof. This completes the proof.

The following corollary concerns a problem raised by Ball ([1], P 888).

**Corollary.** Under the same hypothesis as in Theorem 2, suppose that each element of \(\varGamma X\) is of trivial shape. Then there is a homeomorphism into \(\Lambda : X \rightarrow Y\) such that for every set \(K \subseteq \Box X\)

(3.17) \(Sh(p^{-1}(K)) \subseteq Sh[q^{-1}(A(K))].\) (resp. \(Sh_{wp}(p^{-1}(K)) \subseteq Sh_{wp}(q^{-1}(A(K))).\))

Moreover, if \(Sh(X) = Sh(Y)\) (resp. \(Sh_{wp}(X) = Sh_{wp}(Y)\)), then there is a homeomorphism \(\Lambda : X \rightarrow Y\) and the equality holds in (3.17).

**Proof.** By Theorems 1 and 2 we know that \(Sh(p^{-1}(K)) = Sh(X) = Sh(A(K)) \subseteq Sh(q^{-1}(A(K))).\) This shows the first part of the corollary. If \(Sh(X) = Sh(Y)\), then \(\Lambda\) is a homeomorphism and \(Sh(p^{-1}(a)) = Sh_{wp}(q^{-1}(a))\) for each \(a \in X\). Thus each element of \(\varGamma X\) is of trivial shape. By Theorem 1 \(Sh(a(K)) = Sh(q^{-1}(A(K))).\) This completes the proof.

The following definition was given for compact spaces in [10].

**Definition.** A metric space \(X\) is said to be a 0-space if there is an inverse sequence \(\{K_n, n_{>1}\}\) consisting of simplicial complexes \(K_n\) with metric topology and simplicial maps \(n_{>1} : K_{n+1} \rightarrow K_n\) such that \(\lim_n K_n = X\) (cf. [11]).

**Theorem 3.** Let \(X\) be a finite dimensional and locally compact metric space. Then there exists a 0-space \(Y\) such that \(dim X = dim Y\), \(Sh(X) = Sh_{wp}(X)\) and \(Sh(X) = Sh(Y)\).

**Proof.** Let \(\{K_n\}\) be a sequence of locally finite open covers of \(X\) such that each element of \(\varGamma X\) has a compact closure, order \(K_n \leq dim X + 1\), \(K_{n+1} = K_n\) for \(n = 1, 2, \ldots, \text{mesh} K_n = 0\) (\(n \rightarrow \infty\)), where \(K_0 = \emptyset\). Let \(K_0\) be the nerve of \(\varGamma X\) with metric topology. For each \(n\), let \(\pi_{n_{>1}} : K_n \rightarrow K_{n+1}\) be a simplicial projection of \(K_{n+1}\) into \(K_n\) defined by mapping a vertex \(v\) of \(K_{n+1}\) corresponding to \(v \in \varGamma(X, n_{>1})\) to a vertex \(w\) of \(K_n\) corresponding to \(w \in \varGamma X\), such that \(v = W\). Consider the inverse sequence \(\{K_n, n_{>1}\}\) and put \(Y = \lim_n K_n\). Then \(Y\) is a 0-space and \(dim X = dim Y\). As shown by Kaul [9], there is a perfect map \(f\) onto \(Y\) such that for each \(x \in X = \varGamma X\) is the inverse limit of an inverse sequence consisting of closed simplices and \(f^{-1}(\{x\})\) is of trivial shape. By Theorem 1 and [9], Theorem 2 we know \(Sh(X) = Sh(Y)\) and \(Sh_{wp}(X) = Sh_{wp}(Y)\). This completes the proof.

The assertion in Theorem 3 for weak shape was proved in [11], Theorem 2 by a different way. As shown there, we can obtain two properties of compactness of \(X\) in Theorem 3. To see it, let \(X = \varGamma x\) be a set consisting of all rational numbers in a line. If \(Y\) is a 0-dimensional space, and \(Sh(X) = Sh(Y)\) or \(Sh_{wp}(X) = Sh_{wp}(Y)\), then \(X\) and \(Y\) are homeomorphic by [12], Theorem 1 and [9], Theorem 1 and hence \(Y\) is not completely metrizable. Since every finite dimensional 0-space is completely metrizable, \(Y\) is not a 0-space.

**References**

Some uniformization results

by

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Abstract. Some results on the uniformization of Borel sets are proved in this paper.

1. Introduction. Let $X, Y$ be Polish spaces and $B \subseteq X \times Y$. We say $C$ uniformizes $B$ if $C \subseteq B$ and for all $x \in \text{pr}_Y B$, $C \cap B^x$ is a singleton where $B^x = \{y : (x, y) \in B\}$ and $\text{pr}_Y B$ is the projection of $B$ to $X$. In general, a Borel set $B$ does not have a Borel uniformization (22), (65). However, in some cases, such a uniformization exists, for example, if $B^x$ is $\sigma$-compact for each $x$ [1] or if $\mu(B^x) > 0$ for each $x$, where $\mu$ is a probability measure on the Borel $\sigma$-algebra of $Y$ [3].

The chief aim of this paper is to prove the following:

**Theorem 1.** Let $X, Y$ be Polish spaces and $B \subseteq X \times Y$ be Borel. $B$ has a Borel uniformization if any one of the following is true:

1. For all $x \in \text{pr}_Y B$, $B^x$ contains an isolated point,
2. For all $x \in \text{pr}_Y B$, $B^x$ contains a point which is not its point of condensation,
3. For all $x \in \text{pr}_Y B$, $B^x$ is not meager.

The paper is organized in the following way. Section 2 is devoted to preliminaries. In Section 3, a proof of Theorem 1 is given. In Section 4, a related result is proved.

2. Preliminaries. A set is called meager if it is a countable union of nowhere dense sets. A comeager set is one whose complement is meager. Let $X, Y$ be Polish spaces, $B \subseteq X \times Y$ and $U \subseteq Y$. Following Vaught, we put $B^x_U = \{x : B^x \cap U$ is comeager in $U\}$. It is known that if $B$ is Borel and $U$ open, then $B^x_U$ is Borel [7]. For any set $A$, let $\delta(A)$ denote the diameter of $A$.

If $f$ is a function, put $Z_f = \{y : f^{-1}(y)$ is a singleton$\}$, $I_f = \{y : f^{-1}(y)$ contains an isolated point$\}$, $D_f = \{y : f^{-1}(y)$ is countable and non-empty$\}$, $C_f = \{y : f^{-1}(y)$ contains a point which is not its condensation point$\}$. It is known that if $f$ is a Borel measurable function defined on a Borel subset of a Polish space into a separable metric space, then $Z_f, I_f, D_f, C_f$ are coanalytic [4].

3. Proof of main theorem.

Proof of (1). Let $\{U_a\}$ be a countable open base for $Y$. For any $n$, define $f_n$ on $B \cap (X \times U_n)$ by $f_n(x, y) = n$. Let $Z_n = \{x : B^x \cap U_n$ is a singleton$\}$. Then $Z_n = Z_{f_n}$.