On a certain condition of the monotonicity of functions

by

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Abstract. The generalized derivative (Khintchine's derivative) of a real valued function of a real variable is investigated. The sufficient condition of monotonicity of a function is given.

The classical theorem about the monotonicity of a differentiable function with a non-negative derivative has been generalized in many ways. For example:

TOLSTOV'S THEOREM [5]. *Let f be a function satisfying in the interval (a, b) the following conditions*:

(a) *f is approximately continuous,*

(b) *f'_{ap} exist except perhaps at a countable set of points (i.e. nearly everywhere),*

(c) *$f'_{ap} \geqslant 0$ a.e.*

Then f is continuous and non-decreasing in (a, b).

ZAHORSKI'S THEOREM [6]. *Let f be a function satisfying in the interval (a, b) the following conditions*:

(α) *f is a Darboux function,*

(β) *f' exists n.e.,*

(γ) *$f' \geqslant 0$ a.e.,*

Then f is continuous and non-decreasing in (a, b).

In both of these theorems it is assumed, directly or indirectly, that the function f is a Darboux function of the first class of Baire. In connection with this Zahorski asks in [6] whether the following hypothesis is true.

ZAHORSKI'S HYPOTHESIS. *Let f be a function satisfying in (a, b) the following conditions*:

1) *f is a Darboux function of the first class of Baire,*

2) *f'_{ap} exists n.e.,*

3) *$f'_{ap} \geqslant 0$ a.e.*

Then f is continuous and non-decreasing in (a, b).

Bruckner ([1]) and Świątkowski ([3]) give an affirmative answer to this question.

The three above-mentioned theorems give the characterizations of the same class of functions, namely: the class of continuous and non-decreasing functions which have ordinary derivatives n.e. This follows from Khintchine's theorem ([2]), which says that every point at which a monotonic function f is approximatively differentiable is a point at which that function has an ordinary derivative. This remark suggests the possibility of replacing the ordinary derivative by a generalized derivative which for monotonic function coincides (in the sense of existence and value) with the ordinary derivative. The main theorem (Theorem 2) of this paper is such a generalization of Zahorski's theorem.

Suppose that to every point x of the interval (a, b) there is attached a family $T(x)$ of subsets of (a, b) which satisfies the following conditions:

(a) $x \in E$ for each $E \in T(x)$,

(b) if $E_1 \in T(x)$ and $E_2 \in T(x)$, then $E_1 \cap E_2 \in T(x)$,

(c) if $\delta>0$ and $E \in T(x)$, then the sets $E \cap (x-\delta, x)$ and $E \cap (x, x+\delta)$ are non-empty,

(d) if $\delta>0$, then $(x-\delta, x+\delta) \in T(x)$.

The sets of the family $T(x)$ will be called *T-neighbourhoods of the point x*.

DEFINITION 1. A point x will be called a *T-accumulation point of the set A* $\subset (a, b)$ if each T-neighbourhood of x contains points of the set $A-\{x\}$.

The set of T-accumulation points of A will be denoted by A'_T.

DEFINITION 2. A number g is called the *T-limit of the function f at the point* x_0 if for every $\varepsilon>0$ there exists an $E \in T(x_0)$ such that for every point $x \in E-\{x_0\}$ the following inequality is satisfied:

$$|f(x)-g|<\varepsilon .$$

$T-\lim\limits_{x \to x_0} f(x)$ means the T-limit of f at x_0.

Analogously we define $T-\lim\limits_{x \to x_0} f(x) = \pm\infty$.

DEFINITION 3. The T-derivative of a function f at the point x_0 is the T-limit

$$f'_T(x_0) = T-\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} .$$

One proves that under some additional conditions on $T(x)$, the T-derivative of a monotonic function is its ordinary derivative.

DEFINITION 4. $T(x_0)$ satisfies *Khintchine's condition* if the conditions

(1) $$\lim_{n\to\infty} x_n = x_0 ,$$

(2) $$\delta_n \downarrow 0 ,$$

(3) $$\lim_{n\to\infty} \frac{\delta_n}{|x_n-x_0|} > 0$$

imply that $x_0 \in \left(\bigcup_{n=1}^{\infty} (x_n-\delta_n, x_n+\delta_n)\right)'_T$.

REMARK 1. If conditions (1)-(3) are satisfied, then we have also $x_0 \in \left(\bigcup_{k=1}^{\infty} (x_{n_k}-\delta_{n_k}, x_{n_k}+\delta_{n_k})\right)'_T$, where $\{n_k\}$ is any subsequence of the sequence of natural numbers.

REMARK 2. If $T(x)$ is the family of the sets containing x for which x is a density point, then $f'_T(x) = f'_{ap}(x)$.

THEOREM (Świątkowski, [4]). *$T(x_0)$ satisfies the condition of Khintchine if and only if for every function f which is monotonic in some neighbourhood of x_0 the existence of $f'_T(x_0)$ implies the existence of $f'(x_0)$.*

It will be convenient in the sequel to have

DEFINITION 5. We shall say that the function f and the family $T = \{T(x)\}$ satisfy *condition* (W) in the interval (a, b) if

(1) f is a Darboux function,

(2) f is n.e. continuous,

(3) $T(x)$ satisfies Khintchine's condition for nearly every point $x \in (a, b)$,

(4) f'_T exists n.e.

Furthermore $\{p_n : n \in N\}$ will denote the set of points with the exception of which f is continuous, T satisfies Khintchine's condition and f'_T exists.

LEMMA 1. *Let f and T satisfy condition* (W) *in the interval* (a, b) *and let* α, β *be numbers such that* $\alpha>\beta$. *Then at most one of the sets*

$$A = \{x\colon f'_T(x)>\alpha\} , \quad B = \{x\colon f'_T(x)<\beta\}$$

can be dense in (a, b).

Proof. Without loss of generality we may assume that $\alpha>0>\beta$. Now suppose, on the contrary, that $\bar{A} = \bar{B} = \langle a, b\rangle$. Then there exists an $x_1 \in A-\{p_1\}$. Since $f'_T(x_1)>\alpha$, there is a T-neighbourhood $E_1 \in T(x_1)$ such that

$$\frac{f(x)-f(x_1)}{x-x_1}>\alpha \quad \text{for all } x \in E_1-\{x_1\} .$$

Let δ_1 be such a positive number that $p_1 \notin \langle x_1-\delta_1, x_1+\delta_1\rangle$ and let $x \in E_1 \cap (x_1-\delta_1, x_1)$. Hence $(x, f(x))$ lies under the line $y = \alpha(x-x_1)+f(x_1)$. Because f is a Darboux function in (x, x_1), there is a non-denumerable set of such points z that $f(z)<\alpha(z-x_1)+f(x_1)$. Let x'_1 be such a point in $(x, x_1)-\{p_n : n \in N\}$.

The continuity of the function f in x'_1 implies the existence of such a number $d'_1>0$ that

$$f(x)<\alpha(x-x_1)+f(x_1) \quad \text{for all } x \in \langle x'_1-d'_1, x'_1+d'_1\rangle .$$

Put

$$a_1' = \sup\{x\colon f(t)\leqslant \alpha(t-x_1)+f(x_1) \ \text{for all}\ t\in\langle x_1'-d_1', x\rangle\}.$$

We have of course $a_1'\leqslant x_1$. Let be $0<\sigma_1<\frac{1}{2}(a_1'-x_1'+d_1')$. In the interval $(a_1', a_1'+\sigma_1)$ there are uncountably many points z such that $f(z)>\alpha(z-x_1)+f(x_1)$. Let $x_1''\notin\{p_n\colon n\in N\}$ be one of them. Since the function f is continuous at x_1'', there is a positive number d_1'' such that

$$f(x)>\alpha(x-x_1)+f(x_1) \quad \text{for all}\ x\in\langle x_1''-d_1'', x_1''+d_1''\rangle.$$

Put

$$b_1 = x_1''-d_1'',\quad a_1 = b-\delta_1' \quad\text{where}\quad 0<\delta_1'<\tfrac{1}{2}d_1'' \ \text{and}\ b_1-\delta_1'>a_1',$$

$$A_1=(x_1'-d_1', a_1'),\quad B_1=(b_1, x_1''+d_1'').$$

Then we have

$$\frac{f(x')-f(x'')}{x'-x''}\geqslant\alpha \quad\text{for}\ x'\in A_1 \ \text{and}\ x''\in B_1.$$

Now, since it was assumed that $\bar{B}=\langle a,b\rangle$, we can find $x_2\in(a_1,b_1)\cap B-\{p_2\}$. As before, there is an $E_2\in T\{x_2\}$ such that

$$\frac{f(x)-f(x_2)}{x-x_2}<\beta \quad \text{for}\quad x\in E_2-\{x_2\}.$$

Let δ_2 be such a positive number that $p_2\notin\langle x_2-\delta_2, x_2+\delta_2\rangle$. Because f is a Darboux function, we can find a point $x_2'\in(x_2-\delta_2,x_2)-\{p_n\colon n\in N\}$ such that

$$\frac{f(x_2')-f(x_2)}{x_2'-x_2}<\beta.$$

Since f is continuous at x_2', there is a $d_2'>0$ such that

$$f(x)>\beta(x-x_2)+f(x_2) \quad\text{for all}\ x\in\langle x_2'-d_2', x_2'+d_2'\rangle.$$

Put

$$a_2'=\sup\{x\colon f(t)\geqslant\beta(t-x_2)+f(x_2) \quad \text{for all}\ t\in\langle x_2'-d_2', x\rangle\}.$$

It is obvious that $a_2'\leqslant x_2$ and in every interval $(a_2', a_2'+\eta)$ where $\eta>0$, there are points x such that corresponding points of the graph of the function f lie below the line $y=\beta(x-x_2)+f(x_2)$.

Let

$$0<\sigma_2<\frac{a_2'-x_2'+d_2'}{2^2}.$$

In the interval $(a_2', a_2'+\sigma_2)$ there are uncountably many points x for which the inequality $f(x)<\beta(x-x_2)+f(x_2)$ holds. Let x_2'' be such a point not belonging to

$\{p_n\colon n\in N\}$. Because of the continuity of f at x_2'' there is a positive number d_2'' such that

$$f(x)<\beta(x-x_2)+f(x_2)\quad\text{for all}\ x\in(x_2''-d_2'', x_2''+d_2'').$$

Put

$$b_2=x_2''-d_2'',\ a_2=b_2-\delta_2',\quad\text{where}\quad 0<\delta_2'<\frac{d_2''}{2^2},\ b_2-\delta_2'\geqslant a_2'$$

and

$$A_2=(x_2'-d_2', a_2'),\quad B=(b_2, x_2''+d_2'').$$

Then we have

$$\frac{f(x')-f(x'')}{x'-x''}<\beta\quad\text{for}\quad x'\in A_2 \ \text{and}\ x''\in B_2.$$

Repeating the above argument, we obtain sequences of numbers $\{x_n'\}$, $\{x_n''\}$, $\{d_n''\}$, $\{a_n\}$, $\{b_n\}$, $\{a_n'\}$ and sequences of intervals $\{A_n\}$, $\{B_n\}$ such that

(1) $a_0=a$, $b_0=b$ and for $n\geqslant1$

$$a_{n-1}<x_n'-d_n'<x_n'+d_n'\leqslant a_n'\leqslant a_n<b_n=x_n''-d_n''<x_n''<x_n''+d_n''<b_{n-1},$$

(2) $$x_n''-a_n'<\frac{a_n'-(x_n'-d_n')}{2^n}\quad\text{and}\quad b_n-a_n<\frac{x_n''+d_n''-b_n}{2^n},$$

(3) $$p_n\notin\langle a_n,b_n\rangle,$$

(4) $$\langle a_n,b_n\rangle\supset\langle a_{n+1},b_{n+1}\rangle \quad\text{and}\quad b_n-a_n<\frac{b-a}{2^n},$$

(5) $$A_n=(x_n'-d_n',a_n'),\quad B_n=(b_n,x_n''+d_n''),$$

(6) $$\frac{f(x')-f(x'')}{x'-x''}>\alpha\quad\text{for}\quad x'\in A_{2n-1}\quad\text{and}\quad x''\in B_{2n-1},$$

(7) $$\frac{f(x')-f(x'')}{x'-x''}<\beta\quad\text{for}\quad x'\in A_{2n}\quad\text{and}\quad x''\in B_{2n}.$$

Let $\{x_0\}=\bigcap_{n=1}^{\infty}\langle a_n,b_n\rangle$. From (3) it follows that $x_0\notin\{p_n\colon n\in N\}$. Hence $f_T'(x_0)$ exists. But

$$x_n=\tfrac{1}{2}(x_n'-d_n'+a_n')\in\langle a_{n-1},b_{n-1}\rangle$$

so $x_n\to x_0$ and $y_n=x_n''\in\langle a_{n-1},b_{n-1}\rangle$ and so $y_n\to x_0$ too.

Furthermore

$$\alpha_n=\tfrac{1}{2}(a_n'-x_n'+d_n')\downarrow 0\quad\text{and}\quad d_n''\downarrow 0$$

as well as

$$\frac{\alpha_n}{|x_n-x_0|}\to 1 \quad \text{and} \quad \frac{d_n''}{|y_n-x_0|}\to 1 .$$

From this, and because

$$A_n = (x_n-\alpha_n, x_n+\alpha_n), \quad B_n = (y_n-d_n'', y_n+d_n''),$$

it follows that for every subsequence $\{n_k\}$ of the sequence of natural numbers we have

(8) $$x_0 \in \Big(\bigcup_{k=1}^{\infty} A_{2n_k-1}\Big)_T' \quad \text{and} \quad x_0 \in \Big(\bigcup_{k=1}^{\infty} B_{2n_k-1}\Big)_T'$$

and

(9) $$x_0 \in \Big(\bigcup_{k=1}^{\infty} A_{2n_k}\Big)_T' \quad \text{and} \quad x_0 \in \Big(\bigcup_{k=1}^{\infty} B_{2n_k}\Big)_T' .$$

Hence for every set $E\in T(x_0)$ there exist such numbers n, m that none of the four sets $A_{2n-1}\cap E$, $B_{2n-1}\cap E$, $A_{2m}\cap E$, $B_{2m}\cap E$ is empty. This implies, by (6) and (7), that $f_T'(x_0)$ does not exist. This contradiction proves the lemma.

COROLLARY 1. *Under the assumptions of Lemma* 1 *at most one of the sets*

$$A = \{x:\, f_T'(x)\geqslant \alpha\}, \quad B = \{x:\, f_T'(x)\leqslant \beta\}$$

can be dense in the interval (a, b).

LEMMA 2. *Let f and T satisfy condition* (W) *in the interval* (a, b) *and* $f_T'(x)\geqslant M>0$ *n.e. in* (a, b). *Then there exists a non-empty interval* $(\alpha, \beta)\subset(a, b)$ *such that* $f|_{(\alpha,\beta)}$ *is continuous and non-decreasing.*

Proof. Suppose, on the contrary, that there is no interval $(\alpha, \beta)\subset(a, b)$ in which f is non-decreasing. Put $a_1 = a$, $b_1 = b$. Then there are in (a_1, b_1) two points x_1', x_1'' such that

$$a_1<x_1'<x_1''<b_1 \quad \text{and} \quad f(x_1')>f(x_1'') .$$

We can assume that $p_1\notin\langle x_1', x_1''\rangle$ and $x_1'\notin\{p_n:\, n\in N\}$. (Indeed, if $p_1\in\langle x_1', x_1''\rangle$, then either $f(x_1')>f(p_1)$ or $f(p_1)>f(x_1'')$. If, for example, $f(p_1)>f(x_1'')$, then, since f is a Darboux function, in the interval (p_1, x_1'') there are uncountably many points x satisfying the inequality $f(x)>f(x_1'')$. We can choose one that is different from all p_n and substitute it for x_1'. In the case $f(p_1)<f(x_1')$ the proof proceeds analogously).

Let $0<\varepsilon_1<\frac{1}{2}[f(x_1')-f(x_1'')]$. By the continuity of f in x_1' we have a $d_1>0$ such that

$$f(x)>f(x_1')-\varepsilon_1 \quad \text{for all } x\in\langle x_1', x_1'+d_1\rangle .$$

Let us put

$$a_2' = \sup\{x:\, f(t)\geqslant f(x_1')-\varepsilon_1 \text{ for all } t\in\langle x_1'+d_1, x\rangle\} .$$

It is evident that $a_2'<x_1''$ and in every interval $(a_2', a_2'+\delta)$ $(\delta>0)$ there are uncountably many points x such that $f(x)<f(x_1')-\varepsilon_1$. Let x_1''' denote one of them,

such that $x_1'''\notin\{p_n:\, n\in N\}$ and $a_2'<x_1'''<\min[x_1'', a_2'+\frac{1}{4}(a_2'-x_1')]$. Hence for every positive number $\varepsilon_1'<\frac{1}{2}[f(x_1')-\varepsilon_1-f(x_1''')]$ there exists a $d_1'>0$ such that

$$f(x)<f(x_1''')+\varepsilon_1' \quad \text{for all } x\in\langle x_1'''-d_1', x_1'''\rangle .$$

Setting

$$b_2 = \inf\{x:\, f(t)\leqslant f(x_1''')+\varepsilon_1' \quad \text{for all } t\in\langle x, x_1'''-d_1'\rangle\},$$
$$a_2 = \max[a_2', b_2-\tfrac{1}{8}(x_1'''-b_2)],$$
$$A_1 = (x_1', a_2'), \quad B_1 = (b_2, x_1'''),$$

we have

$$b_2>a_2', \quad p_1\notin\langle a_2, b_2\rangle\subset(x_1', x_1'')$$

and

$$f(x')>f(x'') \quad \text{for} \quad x'\in A_1 \quad \text{and } x''\in B_1$$

because

$$f(x'')\leqslant f(x_1''')+\varepsilon_1'<f(x_1')-\varepsilon_1'-\varepsilon_1\leqslant f(x') .$$

Since we have supposed that there is no interval in which f is non-decreasing, we can define recurrently sequences of numbers $\{x_n'\}$, $\{x_n''\}$, $\{a_n'\}$, $\{b_n\}$, $\{a_n\}$, and sequences of intervals $\{A_n\}$, $\{B_n\}$ such that

(1) $a_1 = a$, $b_1 = b$ and $a_{n-1}<x_{n-1}'<a_n'\leqslant a_n<b_n<x_{n-1}''<b_{n-1}$ and $b_n-a_n<\frac{1}{2}(b_{n-1}-a_{n-1})$ for $n>1$,

(2) $$p_n\notin\langle a_{n+1}, b_{n+1}\rangle,$$

(3) $A_n = (x_n', a_{n+1}') = (\bar{x}_n-\delta_n, \bar{x}_n+\delta_n)$ where $\bar{x}_n = \frac{1}{2}(a_{n+1}'+x_n')$ and $\delta_n = \frac{1}{2}(a_{n+1}'-x_n')$,
$B_n = (b_{n+1}, x_n'') = (\bar{\bar{x}}_n-\sigma_n, \bar{\bar{x}}_n+\sigma_n)$ where $\bar{\bar{x}}_n = \frac{1}{2}(b_{n+1}+x_n'')$ and $\sigma_n = \frac{1}{2}(x_n''-b_{n+1})$,

(4) $$\delta_n\leqslant|\bar{x}_n-b_{n+1}|\leqslant\delta_n+\frac{\delta_n}{2^n}, \quad \sigma_n\leqslant|\bar{\bar{x}}_n-a_{n+1}|\leqslant\sigma_n+\frac{\sigma_n}{2^n},$$

(5) $$\delta_n\downarrow 0, \quad \sigma_n\downarrow 0,$$

(6) $$f(x')>f(x'') \quad \text{for} \quad x'\in A_n \text{ and } x''\in B_n .$$

Conditions (1) and (2) imply the existence of such a point $x_0\notin\{p_n:\, n\in N\}$ that $\{x_0\} = \bigcap_{n=1}^{\infty}\langle a_n, b_n\rangle$. Hence there is a T-neighbourhood E of the point x_0 such that for all $x\in E-\{x_0\}$ the following inequality holds:

(7) $$\frac{f(x)-f(x_0)}{x-x_0}>\frac{1}{2}M .$$

On the other hand, since $\bar{x}_n \to x_0$ and $\bar{\bar{x}}_n \to x_0$, from conditions (4), (5) it follows that

$$x_0 \in \Big(\bigcup_{k=1}^{\infty} A_{n_k}\Big)'_T \quad \text{and} \quad x_0 \in \Big(\bigcup_{l=1}^{\infty} B_{n_l}\Big)'_T$$

for all subsequences $\{n_k\}$, $\{n_l\}$ of the sequence of natural numbers. Hence there is a natural number n such that the sets $A_n \cap E$ and $B_n \cap E$ are non-empty. Thus (6) contradicts (7) and the lemma is proved.

COROLLARY 2. *Under the assumption of Lemma 2, in the interval (a, b) there exists a dense set of intervals of monotonicity of the function f.*

LEMMA 3. *Let f and T satisfy condition (W) in (a, b) and $f'_T(x) \geqslant M > 0$ on a dense set in (a, b). Then there exists an interval $(\alpha, \beta) \subset (a, b)$ such that $f|_{(\alpha,\beta)}$ is non-decreasing.*

Proof. Let $A = \{x: f'_T(x) \geqslant M\}$ and $B = \{x: f'_T(x) \leqslant \frac{1}{2}M\}$. From Corollary 1 it follows that B is not dense in (a, b). Then there exists an interval $(a_1, b_1) \subset (a, b)$ which is disjoint with B. Thus the function $f|_{(a_1,b_1)}$ satisfies the assumptions of Lemma 2 in (a_1, b_1), and so the existence of the interval (α, β) with the required property is proved.

COROLLARY 3. *Under the assumptions of Lemma 3 there exists in (a, b) a dense set of intervals of monotonicity of the function f.*

REMARK 3. If moreover $f'_T \geqslant M > 0$ holds almost everywhere, $\langle \alpha, \beta \rangle \subset (a, b)$ and the function f is monotonic on (α, β), then f is continuous in (α, β) and $f(\beta) - f(\alpha) \geqslant M(\beta - \alpha)$.

LEMMA 4. *Let f and T satisfy condition (W) in (a, b) and let $f'_T(x) \geqslant M > 0$ hold a.e. in (a, b). Let $\beta \in (a, b)$ be such a point that the function f is not monotonic in any interval which contains β as the left end-point. Then for every pair of positive numbers ε and δ there exists such a point x_0 that the following conditions hold:*

(1) $$x_0 \in (\beta, \beta + \delta),$$

(2) $$f(x_0) > f(\beta) - \varepsilon,$$

(3) f is not monotonic in any interval $(x_0 - h, x_0)$,

(4) a) f is monotonic in the interval $(x_0, x_0 + h)$ for certain $h > 0$ or
b) x_0 is the point of continuity of f.

Proof. Suppose that there exist numbers $\varepsilon > 0$ and $\delta > 0$ such that there is no point x_0 satisfying conditions (1)-(4). Since f is a Darboux function, continuous in (a, b) except at most at a countable set of points, we have a point of continuity of f $x_1 \in (\beta, \beta + \delta)$ such that $f(x_1) > f(\beta) - \frac{1}{2}\varepsilon$. The point x_1 satisfies conditions (1), (2) and (4b) and so it can not satisfy (3). Hence x_1 is a left-end point of some interval of monotonicity of the function f. Of course f is non-decreasing in that interval because $f'_T \geqslant M > 0$ a.e. Let (a_1, b_1) denote the maximal open interval of monotonicity of f contained in $(\beta, \beta + \delta)$ and containing x_1. We have $f(b_1) > f(\beta) - \frac{1}{2}\varepsilon$.

Because a_1 satisfies conditions (1), (3) and (4a), it cannot satisfy (2), and so $f(a_1) \leqslant f(\beta) - \varepsilon$.

Repeating this procedure, we conclude that there exists a sequence of intervals (a_n, b_n) such that

(I) $\bigcup_{n=1}^{\infty} (a_n, b_n)$ in dense in $\langle \beta, b_1 \rangle$,

(II) intervals $\langle a_n, b_n \rangle$ are mutually disjoint,

(III) each interval $\langle a_n, b_n \rangle$ is maximal interval of monotonicity,

(IV) $f(a_n) \leqslant f(\beta) - \varepsilon$, $f(b_n) > f(\beta) - \frac{1}{2}\varepsilon$ for every $n \in N$,

(V) the set $A = \langle \beta, a_1 \rangle - \bigcup_{n=2}^{\infty} (a_n, b_n)$ is uncountable and each of its points is an accumulation point of the two sequences $\{a_n\}$ and $\{b_n\}$.

Conditions (IV) and (V) contradict the assumption that the function f is nearly everywhere continuous.

THEOREM 1. *Let f and T satisfy condition (W) and $f'_T \geqslant M > 0$ a.e. in (a, b). Then f is non-decreasing in (a, b).*

Proof. Let T satisfy Khintchine's condition and let f be continuous and f'_T exists in (a, b) except at the points of the set $\{p_n: n \in N\}$.

Suppose that f is not non-decreasing in (a, b). From Lemma 3 it follows that there exists a non-empty interval $(\alpha, \beta) \subset (a, b)$ such that $f|_{(\alpha,\beta)}$ is non-decreasing. Let (α_1, β_1) denote the maximal open interval of monotonicity of f containing (α, β) and contained in (a, b). Since we have supposed that f is not non-decreasing in (a, b), we must have $a \neq \alpha_1$ or $b \neq \beta_1$. Without loss of generality we can assume that $\beta_1 \neq b$. Let δ_1 be such a number that $0 < \delta_1 < \frac{1}{4}(\beta_1 - \alpha_1)$ and $p_1 \notin (\beta_1, \beta_1 + \delta_1)$. Then it follows from Lemma 4 that there exists such a point $x_1 \in (\beta_1, \beta_1 + \delta_1)$ that

(I) $$f(x_1) > f(\beta_1) - \tfrac{1}{16} M(\beta_1 - \alpha_1),$$

(II) the function f is not monotonic in any interval which has x_1 as the right end-point.

(III) a) x_1 is the left end-point of some interval of monotonicity of f or
b) f is continuous in x_1.

From (III) it follows that there exists an $h_1 > 0$ such that for all point $x \in (x_1, x_1 + h_1) \subset (\beta_1, \beta_1 + \delta_1)$ the following inequality holds:

$$f(x) > f(\beta_1) - \tfrac{1}{16}[M(\beta_1 - \alpha_1)].$$

From (II) it follows that there are in the interval $(x_1 - \frac{1}{4}h_1, x_1) \cap (\beta_1, x_1)$ points x_1', x_1'' such that

$$x_1' < x_1'' \quad \text{and} \quad f(x_1') > f(x_1'').$$

Because f is a Darboux function, we can choose x_1'' in such a way that $x_1''\notin\{p_n\colon n\in N\}$. Hence for $\varepsilon_1''=\frac{1}{3}[f(x_1')-f(x_1'')]$ there exists a $\delta_1''>0$ such that

$$f(x)\leqslant f(x_1'')+\varepsilon_1''\quad\text{for}\quad x\in\langle x_1''-\delta_1'',\,x_1''\rangle.$$

Put

$$b_1'=\inf\{x\colon f(t)\leqslant f(x_1'')+\varepsilon_1''\quad\text{for all } t\in\langle x,\,x_1''-\delta_1''\rangle\};$$

of course, $b_1'>x_1'$ and for an arbitrarily small interval $(b_1'-\delta,\,b_1')$ there are uncountably many points x such that $f(x)>f(x_1')+\varepsilon_1''$. Let $x_1'''\in(b_1'-\delta,\,b_1')-\{p_n\colon n\in N\}$, where $0<\delta<\frac{1}{4}\delta_1'$, be such a point. Then there is a $\delta_1'''>0$ such that

$$f(x)>f(x_1''')-\tfrac{1}{3}[f(x_1''')-f(x_1'')]\quad\text{for}\quad x\in(x_1''',\,x_1'''+\delta_1''').$$

Let us put

$$\begin{aligned}
a_1&=\sup\{x\colon f(t)\geqslant f(x_1''')-\tfrac{1}{3}[f(x_1''')-f(x_1'')]\quad\text{for } t\in\langle x_1'''+\delta_1''',\,x\rangle\},\\
A_1&=(\alpha_1,\,\tfrac{1}{2}(\alpha_1+\beta_1))=(y_1'-\sigma_1',\,y_1'+\sigma_1'),\\
B_1&=(x_1,\,x_1+h_1)=(y_1-\sigma_1,\,y_1+\sigma_1),\\
C_1&=(b_1',\,x_1'')=(y_1''-\sigma_1'',\,y_1''+\sigma_1''),\\
D_1&=(x_1''',\,a_1)=(y_1'''-\sigma_1''',\,y_1'''+\sigma_1'''),\\
b_1&=\min(b_1',\,a_1+\tfrac{1}{4}\sigma_1''').
\end{aligned}$$

Then we have

(1) $$p_1\notin\langle a_1,\,b_1\rangle,$$

(2) for every point $x\in\langle a_1,\,b_1\rangle$ the following inequalities hold:

$$|x-y_1'|\leqslant 4\sigma_1',\qquad |x-y_1|\leqslant\tfrac{3}{2}\sigma_1,\qquad |x-y_1''|\leqslant\tfrac{3}{2}\sigma_1'',\qquad |x-y_1'''|\leqslant\tfrac{3}{2}\sigma_1''',$$

(3) for all points x',x'' such that $x'\in A_1$, $x''\in B_1$ we have

$$\frac{f(x')-f(x'')}{x'-x''}\geqslant\frac{1}{2}M\qquad\left(\text{because }\frac{f(x')-f(\frac{1}{2}(\alpha_1+\beta_1))}{x'-\frac{1}{2}(\alpha_1+\beta_1)}\geqslant M\right),$$

(4) $$\frac{f(x')-f(x'')}{x'-x''}<0\quad\text{whenever}\quad x'\in C_1\ \text{and}\ x''\in D_1.$$

Let us notice that $(a_1,\,b_1)$ is not the interval of monotonicity of f. Hence the same arguments allow us to define recurrently sequences of numbers $\{y_n\}$, $\{y_n'\}$, $\{y_n''\}$, $\{y_n'''\}$, $\{\sigma_n\}$, $\{\sigma_n'\}$, $\{\sigma_n''\}$, $\{\sigma_n'''\}$, $\{a_n\}$, $\{b_n\}$ and sequences of intervals $\{A_n\}$, $\{B_n\}$, $\{C_n\}$, $\{D_n\}$ such that

(1′) $$\begin{aligned}
A_n&=(y_n'-\sigma_n',\,y_n'+\sigma_n'),\qquad & B_n&=(y_n-\sigma_n,\,y_n+\sigma_n),\\
C_n&=(y_n''-\sigma_n'',\,y_n''+\sigma_n''),\qquad & D_n&=(y_n'''-\sigma_n''',\,y_n'''+\sigma_n'''),
\end{aligned}$$

(2′) $$\begin{aligned}
a_n&<y_{n+1}'-\sigma_{n+1}'<y_{n+1}'+\sigma_{n+1}'<y_{n+1}'''-\sigma_{n+1}'''<y_{n+1}'''+\sigma_{n+1}'''\\
&\leqslant a_{n+1}<b_{n+1}\leqslant y_{n+1}''-\sigma_{n+1}''<y_{n+1}''+\sigma_{n+1}''<y_{n+1}-\sigma_{n+1}\\
&<y_{n+1}+\sigma_{n+1}<b_n,
\end{aligned}$$

(3′) $$p_n\notin\langle a_n,\,b_n\rangle,$$

(4′) If $x\in\langle a_n,\,b_n\rangle$, then

$$|x-y_n'|\leqslant 4\sigma_n',\quad |x-y_n|\leqslant\tfrac{3}{2}\sigma_n,\quad |x-y_n''|<\tfrac{3}{2}\sigma_n'',\quad |x-y_n'''|\leqslant\tfrac{3}{2}\sigma_n''',$$

(5′) $$\frac{f(x')-f(x'')}{x'-x''}\geqslant\frac{1}{2}M\quad\text{whenever}\quad x'\in A_n\ \text{and}\ x''\in B_n,$$

(6′) $$\frac{f(x')-f(x'')}{x'-x''}<0\quad\text{whenever}\quad x'\in C_n\ \text{and}\ x''\in D_n.$$

Let $\{x_0\}=\bigcap_{n=1}^{\infty}\langle a_n,\,b_n\rangle$. From condition (3′) it follows that $f_T'(x_0)$ exists. On the other hand, from condition (4′) it follows that x_0 is a T-accumulation point of each of the sets $\bigcup_{k=1}^{\infty}A_{n_k}$, $\bigcup_{k=1}^{\infty}B_{n_k}$, $\bigcup_{k=1}^{\infty}C_{n_k}$, $\bigcup_{k=1}^{\infty}D_{n_k}$, where $\{n_k\}$ denotes an arbitrary subsequence of N. Hence for every set $E\in T(x_0)$ there are sequences $\{n_k'\}$ and $\{n_k''\}$ such that

$$A_{n_k'}\cap E\neq 0\neq B_{n_k'}\cap E\quad\text{and}\quad C_{n_k''}\cap E\neq 0\neq D_{n_k''}\cap E.$$

But under conditions (5′) and (6′) this implies that $f_T'(x_0)$ does not exist. This contradiction proves the theorem.

THEOREM 2. *If f and T satisfy condition* (W) *and $f_T'\geqslant 0$ a.e. in (a,b), then f is non-decreasing and continuous in (a,b).*

Proof. If the function f were not non-decreasing, then there would exist points $x_1,x_2\in(a,b)$ such that

$$2M=\frac{f(x_1)-f(x_2)}{x_2-x_1}>0.$$

Hence the function $g(x)=f(x)+Mx$ will not be non-decreasing either, in spite of the fact that it fulfils the assumptions of Theorem 1.

REMARK 4. The assumption that the function f has the Darboux property seems to be too strong because in the proofs we only use the fact that every point of the set $\{x\colon f(x)>a\}$ (or $\{x\colon f(x)<b\}$) is its point of bilateral condensation. But, as was shown by Zahorski in [6], for Baire class 1 functions it is equivalent to the Darboux property.

REMARK 5. Theorem 2 is a generalization of Zahorski's theorem because the assumption (β) in Zahorski's theorem implies continuity nearly everywhere.

It is an interesting question whether the following generalization of the Bruckner–Świątkowski theorem is true:

If a function f is a Baire class 1 function with the Darboux property, T satisfies Khintchine's condition n.e., f'_T exists n.e. and $f'_T \geq 0$ a.e. in (a, b), then f is non-decreasing and continuous in (a, b).

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Decomposition spaces and shape in the sense of Fox

by

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Abstract. It is proved in the paper that if X, Y are finite dimensional metrizable spaces, $f\colon X \to Y$ is a closed continuous map such that $f^{-1}(y)$ is approximatively k-connected for $y \in Y$ and $k = 0, 1, \ldots, \dim Y$, then $\mathrm{Sh}(X) \geq \mathrm{Sh}(Y)$ (in the sense of Fox [5]). By applying the theorem it is shown that for every finite dimensional locally compact metric space X there exists a Δ-space Y such that $\dim X = \dim Y$, $\mathrm{Sh}_W(X) = \mathrm{Sh}_W(Y)$ and $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

§ 1. Introduction. In [5] Fox introduced the notion of shape for metric spaces and proved that for compacta this notion coincides with the notion of shape in the sense of Borsuk [4]. In the previous paper [9] we proved that a certain decomposition map induces a weak shape equivalence. The purpose of this paper is to prove that a similar theorem holds for shape in the sense of Fox. Let X be a finite dimensional metric spaces and let $\mathscr{D}$ be an upper semicontinuous decomposition of X each element of which is a closed set being approximatively k-connected for $k = 0, 1, \ldots, \max(\dim X, \dim Y)$. Then we shall show that the equality $\mathrm{Sh}(X) = \mathrm{Sh}(X_{\mathscr{D}})$ holds, where $X_{\mathscr{D}}$ is the decomposition space of X by $\mathscr{D}$ and $\mathrm{Sh}(X)$ is the shape of X in the sense of Fox. As an application of this theorem we can obtain a generalization of Ball's theorem [1]. Finally, we shall prove that for every finite dimensional and locally compact metric space X there is a Δ-space Y such that $\dim X = \dim Y$, $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ and $\mathrm{Sh}_W(X) = \mathrm{Sh}_W(Y)$, where $\mathrm{Sh}_W(X)$ is the weak shape of X defined by Borsuk [3].

Throughout this paper all of spaces are metrizable and maps are continuous. By an AR-space and an ANR-space we mean always those for metric spaces and by dimension we mean the covering dimension.

§ 2. The shape in the sense of Fox. We first recall the basic notions introduced by Fox [5]. Let X and Y be metric spaces and let M and N be AR-spaces containing X and Y as closed sets respectively. By $U(X, M)$ we mean the inverse system consisting of open neighborhoods U of X in M and all inclusion maps $u\colon U' \to U$, $U' \subset U$. Similarly, by $V(Y, N)$ denote the inverse system of open neighborhoods of U in N. A *mutation* $f\colon U(X, M) \to V(Y, N)$ from $U(X, M)$ to $V(Y, N)$ is defined as a collection of maps $f\colon U \to V$, $U \in U(X, M)$, $V \in V(Y, N)$, such that