On a certain condition of the monotonocity of functions

by

Maria Mastalerz-Wawrzyńczak (Warszawa)

Abstract. The generalized derivative (Khintchine's derivative) of a real valued function of a real variable is investigated. The sufficient condition of monotonocity of a function is given.

The classical theorem about the monotonocity of a differentiable function with a non-negative derivative has been generalized in many ways. For example:

TOLSJOY's Theorem [5]. Let $f$ be a function satisfying in the interval $(a, b)$ the following conditions:

(a) $f$ is approximately continuous,
(b) $f'_e$ exist except perhaps at a countable set of points (i.e. nearly everywhere),
(c) $f'_e \geq 0$ a.e.

Then $f$ is continuous and non-decreasing in $(a, b)$.

ZAHORSKI's Theorem [6]. Let $f$ be a function satisfying in the interval $(a, b)$ the following conditions:

(a) $f$ is a Darboux function,
(b) $f'$ exists n.e.,
(c) $f'' \geq 0$ a.e.,

Then $f$ is continuous and non-decreasing in $(a, b)$.

In both of these theorems it is assumed, directly or indirectly, that the function $f$ is a Darboux function of the first class of Baire. In connection with this Zahorski asks in [6] whether the following hypothesis is true.

ZAHORSKI's Hypothesis. Let $f$ be a function satisfying in $(a, b)$ the following conditions:

1) $f$ is a Darboux function of the first class of Baire,
2) $f'_e$ exists n.e.,
3) $f''_e \geq 0$ a.e.

Then $f$ is continuous and non-decreasing in $(a, b)$.

Bruckner ([1]) and Świątkowski ([3]) give an affirmative answer to this question.
The three above-mentioned theorems give the characterizations of the same class of functions, namely: the class of continuous and non-decreasing functions which have ordinary derivatives n.e. This follows from Khintchine's theorem [21], which says that every point at which a monotonic function $f$ is approximatively differentiable is a point at which that function has an ordinary derivative. This remark suggests the possibility of replacing the ordinary derivative by a generalized derivative which for monotonic function coincides (in the sense of existence and value) with the ordinary derivative. The main theorem (Theorem 2) of this paper is such a generalization of Zahn's theorem.

Suppose that to every point $x$ of the interval $(a, b)$ there is attached a family $T(x)$ of subsets of $(a, b)$ which satisfies the following conditions:

(a) $x \in E$ for each $E \in T(x)$,

(b) if $E_1 \in T(x)$ and $E_2 \in T(x)$, then $E_1 \cap E_2 \in T(x)$,

(c) if $\delta > 0$ and $E \in T(x)$, then the sets $E \cap (x-\delta, x)$ and $E \cap (x, x+\delta)$ are non-empty,

(d) if $\delta > 0$, then $(x-\delta, x+\delta) \in T(x)$.

The sets of the family $T(x)$ will be called $T$-neighbourhoods of the point $x$.

Definition 1. A point $x$ will be called a $T$-accumulation point of the set $A$ if for each $T$-neighbourhood of $x$ contains points of the set $A - \{x\}$.

The set of $T$-accumulation points of $A$ will be denoted by $A_T$.

Definition 2. A number $g$ is called the $T$-limit of the function $f$ at the point $x_0$ if for every $\varepsilon > 0$ there exists an $E \in T(x_0)$ such that for every point $x \in E - \{x_0\}$ the following inequality is satisfied:

$$|f(x) - g| < \varepsilon.$$  

$T$-lim $f(x)$ means the $T$-limit of $f$ at $x_0$.

Analogously we define $T$-lim $f(x) = +\infty$.

Definition 3. The $T$-derivative of a function $f$ at the point $x_0$ is the $T$-limit

$$f'_T(x_0) = T \text{-} \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$  

One proves that under some additional conditions on $T(x)$, the $T$-derivative of a monotonic function is its ordinary derivative.

Definition 4. $T(x_0)$ satisfies Khintchine's condition if the conditions

(1) $x_0 = x_0$,

(2) $\delta_0 > 0$.

implies that $x_0 \in \bigcup_{k=1}^{\infty} (x_0 - \delta_k, x_0 + \delta_k)$.

Remark 1. If conditions (1)-(3) are satisfied, then we have also $x_0 \in \bigcup_{k=1}^{\infty} (x_0 - \delta_k, x_0 + \delta_k)$, where $\{\delta_k\}$ is any subsequence of the sequence of natural numbers.

Remark 2. If $T(x)$ is the family of the sets containing $x$ for which $x$ is a density point, then $f'_T(x) = f'_N(x)$.

Theorem (Świątekowski, [4]). $T(x_0)$ satisfies the condition of Khintchine if and only if for every function $f$ which is monotonic in some neighbourhood of $x_0$ the existence of $f'_T(x_0)$ implies the existence of $f'_N(x_0)$.

It will be convenient in the sequel to have

Definition 5. We shall say that the function $f$ and the family $T = \{T(x)\}$ satisfy condition (W) in the interval $(a, b)$ if

(1) $f$ is a Darboux function,

(2) $f$ is n.e. continuous,

(3) $T(x)$ satisfies Khintchine's condition for nearly every point $x \in (a, b)$,

(4) $f'_T$ exists n.e.

Furthermore $\{x_n: n \in N\}$ will denote the set of points with the exception of which $f$ is continuous, $T$ satisfies Khintchine's condition and $f'_T$ exists.

Lemma 1. Let $f$ and $T$ satisfy condition (W) in the interval $(a, b)$ and let $\alpha, \beta$ be numbers such that $\alpha < \beta$. Then at most one of the sets

$$A = \{x: f'_T(x) = \alpha\}, \quad B = \{x: f'_T(x) > \beta\}$$

can be dense in $(a, b)$.

Proof. Without loss of generality we may assume that $\alpha > 0 > \beta$. Now suppose, on the contrary, that $A = B = (a, b)$. Then there exists an $x_0 \in A - \{x_0\}$. Since $f'_T(x_0) > \alpha$, there is a $T$-neighbourhood $E_1 \in T(x_0)$ such that

$$\frac{f(x) - f(x_0)}{x - x_0} > \alpha$$

for all $x \in E_1 - \{x_0\}$.

Let $\delta_1$ be such a positive number that $p_1 \in (x_1 - \delta_1, x_1 + \delta_1)$ and let $x \in E_1 \cap \cap (x_1 - \delta_1, x_1)$. Hence $\{x, f(x)\}$ lies under the line $y = \alpha(x-x_0) + f(x_0)$. Because $f$ is a Darboux function in $(x, x_1)$, there is a non-denumerable set of such points $x$ that $f(x) < \alpha(x-x_0) + f(x_0)$. Let $x_1$ be such a point in $(x, x_1) - \{p_1\}$.

The continuity of the function $f$ in $x_1$ implies the existence of such a number $d_1 > 0$ that

$$f(x) < \alpha(x-x_1) + f(x_1)$$

for all $x \in (x_1 - d_1, x_1 + d_1)$. 

Put
\[ a'_1 = \sup \{ x : f(0) < \alpha (t-x_1) + f(x_2) \text{ for all } t \in \langle x'_1 - d'_1, x'_1 \rangle \}. \]

We have of course \( a'_1 \leq x_1 \). Let be \( 0 < \sigma_1 < \frac{1}{2} (a'_1 - x'_1 + d'_1) \). In the interval \( (a'_1, a_1 + \sigma_1) \) there are uncountably many points \( z \) such that \( f(z) > \alpha (z-x_1) + f(x_2) \). Let \( x''_1 \notin \langle p_n : n \in N \rangle \) be one of them. Since the function \( f \) is continuous at \( x''_1 \), there is a positive number \( d''_1 \) such that
\[ f(z) > \alpha (z-x_1) + f(x_2) \text{ for all } z \in \langle x'_1 - d'_1, x'_1 + d'_1 \rangle. \]

Put
\[ b_1 = x'_1 - d'_1, \quad a_1 = b_1 - \delta_1, \quad \text{where } 0 < \delta_1 < \frac{1}{2} d'_1 \text{ and } b_1 - \delta_1 > a'_1, \]
\[ A_1 = (x'_1 - d'_1, a'_1), \quad B_1 = (b_1, x'_1 + d'_1). \]

Then we have
\[ \frac{f(x') - f(x'')}{x' - x''} > \beta \quad \text{for } x' \in A_1 \text{ and } x'' \in B_1. \]

Now, since it was assumed that \( B = \langle a, b \rangle \), we can find \( x_2 \in \langle a_1, b_1 \rangle \cap B - \{ p_2 \} \). As before, there is an \( E_2 \in T(x_2) \) such that
\[ \frac{f(x_2) - f(x_2')}{x_2 - x_2'} < \beta \quad \text{for } x \in E_2 \cap \{ x_2 \} \]

Let \( \delta_2 \) be a positive number that \( p_2 \notin \langle x_2 - \delta_2, x_2 + \delta_2 \rangle \). Because \( f \) is a Darboux function, we can find a point \( x_3 \in \langle x_2 - \delta_2, x_2 \rangle \) such that
\[ \frac{f(x_3) - f(x_3')}{x_3 - x_3'} < \beta. \]

Since \( f \) is continuous at \( x'_1 \), there is a \( d'_2 > 0 \) such that
\[ f(z) > \beta (z-x_1) + f(x_2) \text{ for all } z \in \langle x'_1 - d'_2, x'_1 + d'_2 \rangle. \]

Put
\[ a'_2 = \sup \{ x : f(0) > \beta (t-x_1) + f(x_2) \text{ for all } t \in \langle x'_2 - d'_2, x'_2 \rangle \}. \]

It is obvious that \( a'_2 < x_2 \) and in every interval \( (a'_2, a'_2 + \eta) \) where \( \eta > 0 \), there are points \( x \) such that corresponding points of the graph of the function \( f \) lies below the line \( y = \beta (x-x_1) + f(x_2) \).

Let
\[ 0 < \sigma_2 < \frac{a'_2 - x'_2 + d'_2}{2}. \]

In the interval \( (a'_2, a'_2 + \sigma_2) \) there are uncountably many points \( x \) for which the inequality \( f(x) < \beta (x-x_1) + f(x_2) \) holds. Let \( x'_1 \) be such a point not belonging to \( \{ p_n : n \in N \} \). Because of the continuity of \( f \) at \( x'_1 \), there is a positive number \( d''_1 \) such that
\[ f(x) < \beta (x-x_1) + f(x_2) \text{ for all } x \in \langle x'_2 - d''_1, x'_2 + d''_1 \rangle. \]

Put
\[ b_2 = x'_2 - d''_1, \quad a_2 = b_2 - \delta_2, \quad \text{where } 0 < \delta_2 < \frac{d''_1}{2}, \quad b_2 - \delta_2 > a'_2 \]
and
\[ A_2 = (x'_2 - d''_1, a_2), \quad B_2 = (b_2, x'_2 + d''_1). \]

Then we have
\[ \frac{f(x') - f(x'')}{x' - x''} < \beta \quad \text{for } x' \in A_2 \text{ and } x'' \in B_2. \]

Repeating the above argument, we obtain sequences of numbers \( \{ x'_n \}, \{ x''_n \} \)
\( \{ d'_n \}, \{ a_n \}, \{ b_n \}, \{ a'_n \} \) and sequences of intervals \( A_n, \{ B_n \} \) such that
\begin{align*}
(1) & \quad a_0 = a, \quad b_0 = b \quad \text{and for } n \geq 1, \\
& \quad a_{n-1} - \delta_n < x'_n - d'_n < x'_n + d'_n < a_n < b_n = x'_n - d'_n < x'_n < x'_n + d'_n < b_{n-1}, \\
(2) & \quad x'_n - a_n < \frac{d'_n (x'_n - a_n)}{2}, \quad b_n - a_n < \frac{d'_n (b_n - a_n)}{2}, \\
(3) & \quad p_n \notin \langle a_n, b_n \rangle, \\
(4) & \quad \langle a_n, b_n \rangle = \langle a_{n+1}, b_{n+1} \rangle \quad \text{and} \quad b_n - a_n < \frac{b - a}{2}, \\
(5) & \quad A_n = (x'_n - d'_n, a_n), \quad B_n = (b_n, x'_n + d'_n), \\
(6) & \quad \frac{f(x') - f(x'')}{x' - x''} > \alpha \quad \text{for } x' \in A_{2n+1} \text{ and } x'' \in B_{2n-1}, \\
(7) & \quad \frac{f(x') - f(x'')}{x' - x''} < \beta \quad \text{for } x' \in A_{2n} \text{ and } x'' \in B_{2n}. \\
\end{align*}

Let \( \{ x_n \} = \bigcap_{n=1}^{\infty} \langle a_n, b_n \rangle \). From (3) it follows that \( x_0 \notin \{ p_n : n \in N \} \). Hence \( f'(x_0) \) exists. But
\[ x_n = \frac{1}{2} (x'_n - d'_n + a_n) \in \langle a_{n-1}, b_{n-1} \rangle \]
so \( x_n \to x_0 \) and \( y_n = x'_n \in \langle a_{n-1}, b_{n-1} \rangle \) and so \( y_n \to x_0 \) too.

Furthermore
\[ a_n = \frac{1}{2} (a'_n - x'_n + d'_n), \quad a'_n \quad \text{and} \quad d'_n \quad \text{at} \]
as well as
\[ \frac{a_n}{|x_n - x_0|} \to 1 \quad \text{and} \quad \frac{d''_n}{|y_n - x_0|} \to 1. \]

From this, and because
\[ A_n = (x_n - a_n, x_n + a_n), \quad B_n = (y_n - d''_n, y_n + d''_n), \]
it follows that for every subsequence \( \{n_k\} \) of the sequence of natural numbers we have
\[ x_0 \in \bigcap_{k=1}^{\infty} A_{3n_k} \]
and
\[ x_0 \in \bigcap_{k=1}^{\infty} B_{3n_k}. \]

Hence, for every set \( E \subseteq T(x_0) \) there exist such numbers \( n, m \) that none of the four sets \( A_{3n-1} \cap E, B_{3n-1} \cap E, A_{3n} \cap E, B_{3n} \cap E \) is empty. This implies, by (6) and (7), that \( f'_2(x_0) \) does not exist. This contradiction proves the lemma.

**Corollary 1.** Under the assumptions of Lemma 1 at most one of the sets
\[ A = \{x : f'_2(x) > 0\}, \quad B = \{x : f'_2(x) < 0\} \]
can be dense in the interval \((a, b)\).

**Lemma 2.** Let \( f \) and \( T \) satisfy condition (W) in the interval \((a, b)\) and \( f(T(x)) \geq M > 0 \) n.e. in \((a, b)\). Then there exists a non-empty interval \((a, b) \subset (a, b)\) such that \( f(x) \) is continuous and non-decreasing.

Proof. Suppose, on the contrary, that there is no interval \((a, b) = (a, b)\) in which \( f \) is non-decreasing. Put \( a_0 = a, b_1 = b \). Then there are in \((a, b)\) two points \( x', x'' \) such that
\[ a_1 < x'_1 < x''_1 < b_1 \quad \text{and} \quad f(x'_1) > f(x''_1). \]
We can assume that \( p_1 \notin \{x'_1, x''_1\} \) and \( x'_2 \notin \{p_n : n \in \mathbb{N}\} \). (Indeed, if \( p_1 \notin \{x'_1, x''_1\} \), then either \( f(x'_1) > f(p_1) \) or \( f(x''_1) > f(p_1) \). If, for example, \( f(x'_1) > f(p_1) \), then, since \( f \) is a Darboux function, in the interval \([p_1, x'_1]\) there are uncountably many points \( x \) satisfying the inequality \( f(x) > f(x'_1) \). We can choose one that is different from all \( p_n \) and substitute it for \( x'_1 \). In the case \( f(p_1) > f(x'_1) \) the proof proceeds analogously.)

Let \( 0 < r_1 < \frac{1}{4} f(x'_1) - f(x''_1) \). By the continuity of \( f \) in \( x'_1 \) we have \( d_1 > 0 \) such that
\[ f(x) > f(x'_1) - r_1 \quad \text{for all} \quad x \in (x'_1, x'_1 + d_1). \]
Let us put
\[ d'_1 = \sup \{x : f(x) > f(x'_1) - r_1 \quad \text{for all} \quad t \in (x'_1, x'_1 + d_1, x)\}. \]

It is evident that \( d'_1 < x''_1 \) and in every interval \((a'_1, b'_1 + \delta) \) \( (\delta > 0) \) there are uncountably many points \( x \) such that \( f(x) < f(x'_1) - r_1 \). Let \( x_1 \) denote one of them.

Thus \( x_1' \notin \{p_n : n \in \mathbb{N}\} \) and \( a_1 < x''_1 < \min\{x'_1, d'_1 + \frac{1}{2}(a'_1 - x'_1)\} \). Hence for every positive number \( r_1 < \frac{1}{4} f(x''_1) - f(x'_1) \) there exists a \( d'_1 > 0 \) such that
\[ f(x) < f(x'_1) + r_1 \quad \text{for all} \quad x \in (x''_1 - d'_1, x'_1 + d'_1). \]

Setting
\[ b_2 = \inf \{x : f(x) > f(x'_1) + r_1 \quad \text{for all} \quad t \in (x, x''_1 - d'_1)\}, \]
\[ a_2 = \max\{a_1, b_2 + \frac{1}{4}(x''_1 - b_2)\}, \]
we have
\[ A_1 = (x'_1, a_2), \quad B_1 = (b_2, x''_1), \]
and
\[ f(x') > f(x'_1) \quad \text{for} \quad x' \in A_1 \quad \text{and} \quad x' \in B_1 \]
because
\[ f(x') < f(x'_1) + r_1 < f(x'_1) - e_1 < f(x'_1). \]

Since we have supposed that there is no interval in which \( f \) is non-decreasing, we can define recurrently sequences of numbers \( \{x'_1\}, \{x''_1\}, \{a_1\}, \{b_1\}, \{a_2\}, \{b_2\} \) and sequences of intervals \( \{A_n\}, \{B_n\} \) such that
\begin{enumerate}
\item \( e_1 = a, b_1 = b \) and \( a_n-1 < x_n-1 < a_n < b_n < x_n'' < b_n-1 \) and \( b_n-1 < x_n'' < b_n-1 < a_n-1 \) for \( n > 1 \).
\item \( p_n \notin \{a_n, b_n\}. \)
\item \( A_n = (x'_n, a'_n) = (x'_n - \delta_n, x'_n + \delta_n) \) where \( x_n = \frac{1}{2}(a_n + x'_n) \)
and \( x_n = \frac{1}{2}(b_n + x'_n) \)
\item \( \delta_n \leq |x_n - x_{n-1}| < \delta_n = \frac{1}{2} |x_n - x_{n-1}| < |x_n - x_{n-1}| = \frac{1}{2} |x_n - x_{n-1}| \)
\item \( b_n = |x_n - x_{n-1}| = \frac{1}{2} |x_n - x_{n-1}| \)
\item \( f(x') > f(x''_1) \) for \( x' \in A_n \) and \( x' \in B_n \).
\end{enumerate}

Conditions (1) and (2) imply the existence of such a point \( x_0 \notin \{p_n : n \in \mathbb{N}\} \).

Hence there is a \( T \)-neighbourhood \( E \) of the point \( x_0 \) such that for all \( x \in E - \{x_0\} \) the following inequality holds:
\[ \frac{f(x) - f(x_0)}{x - x_0} > \frac{1}{2} M. \]
On the other hand, since \( \varepsilon_1 \rightarrow x_0 \) and \( \varepsilon_n \rightarrow x_n \), from conditions (4), (5) it follows that

\[
x_0 \in \left( \bigcup_{k=1}^{\infty} A_n \right)_{\varepsilon_1} \quad \text{and} \quad x_0 \in \left( \bigcup_{n=1}^{\infty} B_n \right)_{\varepsilon_1}
\]

for all subsequences \( \{\varepsilon_1\}, \{\varepsilon_n\} \) of the sequence of natural numbers. Hence there is a natural number \( n \) such that the sets \( A_n \cap E \) and \( B_n \cap E \) are non-empty. Thus \( (6) \) contradicts \( (7) \) and the lemma is proved.

**Corollary 2.** Under the assumption of Lemma 2, in the interval \((a, b)\) there exists a dense set of intervals of monotonicity of the function \( f \).

**Lemma 3.** Let \( f \) and \( T \) satisfy condition (W) in \((a, b)\) and \( f(x) \geq M > 0 \) on a dense set in \((a, b)\). Then there exists an interval \((a, \beta) \subset (a, b)\) such that \( f(x) \) is non-decreasing.

**Proof.** Let \( A = \{x : f(x) \geq M\} \) and \( B = \{x : f(x) < \frac{1}{2}M\} \). From Corollary 1 it follows that \( B \) is not dense in \((a, b)\). Then there exists an interval \((a_1, \beta_1) \subset (a, b)\) which is disjoint with \( B \). Thus the function \( f \) satisfies the assumptions of Lemma 2 in \((a_1, \beta_1)\), and so the existence of the interval \((a, \beta)\) with the required property is proved.

**Corollary 3.** Under the assumptions of Lemma 3 there exists in \((a, b)\) a dense set of intervals of monotonicity of the function \( f \).

**Remark 3.** If moreover \( f(x) \geq M > 0 \) holds almost everywhere, \( \langle a, \beta \rangle \subset (a, b) \) and the function \( f \) is monotonic on \((a, \beta)\), then \( f \) is continuous in \((a, \beta)\) and \( f(\beta) - f(a) \geq M(\beta - a) \).

**Lemma 4.** Let \( f \) and \( T \) satisfy condition (W) in \((a, b)\) and \( f(x) \geq M > 0 \) hold a.e. in \((a, b)\). Let \( \beta \in (a, b) \) be such a point that the function \( f \) is not monotonic in any interval which contains \( \beta \) as the left end-point. Then for every pair of positive numbers \( \varepsilon \) and \( \delta \) there exists such a point \( x_0 \) that the following conditions hold:

\[
\begin{align*}
(1) & \quad x_0 \in (\beta, \beta + \delta), \\
(2) & \quad f(x_0) > f(\beta) - \varepsilon, \\
(3) & \quad f \text{ is not monotonic in any interval } (x_0-h, x_0), \\
(4) & \quad a) \text{ if } f \text{ is monotonic in the interval } (x_0-h, x_0+h) \text{ for certain } h > 0 \text{ or} \\
& \quad b) \text{ if } x_0 \beta \text{ is the point of continuity of } f.
\end{align*}
\]

**Proof.** Suppose that there exists numbers \( \varepsilon > 0 \) and \( \delta > 0 \) such that there is no point \( x_\beta \) satisfying conditions (1)-(4). Since \( f \) is a Darboux function, continuous in \((a, b)\) except at a countable set of points, we have a point of continuity of \( f \) at \( x_\beta \in (\beta, \beta + \delta) \) such that \( f(x_\beta) > f(\beta) - \frac{1}{2}\varepsilon \). The point \( x_\beta \) satisfies conditions (1), (2) and (4b) and so it cannot satisfy (3). Hence \( x_\beta \) is a left-end point of some interval of monotonicity of the function \( f \). Of course \( f \) is non-decreasing in that interval because \( f(x) \geq M > 0 \) a.e. Let \( (a_1, b_1) \) denote the maximal open interval of monotonicity of \( f \) contained in \((\beta, \beta + \delta)\) and containing \( x_\beta \). We have \( f(b_1) > f(\beta) - \frac{1}{2}\varepsilon \).

Because \( a_\beta \) satisfies conditions (1), (3) and (4a), it cannot satisfy (2), and so \( f(a_\beta) \leq f(b_1) - \varepsilon \).

Repeating this procedure, we conclude that there exists a sequence of intervals \((a_n, b_n)\) such that

\[
\begin{align*}
(1) & \quad \bigcup_{n=1}^{\infty} (a_n, b_n) \text{ dense in } (\beta, \delta), \\
(2) & \quad \text{intervals } (a_n, b_n) \text{ are mutually disjoint}, \\
(3) & \quad \text{each interval } (a_n, b_n) \text{ is maximal interval of monotonicity}, \\
(4) & \quad f(a_n) < f(b_n) - \varepsilon \text{ and } \sum_{n=1}^{\infty} f(b_n) > f(\beta) - \frac{1}{2}\varepsilon \text{ for every } n \in N, \\
(5) & \quad \text{the set } A = \{\beta, a_1\} - \bigcup_{n=1}^{\infty} (a_n, b_n) \text{ is uncountable and each of its points is an accumulation point of the two sequences } (a_n) \text{ and } (b_n). 
\end{align*}
\]

Conditions (4) and (5) contradict the assumption that the function \( f \) is nearly everywhere continuous.

**Theorem 1.** Let \( f \) and \( T \) satisfy condition (W) and \( f(x) \geq M > 0 \) a.e. in \((a, b)\). Then \( f \) is non-decreasing in \((a, b)\).

**Proof.** Let \( T \) satisfy Khintchine's condition and let \( f \) be continuous and \( f(x) \geq M > 0 \) at each of the points of the set \( \{p_n : n \in N\} \).

Suppose that \( f \) is not non-decreasing in \((a, b)\). From Lemma 3 it follows that there exists a non-empty interval \( (a, \beta) \subset (a, b) \) such that \( f(a_\beta) \) is not non-decreasing. Let \( (a_\beta, b_\beta) \) denote the maximal open interval of monotonicity of \( f \) contained in \((a, \beta)\) and containing \( (a_\beta, b_\beta) \). Since we have supposed that \( f \) is not non-decreasing in \((a, b)\), we must have \( a_\beta \neq a_\beta \) or \( b_\beta \neq b_\beta \). Without loss of generality we can assume that \( a_\beta \neq b_\beta \). Let \( p_\delta \) be such a number that \( 0 < \delta < \frac{1}{2}(b_\beta - a_\beta) \) and \( p_\delta \neq (b_\beta, \beta + \delta) \). Then it follows from Lemma 4 that there exists such a point \( x_\beta \in (\beta_1, \beta_1 + \delta_1) \) that

\[
\begin{align*}
(1) & \quad f(x_\beta) > f(\beta_1) - \frac{1}{2}\varepsilon \text{ and } M(\beta_1 - a_\beta), \\
(2) & \quad the function } f \text{ is not monotonic in any interval which has } x_1 \text{ as the right end-point}, \\
(3) & \quad a) \text{ } x_1 \text{ is the left end-point of some interval of monotonicity of } f \text{ or} \\
& \quad b) \text{ } f \text{ is continuous in } x_1.
\end{align*}
\]

From (III) it follows that there exists an \( h_1 > 0 \) such that for all point \( x \in (x_1, x_1 + h_1) \) the following inequality holds:

\[
f(x) > f(\beta_1) - \frac{1}{2}\varepsilon [M(\beta_1 - a_\beta)]
\]

From (II) it follows that there are in the interval \((x_1 - \frac{1}{2}h_1, x_1) \cap (\beta_1, x_1)\) points \( x_1', x_1'' \) such that

\[
x_1' < x_1'' \quad \text{and} \quad f(x_1') > f(x_1'').
\]
Because \( f \) is a Darboux function, we can choose \( x''_i \) in such a way that \( x''_i \neq \{ p_n : n \in N \} \). Hence for \( a'_i = \frac{1}{4} \{ f(x''_i) - f(x''_i) \} \) there exists a \( \delta'_i > 0 \) such that
\[
f(x) \leq f(x''_i) + \delta'_i \quad \text{for} \quad x \in (x''_i - \delta'_i, x''_i).
\]
Put
\[
\delta'_i = \inf \{ x : f(x) < f(x''_i) + \delta'_i \} \quad \text{for each} \ (x, x''_i - \delta'_i).
\]
of course, \( \delta'_i \neq 0 \) and for an arbitrarily small interval \( (x''_i - \delta'_i, x''_i) \) there are uncountably many points \( x \) such that \( f(x) > f(x''_i) + \delta'_i \). Let \( \delta''_i = \delta'_i - \delta'_i - \{ p_n : n \in N \} \), where \( 0 < \delta < \frac{1}{2} \delta'_i \), be such a point. Then there is a \( \delta''_i > 0 \) such that
\[
f(x) > f(x''_i) - \frac{1}{4}(f(x''_i) - f(x''_i)) \quad \text{for} \quad x \in (x''_i - \delta''_i, x''_i).
\]
Let us put
\[
a_i = \sup \{ x : f(x) \geq f(x''_i) - \frac{1}{4}(f(x''_i) - f(x''_i)) \} \quad \text{for} \ i \in (x''_i + \delta''_i, x)
\]
\[
A_i = (x_i, x_i + \delta_i) = (y_i - \sigma_i, y_i + \sigma_i),
\]
\[
B_i = (x_i, x_i + \delta_i) = (y_i - \sigma_i, y_i + \sigma_i),
\]
\[
C_i = (x_i, x_i + \delta_i) = (y_i - \sigma_i, y_i + \sigma_i),
\]
\[
D_i = (x_i, x_i + \delta_i) = (y_i - \sigma_i, y_i + \sigma_i),
\]
\[
\delta_i = \min (\delta_i, \delta_i + \delta_i).
\]
Then we have
\[
(1) \quad p_i \notin (a_i, b_i).
\]
(2) for every point \( x \in (a_i, b_i) \) the following inequalities hold:
\[
|x - y_i| \leq \frac{1}{2} \delta_i, \quad |y_i - x| \leq \frac{1}{2} \delta_i.
\]
(3) for all points \( x', x'' \) such that \( x' \in A_i, x'' \in B_i \) we have
\[
f(x') - f(x'') \leq \frac{1}{2} M
\]
(4) \( \frac{f(x') - f(x'')}{x' - x''} < 0 \) whenever \( x' \in C_i \) and \( x'' \in D_i \).

Let us notice that \( (a_i, b_i) \) is not the interval of monotonicity of \( f \). Hence the same arguments allow us to define recurrently sequences of numbers \( \{ x_n \}, \{ x_n \}, \{ y_n \}, \{ y_n \}, \{ a_n \}, \{ a_n \}, \{ a_n \}, \{ a_n \}, \{ a_n \} \) and sequences of intervals \( \{ A_n \}, \{ B_n \}, \{ C_n \}, \{ D_n \} \) such that
\[
\begin{align*}
A_0 = (y_0 - \sigma_0, y_0 + \sigma_0), & \quad B_0 = (y_0 - \sigma_0, y_0 + \sigma_0), \\
C_0 = (y_0 - \sigma_0, y_0 + \sigma_0), & \quad D_0 = (y_0 - \sigma_0, y_0 + \sigma_0),
\end{align*}
\]
(1) Hence the function \( g(x) = f(x) + Mx \) will not be non-decreasing either, in spite of the fact that it fulfills the assumptions of Theorem 1.

Remark 2. The assumption that the function \( f \) has the Darboux property seems to be too strong because in the proofs we only use the fact that every point of the set \( \{ x : f(x) < a \} \) (or \( \{ x : f(x) < b \} \) is its point of bilateral condensation. But, as was shown by Zaborski in [6], for Baire class 1 functions it is equivalent to the Darboux property.

Remark 3. Theorem 2 is a generalization of Zaborski's theorem because the assumption (b) in Zaborski's theorem implies continuity nearly everywhere.
It is an interesting question whether the following generalization of the Bruckner–Świątkowski theorem is true:

If a function $f$ is a Baire class 1 function with the Darboux property, $T$ satisfies Khintchine’s condition n.e., $f'_T$ exists n.e. and $f'_T \geq 0$ a.e. in $(a, b)$, then $f$ is non-decreasing and continuous in $(a, b)$.

References


Accepted by la Réduction le 23. 6. 1973

Decomposition spaces and shape in the sense of Fox

by

Yukihiro Kodama (Tokyo)

Abstract. It is proved in the paper that if $X$, $Y$ are finite dimensional metrizable spaces, $f : X \to Y$ is a closed continuous map such that $f^{-1}(y)$ is approximately $k$-connected for $y \in Y$ and $k = 0, 1, \ldots, \dim Y$, then $\text{Sh}(X) \geq \text{Sh}(Y)$ (in the sense of Fox [5]). By applying the theorem it is shown that for every finite dimensional locally compact metric space $X$ there exists a $D$-space $Y$ such that $\dim X = \dim Y$, $\text{Shw}(X) = \text{Shw}(Y)$ and $\text{Sh}(X) = \text{Sh}(Y)$.

§ 1. Introduction. In [5] Fox introduced the notion of shape for metric spaces and proved that for compacta this notion coincides with the notion of shape in the sense of Borsuk [4]. In the previous paper [9] we proved that a certain decomposition map induces a weak shape equivalence. The purpose of this paper is to prove that a similar theorem holds for shape in the sense of Fox. Let $X$ be a finite dimensional metric space and let $\mathcal{D}$ be an upper semicontinuous decomposition of $X$ each element of which is a closed set being approximately $k$-connected for $k = 0, 1, \ldots, \max(\dim X, \dim Y)$. Then we shall show that the equality $\text{Sh}(X) = \text{Sh}(X_\mathcal{D})$ holds, where $X_\mathcal{D}$ is the decomposition of $X$ by $\mathcal{D}$ and $\text{Sh}(X)$ is the shape of $X$ in the sense of Fox. An application of this theorem will allow a generalization of the Ball’s theorem [1]. Finally, we shall prove that for every finite dimensional and locally compact metric space $X$ there is a $D$-space $Y$ such that $\dim X = \dim Y$, $\text{Sh}(X) = \text{Sh}(Y)$ and $\text{Shw}(X) = \text{Shw}(Y)$, where $\text{Shw}(X)$ is the weak shape of $X$ defined by Borsuk [3].

Throughout this paper all spaces of are metrizable and maps are continuous.

By an AR-space and an ANR-space we mean always those for metric spaces and by dimension we mean the covering dimension.

§ 2. The shape in the sense of Fox. We first recall the basic notions introduced by Fox [5]. Let $X$ and $Y$ be metric spaces and let $M$ and $N$ be AR-spaces containing $X$ and $Y$ as closed sets respectively. By $U(X, M)$ we mean the inverse system consisting of open neighborhoods $U$ of $X$ in $M$ and all inclusion maps $u : U \to U$, $U \in U$. Similarly, by $V(Y, N)$ denote the inverse system of open neighborhoods of $U$ in $N$. A mutation $f : U(X, M) \to V(Y, N)$ from $U(X, M)$ to $V(Y, N)$ is defined as a collection of maps $f : U \to V$, $U \in U(X, M)$, $V \in V(Y, N)$, such that