family $\mathscr{F} = \mathscr{F}_0 \cup \mathscr{F}_1$ fulfils all the required conditions. This completes the proof of the theorem.

Acknowledgment. The author is grateful to Dr. P. C. Bhakta of Jadavpur University, Calcutta, for his kind suggestions in the preparation of this note.

References

[1] I. P. Natanson, *Theory of Functions of a Real Variable*, Vol. II,
[2] S. Saks, *Theory of the Integral*, Warszawa 1937.

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Accepté par la Rédaction le 18. 4. 1975

On the categorical shape of a functor

by

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Abstract. The concept of shape, first introduced by Borsuk in his study of the homotopy theory of compacta, is extended to an abstract categorical setting. The shape of an arbitrary functor K is defined, and it is proved that the Kan extension is shape-invariant. One then shows that many of the categorical aspects of shape remain valid in this very general setting; others require some restriction on the functor K, and the notion of a rich functor is introduced, which is more general than the notion of a full functor. In addition, it is proved that if K is rich, the iteration of the shape construction produces the same shape category. Finally, the special case when K has a left adjoint is discussed in some detail, and a relation with the categories of fractions is exhibited.

Introduction. Since Borsuk [1] first introduced the concept of shape in his study of the homotopy theory of compacta many authors (e.g. [5], [6], [7], [11], [12], [14], [15], [16], [17]) have contributed to the develepment of shape theory. However the theory has remained almost exclusively confined to a topological context, never very far removed from the setting in which it was originally cast by Borsuk; and further, and arising from this restriction in the scope of the theory, the concept has, in the work cited, related to some category of topological spaces $\mathbf{T}$ and a full subcategory $\mathbf{P}$ of $\mathbf{T}$. However, Holsztyński [10] observed, soon after Borsuk's invention of the concept, that shape could be formulated as an abstract limit, and was thus of more general applicability.

It is the principal purpose of this paper to free shape theory from its restricted scope. Thus we replace the full embedding of a topological category $\mathbf{P}$ in a topological category $\mathbf{T}$ by an arbitrary functor $K\colon \mathbf{P}\to\mathbf{T}$ from the arbitrary category $\mathbf{P}$ to the arbitrary category $\mathbf{T}$. In so doing we are very much inspired by the point of view adopted by LeVan in his thesis [12]. We then find that many of the *categorical* aspects of shape theory (we do not speak of the *topological* aspects) remain valid in this very general setting. Others require some restriction on the functor K, but a restriction far milder than that K should be a full embedding.

In Section 1 we define *shape* and the dual concept *coshape*. Indeed, for $K\colon \mathbf{P}\to\mathbf{T}$,

* The first-named author was partially supported by NSF Grant GP38804, and the second-named author by NSF Grant GP43703, during the preparation of this paper.

the shape of K is the category whose objects are those of $\mathbf{T}$ and whose morphisms are given by

$$\mathbf{S}(X, Y) = \mathrm{Nat}\big(\mathbf{T}(Y, K-), \mathbf{T}(X, K-)\big),\ X, Y \in |\mathbf{T}|,$$

with the obvious law of composition. There is an evident functor $S\colon \mathbf{T}\to\mathbf{S}$ which is the identity on objects. A functor $F\colon \mathbf{T}\to\mathbf{C}$ is said to be *shape-invariant* if $F = \bar{F}S$ for some functor $\bar{F}\colon \mathbf{S}\to\mathbf{C}$; we may also say that $\bar{F}$ *extends* F. The principal result of Section 1 is that right Kan extensions of functors $\mathbf{P}\to\mathbf{C}$ along K are always shape-invariant; this generalizes Holsztyński's observation [10] that Čech cohomology is shape-invariant ([1]). We also give an explicit process, given such a Kan extension $\tilde{F}$, for constructing $\bar{F}\colon \mathbf{S}\to\mathbf{C}$ with $\bar{F}S = \tilde{F}$.

In Section 2 we recall from [3] the notion of a *dominant* functor and then give the definition of a *rich* functor. Thus a functor $U\colon \mathbf{C}\to\mathbf{D}$ is rich if, given any objects C, C' of $\mathbf{C}$ and any morphism $g\colon UC\to UC'$ in $\mathbf{D}$, there is a diagram

$$C \xrightarrow{f_1} A_1 \xleftarrow{f_2} A_2 \to \ldots \xrightarrow{f_{2k-1}} A_{2k-1} \xleftarrow{f_{2k}} C'$$

in $\mathbf{C}$ such that Uf_{2i} is invertible, $1\leqslant i\leqslant k$, and

$$g = (Uf_{2k})^{-1}Uf_{2k-1}\ldots(Uf_2)^{-1}Uf_1 .$$

It turns out that certain results proved (in a restricted topological context) when K is a full embedding retain their validity when K is merely assumed to be rich. If K is rich and dominant then S is an isomorphism of categories; this fact about S in fact characterizes the codense functors K in the sense of [13]. We also take the opportunity in Section 2 to improve some results of [3], replacing a hypothesis of fullness by a hypothesis of richness. Further we prove that the extension $\bar{F}\colon \mathbf{S}\to\mathbf{C}$ of the right Kan extension $\tilde{F}\colon \mathbf{T}\to\mathbf{C}$ of the functor $F\colon \mathbf{P}\to\mathbf{C}$ along K, constructed in Section 1, is the right Kan extension of F along SK, provided that K is rich. With the same hypothesis on K, we give a characterization of the shape functor $S\colon \mathbf{T}\to\mathbf{S}$.

In Section 3 we first point out that, if K is rich, then the shape of $SK\colon \mathbf{P}\to\mathbf{S}$ is isomorphic to $\mathbf{S}$ itself. Thus it is reasonable to ask what happens if we iterate the shape construction when K is not rich, and we obtain a few preliminary results.

In Section 4 we discuss in some detail the special case when K has a left adjoint $L\colon \mathbf{T}\to\mathbf{P}$. It then turns out that the shape of $K\colon \mathbf{P}\to\mathbf{T}$ depends only on the triple generated by the adjunction; precisely, the shape of K is isomorphic to the Kleisli category of $\mathbf{T}$ with respect to the triple T generated by the adjunction $L\dashv K$. From this it follows from results in [3] that, if T is idempotent, the shape of K is isomorphic to the category of fractions of $\mathbf{T}$ with respect to the family S_L of morphisms of $\mathbf{T}$ rendered invertible by L. At this point, the concept of richness again plays a role, since we show, improving a result in [3], that T is idempotent if K is rich.

(1) This observation may be regarded as following directly the original motivation of Borsuk in introducing shape.

It is a pleasure to acknowledge valuable conversations with John Macdonald on the current state of shape theory. We are also grateful to the referee for many helpful remarks, especially with regard to the formulation of shape by means of diagram (1.3) below. We plan to exploit this point of view extensively in a sequel to this paper, *On Borsuk shape and the Grothendieck category of pro-objects* ([1]). As remarked by the referee, we could also have used this formulation of the shape morphisms to provide alternative, and in some cases slicker, proofs of some of the results of this paper, especially those related to Kan extensions (e.g., Theorem 1.4, and the first statement of Theorem 4.4). However, we have preferred to adopt a more explicit approach to the Kan extension, believing that our arguments would thereby be more accessible to the reader more familiar with topology than with category theory. For the slick proofs would involve fairly esoteric theorems of category theory; moreover, we have also wanted to exhibit explicit maps achieving the various relations in question (e.g., S_1 in Proposition 3.1, Γ' in Theorem 4.3) and these maps do emerge directly from the exploitation of (1.1). Nevertheless, we appreciate the value of the referee's observation.

1. Shape and right Kan extensions.

Definition 1.1. Let $K\colon \mathbf{P}\to\mathbf{T}$ be a functor. Then the *shape* of K is the category $\mathbf{S}$ whose objects are the objects of $\mathbf{T}$ and whose morphisms are given by

$$\mathbf{S}(X, Y) = \mathrm{Nat}\big(\mathbf{T}(Y, K-), \mathbf{T}(X, K-)\big). \tag{1.1}$$

The composition of morphisms in $\mathbf{S}$ is the usual composition of natural transformations. The *coshape* $\mathbf{S}'$ of K is the opposite of the shape of $K^{\mathrm{op}}\colon \mathbf{P}^{\mathrm{op}}\to\mathbf{T}^{\mathrm{op}}$, so that

$$\mathbf{S}'(X, Y) = \mathrm{Nat}\big(\mathbf{T}(K-, X), \mathbf{T}(K-, Y)\big). \tag{1.2}$$

The shape category was introduced by LeVan [12] for the particular case that K is a full embedding, and independently by Mardešić [14] for the special case when $\mathbf{T}$ is the homotopy category of topological spaces, $\mathbf{P}$ is the full subcategory of spaces having the homotopy type of CW-complexes, and K is the full embedding. We remark that we may describe the morphisms of $\mathbf{S}$ by means of the *comma category*. Thus, for each object X of $\mathbf{T}$, let $(X\downarrow K)$ denote the comma category of $\mathbf{P}$-objects under X. There is a forgetful functor $D_X\colon (X\downarrow K)\to\mathbf{P}$, given by

$$D_X(X\xrightarrow{f} KP) = P, \quad P\in|\mathbf{P}|,$$

and a morphism $\theta\colon X\to Y$ in $\mathbf{S}$ is just a functor $\theta\colon (Y\downarrow K)\to(X\downarrow K)$ such that the diagram

$$\begin{array}{ccc} (Y\downarrow K) & \xrightarrow{\ \theta\ } & (X\downarrow K) \\ & {\scriptstyle D_Y}\searrow \quad \swarrow{\scriptstyle D_X} & \\ & \mathbf{P} & \end{array} \tag{1.3}$$

commutes (see the closing paragraph of the Introduction).

(1) See [18].

The shape of K may be furnished with the canonical functor $S\colon \mathbf{T}\to\mathbf{S}$ which is the identity on objects and which associates with $f\colon X\to Y$ in $\mathbf{T}$ the induced natural transformation $f^{\#}\colon \mathbf{T}(Y, K-)\to\mathbf{T}(X, K-)$. We will regard S as forming part of the structure of the shape of K; further we will denote by $K_1\colon \mathbf{P}\to\mathbf{S}$ the composition $K_1 = SK$. Now if $\tau\colon X\to Y$ in $\mathbf{S}$ and $P\in|\mathbf{P}|$ we will denote by τ^P the function $\mathbf{T}(Y, KP)\to\mathbf{T}(X, KP)$ determined by τ; the justification for this notation is that if also $\sigma\colon Y\to Z$ in $\mathbf{S}$, then

$$(\sigma\tau)^P = \tau^P\sigma^P. \tag{1.4}$$

We abbreviate $f^{\#P}$ to f^P if $f\colon X\to Y$ in $\mathbf{T}$.

We observe that the definition of shape involves us in the usual set-theoretical difficulties, since $\mathbf{S}(X, Y)$ may well fail to be a set. *Ad hoc* arguments, such as the one given by Mardešić in [14], may be used in individual cases to show that $\mathbf{S}(X, Y)$ is well defined. In general, if $\mathbf{T}$ is a category in a given universe $\mathbf{U}$, then $\mathbf{S}$ is a category in a higher universe containing $\mathbf{U}$ as an element.

We say that a functor $F\colon \mathbf{T}\to\mathbf{C}$ is *shape-invariant* (with respect to K) if it factors through $S\colon \mathbf{T}\to\mathbf{S}$. In the study of shape-invariant functors we are naturally led to the study of right Kan extensions of functors $\mathbf{P}\to\mathbf{C}$. We first consider such extensions when $\mathbf{C} = \mathbf{Ens}$, the category of sets.

PROPOSITION 1.2. *Let $F\colon \mathbf{P}\to\mathbf{Ens}$ be a functor. Then the right Kan extension of F along $K\colon \mathbf{P}\to\mathbf{T}$ is the functor $\tilde{F}\colon \mathbf{T}\to\mathbf{Ens}$, given by*

$$\tilde{F}X = \mathrm{Nat}(\mathbf{T}(X, K-), F), \quad X\in|\mathbf{T}|,$$

$$\tilde{F}f(\xi) = \xi\circ f^{\#}, \quad \text{for } f\colon X\to Y \text{ in } \mathbf{T},\ \xi\colon \mathbf{T}(X, K-)\to F,$$

together with the natural transformation $\varepsilon\colon \tilde{F}K\to F$, given by

$$\varepsilon_P(\xi) = \xi_P(1_{KP}), \quad P\in|\mathbf{P}|, \quad \xi\colon (KP, K-)\to F.$$

Proof. It is plain that $\tilde{F}$ is a functor. To prove that ε is a natural transformation, let $u\colon P\to Q$ in $\mathbf{P}$; we must prove that

$$\begin{array}{ccc} \tilde{F}KP & \xrightarrow{\varepsilon_P} & FP \\ \downarrow{\scriptstyle \tilde{F}Ku} & & \downarrow{\scriptstyle Fu} \\ \tilde{F}KQ & \xrightarrow{\varepsilon_Q} & FQ \end{array} \tag{1.5}$$

commutes. Now if $\xi\colon \mathbf{T}(KP, K-)\to F$, then $(Fu\circ\varepsilon_P)(\xi) = (Fu\circ\xi_P)(1_{KP})$, while

$$(\varepsilon_Q\circ\tilde{F}Ku)(\xi) = (\xi\circ Ku^{\#})_Q(1_{KQ}) = \xi_Q(Ku) = (\xi_Q\circ Ku_{\#})(1_{KP}),$$

and the commutativity of (1.5) follows from the naturality of ξ.

We complete the proof by establishing the universal property of $(\tilde{F}, \varepsilon)$. Assume given a functor $G\colon \mathbf{T}\to\mathbf{Ens}$, and a natural transformation $\tau\colon GK\to F$. We

define a natural transformation $\sigma\colon G\to\tilde{F}$ by the rule

$$\sigma_X(a)_P(f) = (\tau_P\circ Gf)(a), \quad X\in|\mathbf{T}|,\ P\in|\mathbf{P}|,\ a\in GX,\ f\in\mathbf{T}(X, KP). \tag{1.6}$$

That $\sigma_X(a)$ is natural follows immediately from the naturality of τ; that σ is natural follows from the fact, easily proved, that if $h\colon X\to Y$ in $\mathbf{T}$, then

$$(\tilde{F}h\circ\sigma_X)(a)_P(f) = (\tau_P\circ Gf\circ Gh)(a) = (\sigma_Y\circ Gh)(a)_P(f),$$

for $f\colon Y\to KP$ in $\mathbf{T}$. Moreover

$$(\varepsilon\circ\sigma K)_P(a) = \varepsilon_P(\sigma_{KP}(a)) = \sigma_{KP}(a)_P(1_{KP}) = (\tau_P\circ G(1_{KP}))(a) = \tau_P(a), \quad a\in GKP,$$

so that

$$\varepsilon\circ\sigma K = \tau. \tag{1.7}$$

It remains to show that (1.7) characterizes σ. Now, merely by naturality we infer that, in the notation of (1.6),

$$\sigma_X(a)_P(f) = (\tilde{F}f\circ\sigma_X)(a)_P(1_{KP}) = (\sigma_{KP}\circ Gf)(a)_P(1_{KP}) = (\varepsilon_P\circ\sigma_{KP})Gf(a),$$

so that σ is uniquely determined by $\varepsilon\circ\sigma K$, and the proof of the proposition is complete.

COROLLARY 1.3. *For each X in $|\mathbf{T}|$, the right Kan extension of the functor $\mathbf{T}(X, K-)\colon \mathbf{P}\to\mathbf{Ens}$ along K is*

$$\mathbf{S}(X, -)\colon \mathbf{T}\to\mathbf{Ens}$$

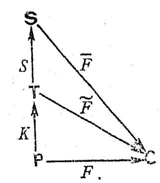
(*together with the natural transformation ε given by Proposition* 1.2).

It is this corollary which we now propose to generalize; the generalization consists of replacing $\mathbf{T}(X, K-)$ by an arbitrary functor $F\colon \mathbf{P}\to\mathbf{C}$. We construct the right Kan extension $\tilde{F}\colon \mathbf{T}\to\mathbf{C}$ of F along K, and prove

THEOREM 1.4. *Let $F\colon \mathbf{P}\to\mathbf{C}$ be a functor and let $\tilde{F}\colon \mathbf{T}\to\mathbf{C}$ be the right Kan extension of F along K. Then $\tilde{F}$ is shape-invariant.*

Proof. We will construct an explicit functor $\bar{F}\colon \mathbf{S}\to\mathbf{C}$ such that $\bar{F}S = \tilde{F}$.

(1.8)



Of course, we must define $\bar{F}$ on objects by $\bar{F}X = \tilde{F}X$, $X \in |\mathbf{S}| = |\mathbf{T}|$. To define $\bar{F}$ on morphisms of $\mathbf{S}$, we need to recall the explicit form of $\tilde{F}$ on objects. Thus if $X, Y \in |\mathbf{T}|$,

$$(1.9) \qquad \tilde{F}X = \varprojlim_{u:\, X \to KP} FP, \qquad \tilde{F}Y = \varprojlim_{v:\, Y \to KQ} FQ.$$

Let $\alpha_u\colon \tilde{F}X \to FP$, $\beta_v\colon \tilde{F}Y \to FQ$ be the canonical morphisms defining the two inverse limits (1.9). Then for any $p\colon P \to P'$, $q\colon Q \to Q'$ in $\mathbf{P}$ we have

$$(1.10) \qquad Fp \circ \alpha_u = \alpha_{u'}, \text{ with } u' = Kp \circ u, \qquad Fq \circ \beta_v = \beta_{v'}, \text{ with } v' = Kq \circ v.$$

Now let $\lambda\colon X \to Y$ in $\mathbf{S}$. We define a morphism $\bar{F}\lambda\colon \tilde{F}X \to \tilde{F}Y$ by the rule

$$(1.11) \qquad \beta_v \circ \bar{F}\lambda = \alpha_{\lambda^Q v}.$$

That (1.11) really does define a morphism $\bar{F}\lambda$ is clear; for, if $q\colon Q \to Q'$ in $\mathbf{P}$, then

$$Fq \circ \alpha_{\lambda^Q v} = \alpha_{Kq \circ \lambda^Q v} = \alpha_{\lambda^{Q'}(Kq \circ v)}.$$

Moreover, if $\mu\colon Y \to Z$ in $\mathbf{S}$, then, in an obvious notation with $w\colon Z \to KR$ in $\mathbf{T}$,

$$\gamma_w \circ \bar{F}(\mu\lambda) = \alpha_{(\mu\lambda)^R w} = \alpha_{\lambda^R \mu^R w} = \beta_{\mu^R w} \circ \bar{F}\lambda = \gamma_w \circ \bar{F}\mu \circ \bar{F}\lambda,$$

so that

$$\bar{F}(\mu\lambda) = \bar{F}\mu \circ \bar{F}\lambda.$$

Since $\bar{F}$ obviously preserves identities, it is obvious that $\bar{F}$ is a functor. Finally we see that, if $f\colon X \to Y$ in $\mathbf{T}$, then

$$\beta_v \bar{F}Sf = \alpha_{vf} = \beta_v \circ \tilde{F}f,$$

so that $\bar{F}S = \tilde{F}$, as desired.

Notice that the natural transformation $\varepsilon\colon \tilde{F}K \to F$, which forms part of the universal characterization of the right Kan extension $\tilde{F}$, may also be interpreted as a natural transformation $\varepsilon\colon \bar{F}K_1 \to F$, where $K_1 = SK$. Thus it is natural to pose the question whether the functor $\bar{F}$ we have constructed is the right Kan extension of F along K_1; we will return to this question in the next section. Meanwhile we observe that if $F = \mathbf{T}(W, K-)\colon \mathbf{P} \to \mathbf{Ens}$, so that $\tilde{F} = \mathbf{S}(W, -)\colon \mathbf{T} \to \mathbf{Ens}$, then indeed $\bar{F} = \mathbf{S}(W, -)\colon \mathbf{S} \to \mathbf{Ens}$, where $W \in |\mathbf{T}|$. For if $u\colon X \to KP$, then $\alpha_u\colon \mathbf{S}(W, X) \to \mathbf{T}(W, KP)$ is given by $\alpha_u(\mu) = \mu^P(u)$, so that

$$(\beta_v \circ \bar{F}\lambda)(\mu) = \alpha_{\lambda^Q v}(\mu) = \mu^Q \lambda^Q v = (\lambda\mu)^Q(v) = \beta_v(\lambda\mu) = (\beta_v \circ \lambda_\#)(\mu),$$

and $\bar{F}\lambda = \lambda_\#$, implying that $\bar{F} = \mathbf{S}(W, -)$. Thus the general construction of $\bar{F}$ in the proof of Theorem 1.4 yields the "natural" extension in the case that $F = \mathbf{T}(W, K-)$.

EXAMPLE 1.5. We do not give an example of a shape category now, preferring to postpone this till we have the stronger theorems of Section 4 which enable us to identify shape in certain familiar cases. However, we are able at this stage to give examples of shape invariance.

In [4] examples were given of the process of extending cohomology functors from a homotopy category $\tilde{\mathbf{T}}_0$ to a larger homotopy category $\tilde{\mathbf{T}}_1$. If we interpret such a cohomology functor as a functor $h\colon \tilde{\mathbf{T}}_0 \to \mathbf{C}$, where $\mathbf{C}$ is the category opposite to the category of graded abelian groups, then the process described in [4] consisted precisely in taking the right Kan extension $h_1\colon \tilde{\mathbf{T}}_1 \to \mathbf{C}$ of h. It thus follows that any cohomology functor h_1 constructed in this way is a shape-invariant of the embedding $K\colon \tilde{\mathbf{T}}_0 \subseteq \tilde{\mathbf{T}}_1$.

A special case of this example would be that in which the objects of $\tilde{\mathbf{T}}_0$ are compact polyhedra and those of $\tilde{\mathbf{T}}_1$ are compact spaces. Then if h is ordinary cohomology (with coefficients in some abelian group G), h_1 is Čech cohomology, and Čech cohomology is shape-invariant. This last very special case constituted a motivation for Borsuk's original introduction of shape.

Of course, the process of extending *homology* functors, described in [2], leads to coshape-invariants.

2. Rich functors. We first recall a definition from [3].

DEFINITION 2.1. A functor $U\colon \mathbf{C} \to \mathbf{D}$ is *dominant* if, given any object D of $\mathbf{D}$ there is an object C of $\mathbf{C}$ and morphisms

$$(2.1) \qquad D \xrightarrow{\alpha} UC \xrightarrow{\beta} D$$

with $\beta\alpha = 1$. We call (2.1) a *domination* of D.

Of course if every object of $\mathbf{D}$ is isomorphic to some U-image then U is dominant. Another example of a dominant functor is provided by a free functor, regarded as mapping to the full subcategory of projective objects.

We now introduce a definition which will play a fundamental role in the sequel.

DEFINITION 2.2. A functor $U\colon \mathbf{C} \to \mathbf{D}$ is *rich* if, given any objects C, C' of $\mathbf{C}$ and any morphism $g\colon UC \to UC'$ of $\mathbf{D}$, there is a diagram

$$C \xrightarrow{f_1} A_1 \xleftarrow{f_2} A_2 \xrightarrow{f_3} \dots \xrightarrow{f_{2k-1}} A_{2k-1} \xleftarrow{f_{2k}} C'$$

in $\mathbf{C}$ such that Uf_{2i} is invertible for each i, $1 \leqslant i \leqslant k$, and

$$g = (Uf_{2k})^{-1} \circ Uf_{2k-1} \circ \dots \circ (Uf_2)^{-1} \circ Uf_1.$$

Of course, a full functor is rich. An example of a rich functor which is not full is provided by the direct limit functor L from the category of direct systems of groups (over arbitrary filtering categories) to the category of groups [8]. The most immediate example of a rich functor (which is not full) is provided by the canonical functor

$$(2.2) \qquad P_\Sigma\colon \mathbf{C} \to \mathbf{C}[\Sigma^{-1}]$$

from a category $\mathbf{C}$ to the category of fractions with respect to a family of morphisms Σ. Indeed, we may characterize rich functors by means of the appropriate functor P_Σ (2.2). For let $U\colon \mathbf{C}\to\mathbf{D}$ be a functor, let Σ be the family of morphisms rendered invertible by U and let

(2.3)
$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{U} & \mathbf{D} \\ & {\scriptstyle P_\Sigma}\searrow \quad \nearrow{\scriptstyle \bar U} & \\ & \mathbf{C}[\Sigma^{-1}] & \end{array}$$

be the canonical factorization of U. Then it is plain that

(2.4) *U is rich iff $\bar U$ is full.*

We will need the following improvement of Proposition 1.4 of [3].

Proposition 2.3. *Let $U\colon \mathbf{C}\to\mathbf{D}$ and $V, W\colon \mathbf{D}\to\mathbf{E}$ be functors and let $U^*\colon \mathrm{Nat}(V,W)\to\mathrm{Nat}(VU,WU)$ be the induced map. Then*

(i) *if U is dominant, U^* is injective,*

(ii) *if U is rich and dominant, U^* is bijective.*

Proof. In Proposition 1.4 of [3] we proved (i) and obtained the conclusion of (ii) under the hypothesis that U was full and dominant. Thus we need only discuss that aspect of the proof of (ii) in which we show it is sufficient to suppose U to be rich.

Given $\tau\colon VU\to WU$ and $D\in|\mathbf{D}|$, choose a domination of D,

$$D\xrightarrow{\alpha} UC\xrightarrow{\beta} D\,,\qquad \beta\alpha=1\,,$$

and define $\sigma_D\colon VD\to WD$ by

(2.5)
$$\sigma_D = W\beta\circ\tau_C\circ V\alpha\,.$$

We must show that σ is natural. To this end, consider $g\colon D\to D'$ in $\mathbf{D}$ and let

$$D'\xrightarrow{\alpha'} UC'\xrightarrow{\beta'} D',\qquad \beta'\alpha'=1\,,$$

be a domination of D'; and consider the diagram

(2.6)
$$\begin{array}{ccccccc} VD & \xrightarrow{V\alpha} & VUC & \xrightarrow{\tau_C} & WUC & \xrightarrow{W\beta} & WD \\ \downarrow{\scriptstyle Vg} & & \downarrow{\scriptstyle V(\alpha' g\beta)} & & \downarrow{\scriptstyle W(\alpha' g\beta)} & & \downarrow{\scriptstyle Wg} \\ VD' & \xrightarrow[V\alpha']{} & VUC' & \xrightarrow[\tau_{C'}]{} & WUC' & \xrightarrow[W\beta']{} & WD' \end{array}$$

The outside squares commute, so we must prove that the inside square commutes. This would follow immediately if U were full (as in Proposition 1.4 of [3]). However, it suffices that U be rich. For then we find

$$C\xrightarrow{f_1} A_1 \xleftarrow{f_2} A_2 \xrightarrow{f_3} \dots \xrightarrow{f_{2k-1}} A_{2k-1} \xleftarrow{f_{2k}} C'$$

in $\mathbf{C}$ with Uf_{2i} invertible and

(2.7)
$$\alpha' g\beta = (Uf_{2k})^{-1}\circ Uf_{2k-1}\circ\dots\circ(Uf_2)^{-1}\circ Uf_1\,.$$

Since τ is natural, $WUf_i\circ\tau_{A_{i-1}} = \tau_{A_i}\circ VUf_i$, i odd, $A_0=C$; and $WUf_i\circ\tau_{A_i} = \tau_{A_{i-1}}\circ VUf_i$, i even, $A_{2k}=C'$. It therefore follows from (2.7) that the middle square of (2.6) commutes, so that the naturality of σ is established.

Notice that our proof shows (by taking $g=1\colon D\to D$) that σ_D is independent of the choice of domination of D. If $D=UC$, we choose the trivial domination $UC\xrightarrow{1} UC\xrightarrow{1} UC$, so that $\sigma_{UC}=\tau_C$. Thus $U^*\sigma=\tau$, establishing that U^* is surjective and completing the proof of the proposition.

Before proceeding to our applications of the notion of rich functors to the theory of shape, we take the opportunity to point out how Proposition 1.5 of [3] may now be strengthened in the light of Proposition 2.3. The corollary below will be exploited in Section 4.

Corollary 2.4. *Consider the triangle of categories and functors*

$$\begin{array}{ccc} \mathbf{L} & \xrightarrow{H} & \mathbf{M} \\ & {\scriptstyle J}\nwarrow \quad \swarrow{\scriptstyle I} & \\ & \mathbf{N} & \end{array}$$

Assume that H is left adjoint to JI with unit $\eta\colon 1\to JIH$ and counit $\varepsilon'\colon HJI\to 1$. Then if I is rich and dominant, there is a unique natural transformation $\varepsilon\colon IHJ\to 1$, given by $\varepsilon I = I\varepsilon'$, and IH is left adjoint to J with unit η and counit ε. Moreover, the two adjunctions generate the same triple on $\mathbf{L}$.

We now pass to further applications of the concept of richness, and of Proposition 2.3.

Corollary 2.5. *Let $K\colon \mathbf{P}\to\mathbf{T}$, let Σ be the family of morphisms rendered invertible by K, and let*

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{K} & \mathbf{T} \\ & {\scriptstyle P_\Sigma}\searrow \quad \nearrow{\scriptstyle \bar K} & \\ & \mathbf{P}[\Sigma^{-1}] & \end{array}$$

be the canonical factorization of K through the category of fractions. Then the shape of K is isomorphic to the shape of $\bar K$.

Proof. We merely observe that P_Σ is rich and dominant. Thus, for $X, Y\in|\mathbf{T}|$,

$$\begin{aligned} &\mathrm{Nat}\big(\mathbf{T}(Y,K-),\mathbf{T}(X,K-)\big)\\ &\qquad = \mathrm{Nat}\big(\mathbf{T}(Y,\bar K-)\circ P_\Sigma,\mathbf{T}(X,\bar K-)\circ P_\Sigma\big)\cong \mathrm{Nat}\big(\mathbf{T}(Y,\bar K-),\mathbf{T}(X,\bar K)\big)\,, \end{aligned}$$

the isomorphism being attested by Proposition 2.3.

Remark. Corollary 2.5 is, of course, only a special case of the fact that the shape of K is isomorphic to the shape of KU for any rich and dominant functor U.

COROLLARY 2.6. *If the functor K is rich and dominant then the canonical functor $S\colon \mathbf{T}\to\mathbf{S}$ is an isomorphism of categories.*

Proof. By the remark above the shape of K is isomorphic to the shape of $\mathrm{Id}\colon \mathbf{T}\to\mathbf{T}$. The isomorphism from the shape of Id to the shape of K is obviously induced by S.

Actually, necessary and sufficient conditions for S to be an isomorphism are provided by a result in [13]. Recall ([13], p. 242) that a functor $K\colon \mathbf{P}\to\mathbf{T}$ is *codense* if, for each $X\in|\mathbf{T}|$,

$$\varprojlim\big((X\downarrow K)\xrightarrow{D_X}\mathbf{P}\xrightarrow{K}\mathbf{T}\big)=X,$$

with canonical morphism $\alpha_f\colon X\to KP$ defined by $\alpha_f=f$, for each $f\colon X\to KP$ in $(X\downarrow K)$. Then Proposition 2 on p. 242 of [13] immediately implies

PROPOSITION 2.7. *The canonical functor $S\colon \mathbf{T}\to\mathbf{S}$ is an isomorphism iff $K\colon \mathbf{P}\to\mathbf{T}$ is codense.*

Thus we have an indirect proof that rich and dominant functors are codense.

EXAMPLE 2.8. (i) The Yoneda embedding $K\colon \mathbf{P}\to\mathrm{Funct}(\mathbf{P},\mathbf{E}\mathrm{ns})^{\mathrm{op}}$, given by $KP=\mathbf{P}(P,-)$, is codense by Corollary 3 on p. 243 of [13]. Thus the shape of K is $\mathrm{Funct}(\mathbf{P},\mathbf{E}\mathrm{ns})^{\mathrm{op}}$ itself.

(ii) Let $\mathbf{G}$ be the category of groups and let $\mathbf{P}$ be the subcategory of $\mathbf{G}$ whose objects are Z and $Z*Z$ (free group on 2 generators) and whose morphisms consist of the identities, the usual embeddings $q_i\colon Z\to Z*Z$, $i=1,2$, and the comultiplication $\mu\colon Z\to Z*Z$. The embedding $K\colon \mathbf{P}\to\mathbf{G}$ may easily be seen to be dense so that the coshape of K is $\mathbf{G}$ itself. This example may, of course, be modified to yield corresponding results in the category of Λ-modules for some fixed unitary ring Λ.

Our next result generalizes the Yoneda Lemma.

PROPOSITION 2.9. *If the functor $K\colon \mathbf{P}\to\mathbf{T}$ is rich, then, for any $P\in|\mathbf{P}|$ and functor $F\colon \mathbf{T}\to\mathbf{E}\mathrm{ns}$, there is a bijection*

$$\mathrm{Nat}\big(\mathbf{T}(KP,K-),FK\big)\cong FKP$$

given by $\tau\mapsto\tau_P(1_{KP})$.

Proof. Given $a\in FKP$, define $\tau^a\colon \mathbf{T}(KP,K-)\to FK$ by $\tau^a_Q(g)=Fg(a)$, $g\colon KP\to KQ$. It is easy to see that τ^a is natural and that $\tau^a_P(1_{KP})=a$. Thus $\tau\mapsto\tau_P(1_{KP})$ is surjective and it remains to show that τ is determined by $\tau_P(1_{KP})$; it is here that we invoke the richness of K. For, given $g\colon KP\to KQ$, we may find

(2.8) $$P\xrightarrow{f_1}A_1\xleftarrow{f_2}A_2\to\dots\xrightarrow{f_{2k-1}}A_{2k-1}\xleftarrow{f_{2k}}Q$$

in $\mathbf{P}$ with Kf_{2i} invertible and $g=(Kf_{2k})^{-1}\circ Kf_{2k-1}\circ\dots\circ(Kf_2)^{-1}\circ Kf_1$. Then (2.8) gives rise to the diagram

$$\begin{array}{ccccccccc}
\mathbf{T}(KP,KP) & \to & \mathbf{T}(KP,KA_1) & \xleftarrow{\sim} & \mathbf{T}(KP,KA_2) & \to\dots\to & \mathbf{T}(KP,KA_{2k-1}) & \xleftarrow{\sim} & \mathbf{T}(KP,KQ)\\
\downarrow{\scriptstyle\tau_P} & & \downarrow{\scriptstyle\tau_{A_1}} & & \downarrow{\scriptstyle\tau_{A_2}} & & \downarrow{\scriptstyle\tau_{A_{2k-1}}} & & \downarrow{\scriptstyle\tau_Q}\\
FKP & \longrightarrow & FKA_1 & \xleftarrow{\sim} & FKA_2 & \longrightarrow\dots\longrightarrow & FKA_{2k-1} & \xleftarrow{\sim} & FKQ
\end{array}$$

which commutes by the naturality of τ. But this shows that

$$\tau_Q(g)=(Fg)(\tau_P)(1_{KP}),$$

so that τ is determined by $\tau_P(1_{KP})$ as required.

As an immediate consequence we have

THEOREM 2.10. *If the functor $K\colon \mathbf{P}\to\mathbf{T}$ is rich then the function*

$$S\colon \mathbf{T}(X,KP)\to\mathbf{S}(X,K_1P)$$

is bijective for all $X\in|\mathbf{T}|$, $P\in|\mathbf{P}|$.

Proof. We apply Proposition 2.9 with $F=\mathbf{T}(X,-)\colon \mathbf{T}\to\mathbf{E}\mathrm{ns}$. Then

$$FKP=\mathbf{T}(X,KP)\cong\mathrm{Nat}\big(\mathbf{T}(KP,K-),\mathbf{T}(X,K-)\big)=\mathbf{S}(X,K_1P).$$

Moreover, the bijection is given by associating with $a\colon X\to KP$ the natural transformation τ^a given by $\tau^a_Q(g)=ga$, $g\colon KP\to KQ$. Thus τ^a is precisely Sa, establishing the theorem.

Now let $S'\colon \mathbf{T}\to\mathbf{S}'$ be any functor which is the identity on objects and let $K_1'=S'K$. We say that S' *has property* $(*)$ if $S'\colon \mathbf{T}(X,KP)\to\mathbf{S}'(X,K_1'P)$ is bijective for all $X\in|\mathbf{T}|$, $P\in|\mathbf{P}|$. Thus Theorem 2.10 asserts that S has property $(*)$ if K is rich. We may immediately deduce.

THEOREM 2.11. *If the functor $K\colon \mathbf{P}\to\mathbf{T}$ is rich then*

(i) *given any functor $S'\colon \mathbf{T}\to\mathbf{S}'$ with property $(*)$ there exists a unique functor $T\colon \mathbf{S}'\to\mathbf{S}$ with $TS'=S$;*

(ii) *the right Kan extension of $K_1\colon \mathbf{P}\to\mathbf{S}$ along itself is the identity.*

Moreover S is characterized among the functors with domain $\mathbf{T}$ having property $(*)$ by (i) *or* (ii) *above.*

Proof. The proof of the statements involving (i) is given in [14]; that of the statements involving (ii) in [12]. Although in [12] and [14] K was assumed to be a full embedding, the proofs remain valid in our more general context since the only use made of the fact that K was a full embedding was to ensure that S has property $(*)$.

We close this section with a further application of the concept of richness. We recall that, in Theorem 1.4, we constructed an extension $\bar{F}$: $\mathbf{S}\to\mathbf{C}$ of the Kan extension $\tilde{F}$: $\mathbf{T}\to\mathbf{C}$ of a functor F: $\mathbf{P}\to\mathbf{C}$ along K. We now prove

THEOREM 2.12. *If K: $\mathbf{P}\to\mathbf{T}$ is rich then $\bar{F}$: $\mathbf{S}\to\mathbf{C}$ is the right Kan extension of F: $\mathbf{P}\to\mathbf{C}$ along K_1, and the right Kan extension of $\tilde{F}$: $\mathbf{T}\to\mathbf{C}$ along S.*

Proof. It follows from Theorem 2.10 that, identifying $\mathbf{T}(X, KP)$ with $\mathbf{S}(X, K_1P)$ under S,

$$\varprojlim_{\substack{u:\, X\to KP\\ \text{in } \mathbf{T}}} FP = \varprojlim_{\substack{\lambda:\, X\to K_1P\\ \text{in } \mathbf{S}}} FP.$$

Moreover if θ: $X\to Y$ in $\mathbf{S}$ then the right Kan extension of F along K_1 associates with θ the morphism f of $\mathbf{C}$ characterized, in the notation of the proof of Theorem 1.4, by $\beta_v\circ f = \alpha_{v\theta}$. But $v\theta = \theta^P(v)$, so that it follows from (1.11) that $f = \bar{F}\theta$, showing that $\bar{F}$ is the right Kan extension of F along K_1. We remark that, obviously, the natural transformation ε: $\bar{F}K_1\to F$ associated with the right Kan extension $\bar{F}$ is precisely the transformation ε: $\tilde{F}K\to F$ associated with the right Kan extension $\tilde{F}$.

To show that $\bar{F}$ is the right Kan extension of $\tilde{F}$, we proceed by proving that, given a natural transformation τ: $GS\to\tilde{F}$, where G: $\mathbf{S}\to\mathbf{C}$ is an arbitrary functor, there is a unique natural transformation σ: $G\to\bar{F}$ with $\sigma S = \tau$. The uniqueness of σ follows from Proposition 2.3 (i), since S is dominant, so it remains to establish the existence of σ.

Consider $\varepsilon\circ\tau K$: $GK_1\to F$. Since $\bar{F}$ is the right Kan extension of F along K_1, there exists σ: $G\to\bar{F}$ with $\varepsilon\circ\sigma K_1 = \varepsilon\circ\tau K$. Since $\tilde{F}$ is the right Kan extension of F along K, this last equation implies that $\sigma S = \tau$, so that the proof of the theorem is complete.

3. Iterated shape. Let $\mathbf{S}$ be the shape of K: $\mathbf{P}\to\mathbf{T}$ with canonical functor S: $\mathbf{T}\to\mathbf{S}$ and let $K_1 = SK$: $\mathbf{P}\to\mathbf{S}$. We may then form the shape $\mathbf{S}_1$ of K_1 with canonical functor S_1: $\mathbf{S}\to\mathbf{S}_1$. We immediately prove

PROPOSITION 3.1. *If K: $\mathbf{P}\to\mathbf{T}$ is rich then S_1: $\mathbf{S}\to\mathbf{S}_1$ is an isomorphism of categories.*

Proof. Consider the function

$$\mathrm{Nat}(\mathbf{T}(Y, K-), \mathbf{T}(X, K-))\to\mathrm{Nat}(\mathbf{S}(Y, K_1-), \mathbf{S}(X, K_1-))$$

given by $\lambda\mapsto\bar{\lambda}$, where $\bar{\lambda}^P = S\circ\lambda^P\circ S^{-1}$, with S: $\mathbf{T}(Z, KP)\to\mathbf{S}(Z, K_1P)$ the bijection of Theorem 2.10 (Z ranging over $|\mathbf{T}|$). It is plain that this function is bijective, so it remains to show that

(3.1) $$\bar{\lambda} = S_1\lambda.$$

Now let $S_1\lambda = \lambda_1$. Then if $\tau\in\mathbf{S}(Y, K_1P)$, $u\in\mathbf{T}(KP, KQ)$ it is straightforward to verify that

$$(\bar{\lambda}^P\tau)^Q(u) = u\circ\lambda^P\tau^P(1_{KP}),\qquad (\lambda_1^P\tau)^Q(u) = \lambda^Q\tau^Q(u).$$

Thus (3.1) is established once we have proved the following lemma [1].

LEMMA 3.2. *Let K: $\mathbf{P}\to\mathbf{T}$ be rich. Then for any ϱ: $X\to Y$ in $\mathbf{S}$ and any f: $Y\to KP$, u: $KP\to KQ$ in $\mathbf{T}$, we have*

$$\varrho^Q(u\circ f) = u\circ\varrho^P f.$$

Proof. We use the richness of K to construct a commutative diagram

$$\begin{array}{ccccccccc}
\mathbf{T}(Y, KP) & \xrightarrow{Kh_1} & \mathbf{T}(Y, KA_1) & \xleftarrow{Kh_2} & \mathbf{T}(Y, KA_2) \to \dots \to & \mathbf{T}(Y, KA_{2k-1}) & \xleftarrow{Kh_{2k}} & \mathbf{T}(Y, KQ)\\
\downarrow{\scriptstyle\varrho} & & \downarrow{\scriptstyle\varrho} & & \downarrow{\scriptstyle\varrho} & \downarrow{\scriptstyle\varrho} & & \downarrow{\scriptstyle\varrho}\\
\mathbf{T}(X, KP) & \xrightarrow{Kh_1} & \mathbf{T}(X, KA_1) & \xleftarrow{Kh_2} & \mathbf{T}(X, KA_2) \to \dots \to & \mathbf{T}(X, KA_{2k-1}) & \xleftarrow{Kh_{2k}} & \mathbf{T}(X, KQ)
\end{array}$$

where each Kh_{2i} is invertible and $u = (Kh_{2k})^{-1}\circ\dots\circ(Kh_2)^{-1}\circ Kh_1$.

Proposition 3.1 shows that the shape of K is stable under iteration if K is rich. Thus we will devote some attention in this section to the situation which arises if K is not rich.

Thus let K: $\mathbf{P}\to\mathbf{T}$ be an arbitrary functor and consider the function

$$\bar{\bar{S}}\colon \mathbf{S}(Y, K_1-)\to\mathbf{T}(Y, K-),$$

given by $\bar{\bar{S}}^P(\tau) = \tau^P(1_{KP})$, τ: $Y\to K_1P$ in $\mathbf{S}$. Examination of the proofs of Proposition 2.9 and Theorem 2.10 shows that $\bar{\bar{S}}S = 1$, where S: $\mathbf{T}(Y, K-)\to\mathbf{S}(Y, K_1-)$.

PROPOSITION 3.3. *$\bar{\bar{S}}$ is a natural transformation of functors $\mathbf{P}\to\mathbf{Ens}$.*

Proof. If u: $P\to Q$ in $\mathbf{P}$, τ: $Y\to K_1P$ in $\mathbf{S}$, then

$$Ku\circ\tau^P(1_{KP}) = \tau^Q(Ku) = \bar{\bar{S}}^Q(K_1u\circ\tau).$$

We now revert to the proof of Proposition 3.1, using $\bar{\bar{S}}$ instead of S^{-1} (of course, the latter will not in general exist). Thus we define functions

$$\Phi\colon \mathbf{S}(X, Y)\to\mathbf{S}_1(X, Y),\qquad \Psi\colon \mathbf{S}_1(X, Y)\to\mathbf{S}(X, Y)$$

by

(3.2) $$\Phi(\tau) = S\tau\bar{\bar{S}},\qquad \Psi(\omega) = \bar{\bar{S}}\omega S$$

(1) A shorter proof of the isomorphism of $\mathbf{S}$ with $\mathbf{S}_1$ is available via the formulation of shape morphisms given by (1.3). However, Lemma 3.2 would still need to be invoked to establish that this isomorphism is achieved by S_1; moreover, Lemma 3.2 seems to us to be interesting in its own right.

and each of these is well-defined by Proposition 3.3. Moreover Φ respects composition and $\Psi\Phi = 1$ since $\bar{S}S = 1$. On the other hand Φ does not respect identities (although Ψ does!).

In the case in which K was rich Φ coincided with S_1. Of course, Φ coincides with S_1 iff Φ is a functor iff $S\colon \mathbf{T}(Y, K-)\to\mathbf{S}(Y, K_1-)$ is bijective for all $Y\in|\mathbf{T}|$ iff S_1 is an isomorphism of categories. In general, we may replace Φ by S_1 and it will remain true that

$$\Psi S_1 = 1\,. \tag{3.3}$$

Thus S_1 embeds $\mathbf{S}$ as a subcategory of $\mathbf{S}_1$ (recall that $\mathbf{S}_1$ has the same objects as $\mathbf{S}$) and Ψ provides a rule for retracting each morphism-set $\mathbf{S}_1(X, Y)$ onto the corresponding $\mathbf{S}(X, Y)$.

4. The shape of right adjoints. In this section we suppose that $K\colon \mathbf{P}\to\mathbf{T}$ is right adjoint to a functor $L\colon \mathbf{T}\to\mathbf{P}$. We first recall from [13], p. 245 a general fact about adjoint functors; we will sketch a proof adapted to our point of view.

PROPOSITION 4.1. *Let $K\colon \mathbf{P}\to\mathbf{T}$ have a left adjoint $L\colon \mathbf{T}\to\mathbf{P}$. Then L is the right Kan extension of the identity functor $1_{\mathbf{P}}$ along K, the natural transformation $LK\to 1$ being the counit ε of the adjunction.*

Proof. We suppose given $G\colon \mathbf{T}\to\mathbf{P}$ and a natural transformation $\tau\colon GK\to 1$. For $X\in|\mathbf{T}|$ we define

$$\sigma_X = \tau_{LX}\circ G\eta_X\colon GX\to LX\,,$$

where η is the unit of the adjunction. Obviously $\sigma\colon G\to L$ is natural and, for $P\in|\mathbf{P}|$,

$$\varepsilon_P\circ\tau_{LKP}\circ G\eta_{KP} = \tau_P\circ GK\varepsilon_P\circ G\eta_{KP} = \tau_P\,,$$

so that

$$\varepsilon\circ\sigma K = \tau\,.$$

Moreover, σ is uniquely determined by this last equation, since, by the naturality of σ,

$$\sigma_X = \varepsilon_{LX}\circ L\eta_X\circ\sigma_X = \varepsilon_{LX}\circ\sigma_{KLX}\circ G\eta_X\,.$$

Henceforth (as in the proposition above) we will always suppose an adjunction $L\dashv K$, where $K\colon \mathbf{P}\to\mathbf{T}$, $L\colon \mathbf{T}\to\mathbf{P}$, with unit η and counit ε. Consider the function $\Gamma\colon \mathbf{S}(X, Y)\to\mathbf{T}(X, KLY)$, given by

$$\Gamma(\tau) = \tau^{LY}(\eta_Y)\,. \tag{4.1}$$

PROPOSITION 4.2. *Γ is bijective.*

Proof. We define $\Delta\colon \mathbf{T}(X, KLY)\to\mathbf{S}(X, Y)$ as follows. First we use primes to denote pairs of morphisms (g, g') connected by the adjunction

$$\mathbf{P}(LX, P)\cong\mathbf{T}(X, KP)\,.$$

(Thus if $g\in\mathbf{P}(LX, P)$, $g'\in\mathbf{T}(X, KP)$; and if $g\in\mathbf{T}(X, KP)$, $g'\in\mathbf{P}(LX, P)$.) Then, given $f\colon X\to KLY$, we define $\Delta(f)\colon X\to Y$ in $\mathbf{S}$ by

$$\Delta(f)^P(u) = Ku'\circ f\,,\quad u\colon Y\to KP \text{ in } \mathbf{T}\,. \tag{4.2}$$

First, $\Delta(f)$ is natural. For if $v\colon P\to Q$ in $\mathbf{P}$, then

$$(Kv\circ\Delta(f)^P)(u) = Kv\circ Ku'\circ f = \Delta(f)^Q(Kv\circ u)\,.$$

Next, $\Delta\Gamma = 1$; for if $f = \tau^{LY}(\eta_Y)$ then

$$\Delta(f)^P(u) = Ku'\circ\tau^{LY}(\eta_Y) = \tau^P(Ku'\circ\eta_Y) = \tau^P(u)\,.$$

Finally, $\Gamma\Delta = 1$; for $\Delta(f)^{LY}(\eta_Y) = f$.

Now define

$$\Gamma'\colon \mathbf{S}(X, Y)\to\mathbf{P}(LX, LY)$$

by

$$\Gamma'(\tau) = \Gamma(\tau)'\,. \tag{4.3}$$

We note immediately that

$$\Gamma'S = L\,, \tag{4.4}$$

for, if $f\colon X\to Y$ in $\mathbf{T}$, then $\Gamma(Sf) = f^{LY}(\eta_Y) = \eta_Y\circ f$, so that $\Gamma'(Sf) = (\eta_Y\circ f)' = Lf$.

THEOREM 4.3. *$\Gamma'\colon \mathbf{S}(X, Y)\to\mathbf{P}(LX, LY)$ is bijective and respects identities and composition. Thus $\mathbf{S}$ is isomorphic to the Kleisli category of $\mathbf{T}$ with respect to the triple T generated by the adjunction $L\dashv K$.*

Proof. We know from Proposition 4.2 that Γ' is bijective, and from (4.4) that Γ' respect identities. Now let $\tau\colon X\to Y$, $\sigma\colon Y\to Z$ in $\mathbf{S}$, and let $f = \Gamma'(\tau)$, $g = \Gamma'(\sigma)$. Thus $f' = \tau^{LY}(\eta_Y)$, $g' = \sigma^{LZ}(\eta_Z)$, and

$$(\sigma\tau)^{LZ}(\eta_Z) = \tau^{LZ}\sigma^{LZ}(\eta_Z) = \tau^{LZ}(g')\,.$$

From the naturality of τ we infer that

$$\tau^{LZ}(g') = \tau^{LZ}(Kg\circ\eta_Y) = Kg\circ\tau^{LY}(\eta_Y) = Kg\circ f' = (gf)'\,,$$

so that

$$\Gamma'(\sigma\tau) = \Gamma'(\sigma)\Gamma'(\tau)\,,$$

and the first statement of the theorem is proved. The second statement is then an immediate consequence of the explicit definition of the Kleisli category (see [13], p. 143).

Remark. LeVan [12] established the bijection $\mathbf{S}(X, Y)\cong\mathbf{P}(LX, LY)$ when K is a full embedding of a reflective subcategory. We may apply the theorem to any adjoint pair. Thus, for example, if $\mathbf{H}$ is the homotopy category of pointed spaces.

of the homotopy type of CW-complexes then the shape of the loop functor Ω: **H**→**H** is given by

$$\mathbf{S}(X, Y) = \mathbf{H}(\Sigma X, \Sigma Y),$$

where Σ is the suspension. Thus, effectively, we may identify (under Σ) the shape of Ω with the full subcategory of **H** whose objects are suspensions.

THEOREM 4.4. *Let K: **P**→**T** with left adjoint L and let F: **P**→**C** be an arbitrary functor. Then the right Kan extension $\tilde{F}$: **T**→**C** of F along K is FL, with natural transformation $F\varepsilon$: $FLK \to F$, where ε is the counit of the adjunction. If $\bar{F}$ is the extension of $\tilde{F}$ to **S**, given in the proof of Theorem 1.4, then*

$$\bar{F} = F\Gamma'. \tag{4.5}$$

Thus $\bar{F}$ is full (faithful) if F is full (faithful).

Proof. The first statement of the theorem is proved by a trivial modification of the argument establishing the special case of Proposition 4.1; it is, in any case, given in [13], Proposition 3, p. 245. To establish (4.5) we observe that, in the notation of the proof of Theorem 1.4,

$$\alpha_u = Fu'\colon \tilde{F}X \to FP \quad \text{where } u\colon X \to KP \text{ in } \mathbf{T},$$

$$\beta_v = Fv'\colon \tilde{F}Y \to FQ \quad \text{where } v\colon Y \to KQ \text{ in } \mathbf{T}.$$

Thus to establish (4.5) we must show that

$$Fv' \circ F\Gamma'\lambda = F(\lambda^Q v)', \tag{4.6}$$

for all $\lambda \in \mathbf{S}(X, Y)$ and all v: $Y \to KQ$. Now, by the naturality of λ (see the proof of Theorem 4.3),

$$Kv' \circ \lambda^{LY}(\eta_Y) = \lambda^Q v.$$

Thus by (4.1), (4.3)

$$v' \circ \Gamma'\lambda = (\lambda^Q v)',$$

so that (4.6), and hence (4.5), is established.

The final statement of the theorem now follows immediately from Theorem 4.3.

We now discuss the relation between shape categories and categories of fractions. In the course of doing so, we will have occasion to improve certain results in [2] and [3]. We first improve Proposition 2.2 of [3].

PROPOSITION 4.5. *Let the adjoint pair K: **P**→**T**, L: **T**→**P**, with $L \dashv K$, generate the triple T on **T**. Then if K is rich T is idempotent.*

Our proof will depend on the following lemma.

LEMMA 4.6. *Let Σ be the family of morphisms rendered invertible by K, let P_Σ: **P**→**P**$[\Sigma^{-1}]$ be the canonical functor to the category of fractions and let $\bar{L}$: **T** →**P**$[\Sigma^{-1}]$, $\bar{K}$: **P**$[\Sigma^{-1}]$→**T** be given by*

$$\bar{L} = P_\Sigma L, \qquad \bar{K}P_\Sigma = K.$$

Then $\bar{L} \dashv \bar{K}$ and the adjunction generates the same triple T.

Proof. We apply Corollary 2.4 (with $H = L$, $I = P_\Sigma$, $J = \bar{K}$), noting that P_Σ is, of course, rich and dominant. We remark that, in Lemma 4.6, no use is made of the richness of K.

Proof of Proposition 4.5. Assume now that K is rich. By (2.4) $\bar{K}$ is full and the adjunction $\bar{L} \dashv \bar{K}$ generates T. Thus Proposition 4.5 follows from Proposition 2.2 of [3].

We now improve Propositions 2.4 and 2.5 of [2]. We assume as before L: **T**→**P**, K: **P**→**T** with $L \dashv K$, generating the triple T on **T**. Let S_L be the family of morphisms rendered invertible by L.

PROPOSITION 4.7. *Given f: $X \to Y$ in S_L, there exists a unique g: $Y \to KLX$ in **T** with $gf = \eta_X$, where η is the unit of the adjunction.*

Proof. We have $\eta_Y \circ f = KLf \circ \eta_X$. But KLf is invertible so we may take $g = (KLf)^{-1} \circ \eta_Y$. If $gf = \eta_X$, then $1_X = (gf)' = g' \circ Lf$. But Lf is invertible so g' is uniquely determined; so therefore is g.

PROPOSITION 4.8. *If T is idempotent, then* (i) *$\eta_X \in S_L$ for all $X \in |\mathbf{T}|$,* (ii) *S_L admits a calculus of left fractions.*

Proof. (i) forms part of Proposition 2.1 of [3]. The proof of Proposition 2.5 of [2] serves to prove (ii), as it depended only on the fact that η_X is in S_L.

Remark. Proposition 4.7 and Proposition 4.8 (i) constitute the essential content of Theorem 2.9 of [3]. Proposition 4.8 (ii) appears in the proof, but not the statement, of Corollary 2.10 of [3]. That corollary is what we now need.

THEOREM 4.9. *Let K: **P**→**T** with left adjoint L: **T**→**P** generating an idempotent triple, and let **S** be the shape of K. Then $\mathbf{S} \cong T[S_L^{-1}]$, where S_L is the family of morphisms rendered invertible by L (that is, by KL).*

Proof. By Corollary 2.10 of [3] (or Theorem 1.2 of [2]) we infer that there is a natural equivalence of functors

$$\mathbf{T}[S_L^{-1}](-, Y) \cong \mathbf{T}(-, KLY). \tag{4.7}$$

Moreover, if we interpret (4.7) as a bijection

$$\theta\colon \mathbf{T}[S_L^{-1}](X, Y) \cong \mathbf{P}(LX, LY), \tag{4.8}$$

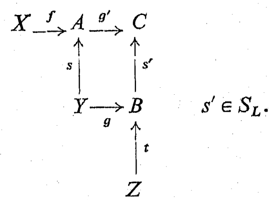
then θ is given by

$$\theta(s^{-1}f) = (Ls)^{-1} \circ Lf, \qquad X \xrightarrow{f} A \xleftarrow{s} Y, \qquad s \in S_L.$$

In the light of Theorem 4.3 it remains only to verify that θ preserves composition (since θ obviously preserves identities). But, given also

$$Y \xrightarrow{g} B \xleftarrow{t} Z, \qquad t \in S_L,$$

then $(t^{-1}g)\circ(s^{-1}f) = (s't)^{-1}g'f$, where

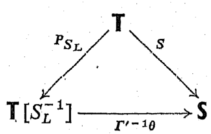


Thus

$$\theta(t^{-1})g\circ\theta(s^{-1}f) = (Lt)^{-1}\circ Lg\circ(Ls)^{-1}\circ Lf = (Lt)^{-1}\circ(Ls')^{-1}\circ Lg'\circ Lf$$
$$= L(s't)^{-1}\circ L(g'f) = \theta\big((t^{-1}g)\circ(s^{-1}f)\big),$$

and the theorem is proved.

Remark. The isomorphism established by this theorem is $\Gamma'^{-1}\theta\colon \mathbf{T}[S_L^{-1}]\cong\mathbf{S}$. We note that this isomorphism renders the triangle



commutative. For (see (4.4)) $\Gamma'S = L$ and if $f\colon X\to Y$ in $\mathbf{T}$ then $\theta P_{S_L}f = Lf$. Thus the shape of K is $\mathbf{T}[S_L^{-1}]$, with canonical functor P_{S_L}.

COROLLARY 4.10. *Let $K\colon \mathbf{P}\to\mathbf{T}$ be rich and admit the left adjoint $L\colon \mathbf{T}\to\mathbf{P}$. Then the shape of K is the category of fractions $\mathbf{T}[S_L^{-1}]$, where S_L is the family of morphisms rendered invertible by L. If K is, in fact, full and faithful then the shape of K is equivalent to $\mathbf{P}$.*

The last statement follows from Theorem 4.3 since, K being full and faithful, $P\cong LKP$ for every $P\in|\mathbf{P}|$.

Examples 4.11. (i) We first give an example of the dual form of Corollary 4.10. Let $\mathbf{T}$ be the homotopy category of topological spaces and let $\mathbf{P}$ be the homotopy category of spaces of the homotopy type of a CW-complex. Then the embedding functor $K\colon \mathbf{P}\to\mathbf{T}$ has a *right* adjoint R which associates with the space X the geometrical realization of its singular complex. The morphisms of $\mathbf{T}$ rendered invertible by R are precisely the weak homotopy equivalences (that is, the maps inducing isomorphisms of homotopy groups in the connected case). Thus the coshape of K is the category of fractions of $\mathbf{T}$ with respect to weak homotopy equivalences, and is equivalent to $\mathbf{P}$ itself.

(ii) The semisimplicial analog of the situation described in (i) yields a further example of Corollary 4.10. If $\mathbf{T}$ is the homotopy category of simplicial sets and $\mathbf{P}$ the full subcategory of Kan complexes, then the embedding $K\colon \mathbf{P}\to\mathbf{T}$ has a left adjoint which is the singular complex of the geometric realization. Thus the shape of K is the category of fractions of $\mathbf{T}$ with respect to weak homotopy equivalences. These are again to be understood as simplicial maps rendered invertible by L and are indeed, in the connected case, characterized as those maps inducing isomorphisms of homotopy groups. The shape of K is equivalent to $\mathbf{P}$ itself.

(iii) Let P be a family of primes and let $\mathbf{N}_P$ be the full subcategory of the category $\mathbf{N}$ of nilpotent groups which consists of nilpotent P-local groups (see [9]). Then there is a P-localizing functor $L\colon \mathbf{N}\to\mathbf{N}_P$ which is left adjoint to the embedding $K\colon \mathbf{N}_P\to\mathbf{N}$. The family S_L of morphisms of $\mathbf{N}$ rendered invertible by L consists precisely of the *P-bijections* [9], so that the shape of K is the category of fractions of $\mathbf{N}$ with respect to P-bijections. We may also use Theorem 4.3 (as we might have done in the previous two examples) to give an explicit description of the shape category via the Kleisli category of the triple generated by the adjunction $L\dashv K$. Let us write G_P, φ_P for KLG, $KL\varphi$ as in [9]. Then a shape morphism from G to H is a homomorphism $\alpha\colon G\to H_P$; and given two shape morphisms $\alpha\colon G\to H$, $\beta\colon H\to M$, their composite is

$$\beta\circ\alpha = \bar\beta\alpha,$$

where $\bar\beta\colon H_P\to M_P$ is determined by $\bar\beta\eta_H = \beta$, η being the unit of the adjunction (we may also write $\bar\beta = \beta_P$ if we take $LK = 1$, as we may).

(iv) We obtain an example closely related to (iii) by replacing $\mathbf{N}$ by the homotopy category $\mathbf{NH}$ of nilpotent spaces [9].

References

[1] K. Borsuk, *Concerning homotopy properties of compacta*, Fund. Math. 62 (1968), pp. 223–254.
[2] A. Deleanu, A. Frei and P. Hilton, *Generalized Adams completion*, Cahiers de Topologie et Géometrie Differentielle 15 (1974), pp. 61–82.
[3] — — — *Idempotent triples and completion*, Math. Z. 143 (1975), pp. 91–104.
[4] — and P. Hilton, *On Kan extensions of cohomology theories and Serre classes of groups*, Fund. Math. 73 (1971), pp. 143–165.
[5] D. Edwards and R. Geoghegan, *Shapes of complexes, ends of manifolds, homotopy limits and the Wall obstruction*, (1974) (preprint).
[6] R. Geoghegan and R. Lacher, *Compacta with the shape of finite complexes*, Fund. Math. 92 (1976), pp. 25–27.
[7] — and R. Summerhill, *Concerning the shapes of finite-dimensional compacta*, Trans. Amer. Math. Soc. 179 (1973), pp. 281–292.
[8] P. Hilton, *On filtered systems of groups, colimits and Kan extensions*, J. Pure App. Algebra 1 (1971), pp. 199–217.
[9] — G. Mislin and J. Roitberg, *Localization of Nilpotent Groups and Spaces*, Mathematics Studies 15, North-Holland 1975.
[10] W. Holsztyński, *An extension and axiomatic characterization of Borsuk's theory of shape*, Fund. Math. 70 (1971), pp. 157–168.
[11] G. Kozlowski, *Polyhedral limitations on shape* (1974) (preprint).

[12] J. LeVan, *Shape Theory*, Dissertation, University of Kentucky 1973.
[13] S. MacLane, *Categories for the Working Mathematician*, Springer Verlag 1971.
[14] S. Mardešić, *Shapes for topological spaces*, Gen. Top. and Appl. 3 (1973), pp. 265–282.
[15] — and J. Segal, *Shapes of compacta and* ANR-*systems*, Fund. Math. 72 (1971), pp. 41–59.
[16] — — *Equivalence of the Borsuk and the* ANR-*system approach to shape*, ibid., pp. 61–68.
[17] T. Porter, *Generalized shape theory*, Proc. Roy. Irish Acad. 74 (1974), pp. 33–48.
[18] A. Deleanu and P. Hilton, *Borsuk shape and a generalization of Grothendieck's definition of pro-category*, Math. Proc. Cam. Phil. Soc. 79 (1976), pp. 473–482.

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Accepté par la Rédaction le 26. 5. 1975

Some algebraic properties of weakly compact and compact cardinals

by

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Abstract. A combinatorial property $[\varkappa, \lambda, \varrho]$ of cardinals is introduced and studied. Work of Jech shows that $\varkappa$ inaccessible and $\varkappa$ weakly compact implies $[\varkappa, \varkappa, 3]$. $[\varkappa, \varkappa, 3]$ is used to establish an algebraic embedding theorem for certain classes of universal algebras. One corollary of this embedding theorem is: if $\varkappa$ is inaccessible and weakly compact and G is a group with $|G| = \varkappa$ and every subgroup of G of smaller cardinality is free, then G is free.

In 1949 R. Rado published the following: [9].

SELECTION LEMMA. *Let A and N be sets and let A_ν be a finite subset of A for each $\nu \in N$. Suppose that for each finite $L \subseteq N$ we are given a function $f_L : L \to A$ such that $f_L(\nu) \in A_\nu$ for each $\nu \in L$. Then there is a function $f: N \to A$ such that given any finite $L \subseteq N$ there is a finite $M \subseteq N$ with $L \subseteq M$ and $f|L = f_M|L$.*

Through the years other have discovered versions of this lemma (see [4], [6], [7], [10]) and several have explored its connection with logical compactness (see [5], [7], [10]). It is natural to ask about possible generalizations of this lemma. Rado in [9] gave an example to show that "finite" could not be replaced by "denumerable." In [7] Jech defined "$\varkappa$ is λ-compact" for infinite cardinals $\varkappa \leqslant \lambda$ with $\varkappa$ regular, and in this same paper he gave a generalization of the Selection lemma for such $\varkappa$ and λ which we denote by $[\varkappa, \lambda, 3]$ (we define this notation in § 0). Jech showed ([7], Theorem 2.2) that weakly compact inaccessible cardinals $\varkappa$ satisfy $[\varkappa, \varkappa, 3]$, and conversely that if $[\varkappa, \varkappa, 3]$ holds then $\varkappa$ is weakly compact. Further he in effect showed that $\varkappa$ is compact if and only if $[\varkappa, \lambda, 3]$ holds for all $\lambda \geqslant \varkappa$.

In this paper we study $[\varkappa, \varkappa, 3]$ and some related properties $[\varkappa, \lambda, \varrho]$. We assume their validity and derive some of their consequences, both set theoretical (§ 1 and § 2) and algebraic (§ 3). In § 1 we show that $[\varkappa, \varkappa, 3]$ implies that $\varkappa$ is a regular limit cardinal without appealing to weak compactness. In § 2 we use inverse limit systems to give a measurability criterion. Our main result, Theorem 3.1, uses $[\varkappa, \varkappa, 3]$ to prove an algebraic embedding theorem. Because of Jech's work this gives an algebraic property of weakly compact inaccessible cardinals and of compact cardinals, special cases of which have been proved by Mekler and Gregory ([1]).

([1]) We wish to thank Paul Eklöf for informing us of the work of Mekler and Gregory.