Table des matières du tome XCVII, fascicule 3

	Pages
D. N. Sarkhel, A generalization of the Vitali covering theorem	151–156
A. Deleanu and P. Hilton, On the categorical shape of a functor	157–176
K. K. Hickin and J. M. Plotkin, Some algebraic properties of weakly compact and compact cardinals	177–185
M. Mastalerz-Wawrzyńczak, On a certain condition of the monotonicity of functions	187–198
Y. Kodama, Decomposition spaces and shape in the sense of Fox	199–208
H. Sarbadhikari, Some uniformization results	209–214
G. M. Huckabay, On the classification of locally compact separable metric spaces	215–219
V. -T.-Liem, Certain continua in S^n with homeomorphic complements have the same shape	221–228
J. Reineke, Commutative rings in which every proper ideal is maximal	229–231

Les FUNDAMENTA MATHEMATICAE publient, en langues des congrès internationaux, des travaux consacrés à la *Théorie des Ensembles, Topologie, Fondements de Mathématiques, Fonctions Réelles, Théorie Descriptive des Ensembles, Algèbre Abstraite*

Chaque volume paraît en 3 fascicules

Adresse de la Rédaction:
FUNDAMENTA MATHEMATICAE, Śniadeckich 8, 00-950 Warszawa (Pologne)

Adresse de l'Échange:
INSTITUT MATHÉMATIQUE, ACADÉMIE POLONAISE DES SCIENCES
Śniadeckich 8, 00-950 Warszawa (Pologne)

Tous les volumes sont à obtenir par l'intermédiaire de
ARS POLONA, Krakowskie Przedmieście 7, 00-068 Warszawa (Pologne)

Correspondence concerning editorial work and manuscripts should be addressed to:
FUNDAMENTA MATHEMATICAE, Śniadeckich 8, 00-950 Warszawa (Poland)

Correspondence concerning exchange should be addressed to:
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, Exchange,
Śniadeckich 8, 00-950 Warszawa (Poland)

The Fundamenta Mathematicae are available at your bookseller or at
ARS POLONA, Krakowskie Przedmieście 7, 00-068 Warszawa (Poland)

DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

A generalization of the Vitali covering theorem

by

D. N. Sarkhel (Kalyani)

Abstract. Let μ be an outer measure defined on the power-set of a set X endowed with a pseudo-metric ϱ such that $\mu(E)<+\infty$ for bounded E. For $E\subset X$ and $r\geqslant 0$ let

$$S[E;\,r]=\{x\in X|\ \varrho(x,y)\leqslant r \text{ for some } y\in E\}.$$

Definition. A sequence $\{V_n\}$ of subsets of X is a *μ-regular sequence converging to a point* $x\in X$ if the diameter $d(V_n)\to 0$, $x\in\bigcap_{n=1}^{\infty}V_n$ and there exists $p>0$ such that

$$\mu(V_n)>p\mu(S[V_n;\,(1+p)d(V_n)])\quad \text{for all } n;$$

also then p is a parameter of regularity of the sequence $\{V_n\}$. A family $\mathcal{V}$ of subsets of X is a *Vitali μ-covering* of a set $E\subset X$ if, for every $x\in E$, there exists a μ-regular sequence of sets from $\mathcal{V}$ converging to x.

The following theorem is proven: *Let μ be a metric outer measure and let a family $\mathcal{V}$ of closed subsets of X be a Vitali μ-covering of a set $E\subset X$. Then there exists a countable family $\mathcal{F}$ of pairwise disjoint sets in $\mathcal{V}$ such that* $\mu(E\setminus\bigcup\mathcal{F})=0$.

Let (X,ϱ) be a pseudo-metric space endowed with the pseudo-metric topology induced by ϱ, and let μ be an outer measure defined on the power-set of X with $\mu(E)<+\infty$ for every bounded subset $E\subset X$ (this condition implies that μ is σ-finite). We denote the closure and the diameter of a subset $E\subset X$ by $\overline{E}$ and $d(E)$, respectively. Given a subset $E\subset X$ and a real number $r\geqslant 0$, we write

$$S[E;\,r]=\{x\colon x\in X,\ \varrho(x,y)\leqslant r \text{ for some } y\in E\}.$$

We now introduce the following definition.

Definition. A sequence $\{V_n\}$ of subsets of X is a *μ-regular sequence converging to a point* $x\in X$ if $d(V_n)\to 0$, $x\in\bigcap_{n=1}^{\infty}V_n$ and there exists a real number $p>0$ such that

$$\mu(V_n)>p\,\mu(S[V_n;\,(1+p)d(V_n)])\quad \text{for all } n;$$

also then p is a *parameter of regularity* of the sequence $\{V_n\}$. A family $\mathcal{V}$ of subsets of X is a *Vitali μ-covering* of a set $E\subset X$ if, for every $x\in E$, there exists a μ-regular sequence of sets from $\mathcal{V}$ converging to x.

In the particular case when X is some Euclidean space, ϱ is the usual metric for X and μ is the outer Lebesgue measure in X, it can be easily verified that our definition of Vitali μ-covering is equivalent to the classical one ([2], p. 109).

The purpose of this note is to prove the following generalization of the classical Vitali covering theorem. In our proof we shall use a modification of the well-known idea of Banach (cf. [2], p. 109–111).

THEOREM. *Let μ be a metric outer measure and let a family $\mathscr{V}$ of closed subsets of X be a Vitali μ-covering of a set $E\subset X$. Then there exists a countable family $\mathscr{F}$ of pairwise disjoint sets from $\mathscr{V}$ such that $\mu(E\setminus\bigcup\mathscr{F})=0$.*

Proof. Note that the Borel subsets of X are measurable (cf. [2], p. 52, (7.4)) for the metric outer measure μ. Now, first consider two particular cases.

Case I. In this case we additionally assume that (i) $d(V)>0$ for every $V\in\mathscr{V}$, (ii) E is bounded, say $E\subset G$, where G is open and bounded, and (iii) for every $x\in E$ there is a μ-regular sequence of sets from $\mathscr{V}$ converging to x with a parameter of regularity exceeding a fixed $p>0$.

Suppose, to get a contradiction, that for every countable family $\mathscr{F}$ of pairwise disjoint sets from $\mathscr{V}$ we have

(1) $$\mu(E\setminus\bigcup\mathscr{F})>0\,.$$

Let us write

$$\mathscr{V}_0=\{V\colon V\in\mathscr{V},\ V\subset G,\ \mu(V)>p\,\mu(S[V;\,(1+p)d(V)])\}\,.$$

Then $\mathscr{V}_0$ is still a Vitali μ-covering of E. We note, in particular, that $\mu(V)>0$ for every $V\in\mathscr{V}_0$. Let

$$l_0=\sup\{d(V)\colon\ V\in\mathscr{V}_0\}\,.$$

Then $0<l_0\leqslant d(G)<+\infty$, and so we can choose a $V_0\in\mathscr{V}_0$ with

$$d(V_0)>(1+p)^{-1}l_0\,.$$

Now let β be an ordinal number such that $0<\beta<\Omega$ ([1], p. 119). Suppose that we have already determined, corresponding to each ordinal $\alpha<\beta$, a set $V_\alpha\in\mathscr{V}_0$ subject to the conditions (2)-(4):

(2) $$\mu(E\cap\overline{H}_\alpha\setminus H_\alpha)=0\,,\quad\text{where}\quad H_\alpha=\bigcup_{0\leqslant\lambda<\alpha}V_\lambda\,,$$

(3) $$V_\alpha\cap H_\alpha=\varnothing\,,$$

(4) $$d(V_\alpha)>(1+p)^{-1}l_\alpha\,,$$

where

(5) $$l_\alpha=\sup\{d(V)\colon\ V\in\mathscr{V}_0,\ V\cap H_\alpha=\varnothing\}\,.$$

Now we shall show that (2) holds with $\alpha=\beta$, that is

(6) $$\mu(E\cap\overline{H}_\beta\setminus H_\beta)=0\,.$$

If β has an immediate predecessor, γ say, then (6) follows easily by taking $\alpha=\gamma$ in (2) and by noting that V_γ is closed. So let β be a limit number, and suppose that

(6′) $$\mu(E\cap\overline{H}_\beta\setminus H_\beta)>0\,.$$

Let $\{\alpha_n\}_{n=1}^{\infty}$ be an enumeration of the ordinals less than β, and let us write, for brevity,

$$S_n=S[V_{\alpha_n};\,(1+p)d(V_{\alpha_n})]\qquad(n=1,2,\ldots)\,.$$

Then, by the definition of $\mathscr{V}_0$ and (3), we have

$$\sum_{n=1}^{\infty}\mu(S_n)<p^{-1}\sum_{n=1}^{\infty}\mu(V_{\alpha_n})=p^{-1}\mu(\bigcup_{n=1}^{\infty}V_{\alpha_n})\leqslant p^{-1}\mu(G)<+\infty\,.$$

Hence, by (6′), there exists a positive integer N such that

(7) $$\sum_{n=N+1}^{\infty}\mu(S_n)<\mu(E\cap\overline{H}_\beta\setminus H_\beta)\,.$$

Since β is a limit number, there is an ordinal η such that

$$\alpha_n<\eta<\beta\qquad(n=1,2,\ldots,N)\,.$$

Then we have

(8) $$\bigcup_{\eta\leqslant\alpha_n<\beta}S_n\subset\bigcup_{n=N+1}^{\infty}S_n\,.$$

Taking $\alpha=\eta$ in (2) and using (7) and (8), we get

$$\mu(\{[E\cap\overline{H}_\beta\setminus H_\beta]\setminus[E\cap\overline{H}_\eta\setminus H_\eta]\}\setminus\bigcup_{\eta\leqslant\alpha_n<\beta}S_n)>0\,.$$

Therefore, there is at least one point y such that

(9) $$y\in E\cap\overline{H}_\beta\setminus H_\beta$$

and

(10) $$y\notin\overline{H}_\eta\cup(\bigcup_{\eta\leqslant\alpha_n<\beta}S_n)\,.$$

Since, by (10), $y\notin\overline{H}_\eta$, there is a $V\in\mathscr{V}_0$ such that

(11) $$V\cap H_\eta=\varnothing\,,\quad y\in V.$$

Since $d(V)>0$ and $y \notin \bar{H}_\eta$, we can choose an open sphere B with centre y such that

(12) (a) $d(B)<d(V)$ and (b) $B \cap H_\eta = \emptyset$.

Now, by (9), $y \in \bar{H}_\beta$. So, for some α_m, we shall have

(13) $$B \cap V_{\alpha_m} \neq \emptyset .$$

Then, by (12) (b), $\alpha_m \geqslant \eta$, and hence, by (10),

(14) $$y \notin S_m = S[V_{\alpha_m}; (1+p)d(V_{\alpha_m})] .$$

From (12) (a), (13), (14) and (4) we deduce that

$$d(V)>d(B)>(1+p)d(V_{\alpha_m})>l_{\alpha_m} .$$

Consequently, by (5), $V \cap H_{\alpha_m} \neq \emptyset$. Hence, for some $\alpha<\alpha_m$, V intersects V_α. There exists, therefore, a smallest one, ξ say, among the ordinals α for which we have $V \cap V_\alpha \neq \emptyset$. Now, by (11), $\xi \geqslant \eta$, and hence, by (10),

$$y \notin S[V_\xi; (1+p)d(V_\xi)] .$$

But V intersects V_ξ and contains the point y. Hence it follows, using (4), that $d(V) >(1+p)d(V_\xi)>l_\xi$. On the other hand, by the definition of ξ, $V \cap H_\xi = \emptyset$, and hence, by (5), $d(V)\leqslant l_\xi$. Thus we arrive at a contradiction, proving the truth of (6).

Now, the family $\{V_\alpha\}_{\alpha<\beta}$ being countable, (1) gives $\mu(E\setminus H_\beta)>0$, whence it follows, by (6), that

$$\mu([E\setminus H_\beta]\setminus[E \cap \bar{H}_\beta\setminus H_\beta])>0 .$$

Since, moreover, $[E\setminus H_\beta]\setminus[E \cap \bar{H}_\beta\setminus H_\beta] = E\setminus\bar{H}_\beta$, there are sets $V \in \mathscr{V}_0$ such that $V \cap H_\beta = \emptyset$. So, if l_β is obtained by taking $\alpha = \beta$ in (5), then $0<l_\beta\leqslant d(G)<+\infty$, and hence we can choose a $V_\beta \in \mathscr{V}_0$ satisfying conditions (3) and (4). Hence, by transfinite induction, an uncountable family $\{V_\alpha\}_{\alpha<\Omega}$ ([1], p. 120, Corollary) of sets in $\mathscr{V}_0$ is thus generated. This, however, contradicts the σ-finiteness of μ, and hence the theorem is proved in this case.

Case II (cf. [2], p. 110, b)). In this case we retain only the additional hypothesis (i) of Case I. First choose a point $x_0 \in E$, and denote, for any positive integer n, by G_n the open sphere of radius n and centre x_0. Then denote by E_n the set of the points $x \in E \cap G_n$ for which there exists a μ-regular sequence of sets from $\mathscr{V}$ converging to x with a parameter of regularity exceeding n^{-1}. Since $E_1 \subset G_1$, there exists, by Case I, a countable family $\{V_i\}$ of pairwise disjoint sets in $\mathscr{V}$ such that

$$\mu(E_1\setminus \bigcup_i V_i) = 0 \quad \text{and} \quad \bigcup_i V_i \subset G_1 .$$

It follows that

$$\sum_i \mu(V_i) = \mu(\bigcup_i V_i)\leqslant\mu(G_1)<+\infty .$$

Therefore, given $\varepsilon>0$, there is a positive integer N such that

$$\mu(E_1\setminus \bigcup_{i=1}^{N} V_i)<\varepsilon .$$

Then $E_2\setminus \bigcup_{i=1}^{N} V_i$ is contained in the bounded open set $G_2\setminus \bigcup_{i=1}^{N} V_i$. Now it is clear that by simple induction we can define a sequence $\{\mathscr{E}_n\}$ subject to the following conditions:

(c_1) $\bigcup_{i=1}^{n} \mathscr{E}_i$ consists of a finite number of pairwise disjoint sets in $\mathscr{V}$ $(n = 1, 2, \ldots)$;

(c_2) if T_i denotes the union of the sets in $\mathscr{E}_i$, then

$$\mu(E_n\setminus \bigcup_{i=1}^{n} T_i)<n^{-1} \quad (n = 1, 2, \ldots) .$$

Since $E_n \subset E_{n+1}$ for all n, it follows, by (c_2), that

(15) $$\mu(E_n\setminus \bigcup_{i=1}^{\infty} T_i) = 0 \quad (n = 1, 2, \ldots) .$$

But $E = \bigcup_{n=1}^{\infty} E_n$. Hence, by (15), it follows that

$$\mu(E\setminus \bigcup_{i=1}^{\infty} T_i) = 0 ,$$

which proves the theorem in this case.

For the general case, let E_0 denote the set of the points $x \in E$ for which there is a μ-regular sequence $\{V_n\}$ of sets from $\mathscr{V}$ converging to x with $d(V_n)>0$ for all n. Then, by Case II, there is a countable family $\mathscr{F}_0$ of pairwise disjoint sets in $\mathscr{V}$ such that

$$\mu(E_0\setminus \bigcup \mathscr{F}_0) = 0 .$$

Let

$$E' = (E\setminus E_0)\setminus \bigcup \mathscr{F}_0 ,$$

and let $\mathscr{F}_1$ denote, under the equivalence relation $\varrho(x, y) = 0$ defined on X, the family of the equivalence sets which intersect E'. For every $x \in E'$ there is a set $V_x \in \mathscr{V}$ such that $x \in V_x$, $d(V_x) = 0$ and $\mu(V_x)>0$. It is easy to see that V_x is in fact the equivalence set containing x. Since closed sets are measurable (μ), it follows from the σ-finiteness of μ that the family $\mathscr{F}_1$ is countable. Also, the sets in $\mathscr{F}_0$ being closed, no member of $\mathscr{F}_1$ can intersect any member of $\mathscr{F}_0$. Therefore, the family $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ fulfils all the required conditions. This completes the proof of the theorem.

Acknowledgment. The author is grateful to Dr. P. C. Bhakta of Jadavpur University, Calcutta, for his kind suggestions in the preparation of this note.

References

[1] I. P. Natanson, *Theory of Functions of a Real Variable*, Vol. II,
[2] S. Saks, *Theory of the Integral*, Warszawa 1937.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KALYANI
West Bengal, India

Accepté par la Rédaction le 18. 4. 1975

On the categorical shape of a functor

by

Aristide Deleanu and **Peter Hilton** * (Syracuse, N. Y. and Seattle, Wa.)

Abstract. The concept of shape, first introduced by Borsuk in his study of the homotopy theory of compacta, is extended to an abstract categorical setting. The shape of an arbitrary functor K is defined, and it is proved that the Kan extension is shape-invariant. One then shows that many of the categorical aspects of shape remain valid in this very general setting; others require some restriction on the functor K, and the notion of a rich functor is introduced, which is more general than the notion of a full functor. In addition, it is proved that if K is rich, the iteration of the shape construction produces the same shape category. Finally, the special case when K has a left adjoint is discussed in some detail, and a relation with the categories of fractions is exhibited.

Introduction. Since Borsuk [1] first introduced the concept of shape in his study of the homotopy theory of compacta many authors (e.g. [5], [6], [7], [11], [12], [14], [15], [16], [17]) have contributed to the develepment of shape theory. However the theory has remained almost exclusively confined to a topological context, never very far removed from the setting in which it was originally cast by Borsuk; and further, and arising from this restriction in the scope of the theory, the concept has, in the work cited, related to some category of topological spaces $\mathbf{T}$ and a full subcategory $\mathbf{P}$ of $\mathbf{T}$. However, Holsztyński [10] observed, soon after Borsuk's invention of the concept, that shape could be formulated as an abstract limit, and was thus of more general applicability.

It is the principal purpose of this paper to free shape theory from its restricted scope. Thus we replace the full embedding of a topological category $\mathbf{P}$ in a topological category $\mathbf{T}$ by an arbitrary functor $K\colon \mathbf{P}\to\mathbf{T}$ from the arbitrary category $\mathbf{P}$ to the arbitrary category $\mathbf{T}$. In so doing we are very much inspired by the point of view adopted by LeVan in his thesis [12]. We then find that many of the *categorical* aspects of shape theory (we do not speak of the *topological* aspects) remain valid in this very general setting. Others require some restriction on the functor K, but a restriction far milder than that K should be a full embedding.

In Section 1 we define *shape* and the dual concept *coshape*. Indeed, for $K\colon \mathbf{P}\to\mathbf{T}$,

* The first-named author was partially supported by NSF Grant GP38804, and the second-named author by NSF Grant GP43703, during the preparation of this paper.