$W = \{C_i: i \in \omega\}$ is an example to show that $\mathrm{MC}^{\mathrm{LO}}_{\mathrm{LO}}$ is false in M_6. This completes the proof of Theorem 3.6 and therefore the proof of the non-implications given in Figure 3.

To summarize the results of this section we include Figure 4. It shows for each of the models M_1-M_6, which of our statements are true (T) and which are false (F).

	M_1	M_2	M_3	M_4	M_5	M_6
AC	F	F	F	F	F	F
$\mathrm{AC}^{\mathrm{LO}}$	F	F	F	F	T	F
MC	F	F	F	T	F	F
$\mathrm{AC}_{\mathrm{LO}}$	F	F	F	F	F	F
$\mathrm{MC}^{\mathrm{LO}}$	F	F	F	T	T	F
$\mathrm{AC}^{\mathrm{LO}}_{\mathrm{LO}}$	F	T	T	F	T	F
A	F	F	T	T	F	T
$\mathrm{MC}_{\mathrm{LO}}$	F	T	T	T	F	F
$\mathrm{MC}^{\mathrm{LO}}_{\mathrm{LO}}$	F	T	T	T	T	F
$\mathrm{MC}_{\mathrm{DLO}}$	F	T	T	T	F	T
LW	F	T	T	T	T	T
PW	T	T	T	T	T	T

Fig. 4

References

[1] U. Felgner and T. Jech, *Variants of the axiom of choice in set theory with atoms*, Fund. Math. 79 (1973), pp. 77–85.

[2] T. Jech, *The Axiom of Choice*, North Holland 1973.

[3] A. Mostowski, *Über die Unabhängigkeit des Wohlordnungssatzes vom Ordnungsprinzip*, Fund. Math. 32 (1939), pp. 201–252.

[4] H. Rubin, *Two propositions equivalent to the axiom of choice only under both the axioms of extensionality and regularity*, Notices Amer. Math. Soc. 7 (1960), p. 381.

[5] — and J. Rubin, *Equivalents of the Axiom of Choice*, North Holland 1963.

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The hereditary classes of mappings

by

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Abstract. If $\mathscr{C}$ is an arbitrary class of mappings, then a mapping $f\colon X\to Y$ is hereditarily $\mathscr{C}$ if for each continuum $K\subset X$ the partial mapping $f|K$ is in $\mathscr{C}$. In the paper we study some properties of hereditarily monotone, hereditarily confluent, hereditarily weakly confluent and hereditarily atriodic mappings (for the definition see § 3). In particular, it is proved that a continuum X is hereditarily unicoherent if and only if any monotone mapping of X is hereditarily monotone. We give also other mapping characterizations of some classes of continua. Namely, we prove that a continuum X is hereditarily indecomposable (atriodic) if and only if any confluent (atriodic) mapping of a continuum onto X is hereditarily confluent (hereditarily atriodic). Using these results, we characterize hereditarily decomposable snake-like continua and an arc by hereditarily weakly confluent mappings. These results are connected with the problem posed in [12], and imply some partial solutions of this problem.

Further, it is proved that any (irreducible) hereditarily confluent mapping of an arcwise connected continuum (onto a locally connected continuum, respectively) is monotone. We discuss also some invariance properties of the above mappings. In particular, we show that if a continuum X is hereditarily decomposable, then the hereditary unicoherence of X as well as the atriodicity of X is an invariant under hereditarily weakly confluent mappings.

§ 1. Introduction. The topological spaces under consideration are assumed to be metric and compact, and the mappings — to be continuous and surjective. A continuum means a compact connected space.

Pseudo-monotone mappings have been introduced in [20], p. 13, by L. E. Ward, Jr. Namely, we call a mapping $f\colon X\to Y$ *pseudo-monotone* if, for each pair of closed connected sets $A\subset X$ and $B\subset f(A)$, some component of $A\cap f^{-1}(B)$ is mapped by f onto B. Simple examples show that the pseudo-monotoneity of f neither implies nor is implied by its monotoneity. We describe below a monotone mapping which is not pseudo-monotone. This example will be used in further considerations.

(1.1) Example. There exists a monotone mapping f of a circle S onto itself such that f is not pseudo-monotone.

Let (r,φ) denote a point of the Euclidean plane having r and φ as its polar coordinates. Take the unit circle $S = \{(r,\varphi)\colon r=1 \text{ and } 0\leqslant\varphi\leqslant 2\pi\}$. We define

$$f(r,\varphi) = \begin{cases} (r, 2\varphi) & \text{if} \quad 0\leqslant\varphi\leqslant\pi, \\ (r, 0) & \text{if} \quad \pi\leqslant\varphi\leqslant 2\pi. \end{cases}$$

Observe that a mapping $f\colon S\to S$ is monotone but it is not pseudo-monotone.

Recall that a mapping $f\colon X\to Y$ is called *confluent* (*weakly confluent*) if for every subcontinuum Q of Y each (at least one, respectively) component of the inverse image $f^{-1}(Q)$ is mapped by f onto Q (see [3], p. 213 and [13], Sections 4 and 5). The following proposition is an immediate consequences of the definitions.

(1.2) PROPOSITION. *A mapping $f\colon X\to Y$ is pseudo-monotone if and only if for each continuum K in X the partial mapping $f|K$ is weakly confluent.*

Let $\mathscr{C}$ be a class of mappings. We shall call a mapping $f\colon X\to Y$ *hereditarily $\mathscr{C}$* if for each continuum $K\subset X$ the partial mapping $f|K$ is in $\mathscr{C}$. Taking the class of monotone, confluent or weakly confluent mappings for $\mathscr{C}$, we get in this way the classes of hereditarily monotone, hereditarily confluent and hereditarily weakly confluent mappings, respectively.

This use of the term "hereditarily weakly confluent" instead of the term "pseudo-confluent" is well grounded by Proposition 1.2 in view of the terminology mentioned above.

In the present paper we consider the hereditary classes of the above mappings. In § 2 we study some properties of those mappings, which are used in further considerations. We characterize hereditarily unicoherent continua, hereditarily indecomposable continua and atriodic continua in terms of hereditarily monotone mappings, hereditarily confluent mappings and hereditarily weakly confluent mappings, respectively (§ 3). Further, in § 4, we prove that hereditarily confluent mappings of arcwise connected continua are monotone and that those mappings onto locally connected continua are also monotone with some additional assumptions. It is proved, in § 5, that if a continuum X is hereditarily decomposable, then the hereditary unicoherence of X as well as the atriodicity of X are invariants under hereditarily weakly confluent mappings, and we discuss these invariance properties for other classes of mappings.

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§ 2. Preliminaries. We have the following easy consequences of the definitions in § 1.

(2.1) PROPOSITION. *Any hereditarily monotone (hereditarily confluent, hereditarily weakly confluent) mapping defined on a continuum is monotone (confluent, weakly confluent, respectively).*

(2.2) PROPOSITION. *Any hereditarily monotone mapping is hereditarily confluent and any hereditarily confluent mapping is hereditarily weakly monotone.*

Since a composition of two monotone (confluent, weakly confluent) mappings is monotone (confluent, weakly confluent, respectively) (see [3], III, p. 214 and [13], 4.4), we have

(2.3) PROPOSITION. *A composition of two hereditarily monotone (hereditarily confluent, hereditarily weakly confluent) mappings is hereditarily monotone (hereditarily confluent, hereditarily weakly confluent, respectively).*

Moreover (cf. [3], I, p. 213 and [13], 4.7)

(2.4) PROPOSITION. *If a mapping $f\colon X\to Y$ is hereditarily monotone (hereditarily confluent, hereditarily weakly confluent), and A is a subcontinuum of X, then the partial mapping $f|A$ is hereditarily monotone (hereditarily confluent, hereditarily weakly confluent, respectively).*

W now prove an analogous theorem to IV in [3], p. 214, for hereditarily monotone, hereditarily confluent and hereditarily weakly confluent mappings.

(2.5) THEOREM. *Let a mapping $f\colon X\to Y$ be weakly confluent. If a mapping $g\colon Y\to Z$ is such that gf is hereditarily monotone (hereditarily confluent, hereditarily weakly confluent), then the mapping g is hereditarily monotone (hereditarily confluent, hereditarily weakly confluent, respectively).*

Indeed, it suffices to prove that for each continuum K in Y the partial mapping $g|K$ is monotone (confluent, weakly confluent, respectively). Let K be an arbitrary subcontinuum of Y. Since f is weakly confluent by the assumption, there is a continuum C in X such that $f(C)=K$. It follows from the definition of hereditary monotoneity (hereditary confluence, hereditary weak confluence) that $(gf)|C$ is monotone (confluent, weakly confluent, respectively). Since $(gf)|C=(g|K)(f|C)$, we infer that $g|K$ is monotone (confluent, weakly confluent, respectively) (cf. [3], IV, p. 214).

The hypothesis of weak confluence of f in Theorem (2.5) is essential. This can be proved by the following

(2.6) EXAMPLE. There are mappings f and g such that gf is hereditarily monotone, f is not weakly confluent and g is not hereditarily weakly confluent.

Let C denote the Cantor standard ternary set lying in the closed unit interval $I=[0,1]$. Given a point $x\in C$, we have

$$x=\frac{c_1}{3}+\frac{c_2}{3^2}+\ldots+\frac{c_m}{3^m}+\ldots\qquad (c_m=0\text{ or }2),$$

and we put

$$h(x)=\frac{c_1}{2^2}+\frac{c_2}{2^3}+\ldots+\frac{c_m}{2^{m+1}}+\ldots$$

Thus h is the well-known "step-function", which maps C continuously onto I (see e.g. [10], § 16, II, p. 150).

Let (r,φ) denote a point of the Euclidean plane having r and φ as its polar coordinates. Consider the set K and the mapping $k\colon K\to k(K)$ defined as follows:

$$K=\{(r,\varphi)\colon r\in C\text{ and }0\leqslant r\leqslant\tfrac{1}{2},\ 0\leqslant\varphi\leqslant\pi;\text{ or }r\in C\text{ and }\tfrac{1}{2}\leqslant r\leqslant 1\text{ and }0\leqslant\varphi\leqslant\tfrac{1}{3}\pi\text{ or }\tfrac{2}{3}\pi\leqslant\varphi\leqslant\pi\},$$

$$k(r,\varphi)=(h(r),\varphi)\quad\text{for each }(r,\varphi)\in K.$$

Put

$$L=K\cup\{(r,\tfrac{1}{3}\pi i)\colon\tfrac{1}{3}\leqslant r\leqslant 1\text{ and }i=1,2\}\cup\{(r,\tfrac{1}{2}\pi)\colon 0\leqslant r\leqslant\tfrac{1}{3}\}.$$

The set L is a continuum. Let f be a monotone extension of k onto the whole L. Since $M = \{(r, \varphi): 0 \leqslant r \leqslant 1,\ \varphi = 0\}$ is a continuum which is contained in $f(L)$, and since any component of the set $f^{-1}(M)$ is degenerate, we conclude that f is not weakly confluent. We define a mapping $g: f(L) \to gf(L)$ as follows:

$$g(r, \varphi) = \begin{cases} (-2r+1, \tfrac{1}{2}\pi) & \text{if } r \leqslant \tfrac{1}{2}, \\ (2r-1, 0) & \text{if } \tfrac{1}{2} < r \text{ and } 0 \leqslant \varphi \leqslant \tfrac{1}{3}\pi, \\ (2r-1, \pi) & \text{if } \tfrac{1}{2} < r \text{ and } \tfrac{2}{3}\pi \leqslant \varphi \leqslant \pi. \end{cases}$$

It is easy to observe that the mapping g is monotone, but it is not hereditarily weakly confluent. Moreover, the mapping gf is hereditarily monotone.

§ 3. Mapping characterizations of some classes of continua. Recall that a continuum is said to be *unicoherent* if for any its decomposition into two subcontinua the intersection of those subcontinua is connected. A continuum is said to be *hereditarily unicoherent* if each of its subcontinua is unicoherent. It is known that a continuum X is hereditarily unicoherent if and only if for any two points $x, y \in X$ there exists a unique subcontinuum $I(x, y)$ which is irreducible between x and y. The proof of the following theorem has been suggested by D. Zaremba.

(3.1) THEOREM. *If any monotone mapping of the continuum X is hereditarily monotone, then X is hereditarily unicoherent.*

Proof. Suppose, on the contrary, that X is not hereditarily unicoherent. Then there are subcontinua A and B of X such that $A \cap B$ is not connected. We define the following equivalence relation ϱ. Let $x, y \in X$. Then

$$x \varrho y \quad \text{if and only if} \quad x = y \text{ or } x, y \in B.$$

Since any equivalence class is a continuum, the canonical mapping $\varphi: X \to X/\varrho$ is monotone (cf. [21], p. 127). But the partial mapping $\varphi|A$ is not monotone. In fact, if a is a point of $A \cap B$, then

$$(\varphi|A)^{-1}(\varphi|A)(a) = \varphi^{-1}\varphi(a) \cap A = A \cap B,$$

by the definition of ϱ. But by the assumption the set $A \cap B$ is not connected, and thus $\varphi|A$ is not monotone, a contradiction.

It is easy to prove that any monotone mapping of the hereditarily unicoherent continuum is hereditarily monotone. Therefore, by Theorem (3.1), we have

(3.2) COROLLARY. *A continuum X is hereditarily unicoherent if and only if any monotone mapping of X is hereditarily monotone.*

Recall that a continuum X is said to be *decomposable* (*indecomposable*) if there exists (there does not exist) a decomposition of X into two proper subcontinua A and B such that $X = A \cup B$. A continuum is said to be *hereditarily decomposable* (*hereditarily indecomposable*) if any of its subcontinua is decomposable (indecomposable).

Let I be the unit closed interval $[0, 1]$ and let $p: X \times I \to X$ be the projection mapping, i.e., such that $p(x, t) = x$ for each $x \in X$ and $t \in I$. We have

(3.3) THEOREM. *Let X be a continuum. If a projection mapping $p: X \times I \to X$ is hereditarily confluent, then X is hereditarily indecomposable.*

Proof. Suppose that X is not hereditarily indecomposable. Then there are proper subcontinua A and B of X such that $A \cap B \neq \emptyset$ and $A \neq A \cup B \neq B$. Let $a \in A \setminus B$. Take the continuum $Q \subset X \times I$ defined by: $Q = (A \cup B) \times \{0\} \cup \cup A \times \{1\} \cup \{a\} \times I$. The partial mapping $p|Q$ is not confluent. In fact, the set $Q \cap p^{-1}(B)$ has at least two components, namely $B \times \{0\}$ and at least one component C which is contained in the set $(A \cap B) \times \{1\}$. We have $p(C) \subset p((A \cap B) \times \times \{1\}) \subset A \cap B \neq B$; thus the mapping $p|Q$ is not confluent, i.e., p is not hereditarily confluent.

(3.4) COROLLARY. *The following conditions are equivalent:*

(i) *X is hereditarily indecomposable,*

(ii) *any mapping of a continuum X is confluent,*

(iii) *any mapping of a continuum onto X is hereditarily confluent,*

(iv) *any confluent mapping of a continuum onto X is hereditarily confluent,*

(v) *the projection mapping $p: X \times I \to X$ is hereditarily confluent.*

Proof. (i) is equivalent to (ii) (see [13], 5.7, cf. [6], p. 243). (i) and (ii) imply (iii). Indeed, if f maps a continuum M onto X, and Q is a subcontinuum of M, then $f(Q)$ is hereditarily indecomposable by (i), because any subcontinuum of a hereditarily indecomposable continuum is hereditarily indecomposable. We infer from (ii) that $f|Q$ is confluent, and thus f is hereditarily confluent. The implication from (iii) to (iv) is obvious. Since the projection mapping $p: X \times I \to X$ is confluent, (iv) implies (v). Finally, (v) implies (i) by Theorem (3.3).

Remark. One can find other mapping characterizations of hereditarily indecomposable continua, e.g. in [8], Theorem 4, p. 51, by monotone and atomic mappings.

Recall that a continuum T is called a *triod* provided that there are three subcontinua A, B and C of T such that $T = A \cup B \cup C$, $A \cap B \cap C = A \cap B = A \cap C = B \cap C$ and this common part is a proper subcontinuum of each of them. A continuum X is said to be *atriodic* if it fails to contain a triod.

We say that a mapping $f: X \to Y$ is *atriodic* (the name was suggested by Professor B. Knaster) if for each continuum Q in Y there are two components C' and C'' of the set $f^{-1}(Q)$ such that

(i) $f(C') \cup f(C'') = Q$,

(ii) for each component C of the set $f^{-1}(Q)$ we have either $f(C) = Q$ or $f(C) \subset f(C')$ or $f(C) \subset f(C'')$.

We say (according to the general rule — see Section 1) that a mapping $f: X \to Y$ is *hereditarily atriodic* provided for each subcontinuum K of X the partial mapping $f|K$ is atriodic. Obviously

(3.5) PROPOSITION. *Any weakly confluent mapping is atriodic, and any hereditarily weakly confluent mapping is hereditarily atriodic.*

Now we prove the following

(3.6) THEOREM. *If a mapping f maps a continuum M onto a hereditarily unicoherent and atriodic continuum X, then f is atriodic.*

Proof. Let Q be an arbitrary subcontinuum of X. We shall prove three properties of $f^{-1}(Q)$, which are needed in the sequel.

(1) For each point $q\in Q$ there is a component C_q of $f^{-1}(Q)$ such that $q\in f(C_q)$ and such that for each component C of the set $f^{-1}(Q)$ it is not true that $f(C_q)\subset f(C)\neq f(C_q)$.

Let C_1 be an arbitrary component of $f^{-1}(Q)$ such that $f^{-1}(q)\cap C_1\neq\emptyset$, and let $\mathscr{C}=\{H: f(C_1)\subset H$ and there is a component K of $f^{-1}(Q)$ such that $f(K)=H\}$. Denote by $\mathscr{D}$ a maximal subcollection of $\mathscr{C}$ totally ordered (by inclusion) and put $D=\overline{\bigcup\{H\colon H\in\mathscr{D}\}}$. It is easy to check that $D\in\mathscr{C}$, and thus there is a component C_q of $f^{-1}(Q)$ such that $f(C_q)=D$. Then C_q satisfies the required conditions.

(2) There are no three components C_1, C_2 and C_3 of $f^{-1}(Q)$ such that sets $f(C_1)$, $f(C_2)$ and $f(C_3)$ are pairwise disjoint.

In fact, suppose, on the contrary, that C_1, C_2 and C_3 are components of the set $f^{-1}(Q)$ such that the sets $f(C_1)$, $f(C_2)$ and $f(C_3)$ are pairwise disjoint. Then there are continua K_1, K_2 and K_3 such that $C_i\subset K_i\neq C_i$ and $f(K_i)\cap f(K_j)=\emptyset$ for each $i\neq j$ and $i,j=1,2,3$. Since

$$Q=\bigcap_{n=1}^{3}(Q\cup f(K_n))=(Q\cup f(K_j))\cap(Q\cup f(K_i))$$

for each $i\neq j$ and $i,j=1,2,3$, and $f(K_i)\setminus Q\neq\emptyset$ for each $i=1,2,3$, we infer that the set $T=Q\cup\bigcup_{i=1}^{3}f(K_i)$ is a triod. But X is atriodic, a contradiction.

(3) There are no components C_1, C_2 and C_3 of $f^{-1}(Q)$ such that the sets $f(C_1)\cap f(C_i)$ and $f(C_i)\setminus(f(C_1)\cup f(C_j))$ are non-empty for $i\neq j$ and $i,j=2,3$.

In fact, suppose, on the contrary, that there are such components C_1, C_2 and C_3 of the set $f^{-1}(Q)$. Then there is a continuum K' in M such that $C_1\subset K'\neq C_1$ and such that the sets $f(C_i)\setminus(f(K')\cup f(C_j))$ are non-empty for $i\neq j$ and $i,j=2,3$. Put

$$R=f(C_1)\cup(f(C_2)\cap f(C_3))\cup(f(K')\cap f(C_2))\cup(f(K')\cap f(C_3))$$

and $P_1=f(K')\cup R$, $P_2=f(C_2)\cup R$ and $P_3=f(C_3)\cup R$. Since X is hereditarily unicoherent, we infer that the sets R and P_i for $i=1,2,3$ are continua. It is easy to check that $R=P_1\cap P_2\cap P_3=P_i\cap P_j$ for $i\neq j$ and $i,j=1,2,3$. Moreover, since

$$\emptyset\neq f(K')\setminus Q\subset P_1\setminus(P_2\cup P_3)\quad\text{and}\quad\emptyset\neq f(C_i)\setminus(f(K')\cup f(C_j))\subset P_i\setminus(P_1\cup P_j)$$

for $i\neq j$ and $i,j=2,3$, we conclude that the set $P_1\cup P_2\cup P_3$ is a triod. But X is atriodic, a contradiction.

Now, let p be an arbitrary point of Q and let C_p be a component of $f^{-1}(Q)$ which is determined by (1). If $f(C_p)=Q$, then putting $C'=C''=C_p$ we complete the proof of the theorem. Suppose that $f(C_p)\neq Q$. Then there is a point q such that $q\in Q\setminus f(C_p)$. Let C_q be a component of $f^{-1}(Q)$ which is defined by (1). Then

$$f(C_p)\cup f(C_q)=Q\,. \tag{4}$$

In fact, suppose, on the contrary, that $f(C_p)\cup f(C_q)\neq Q$. Then there is a point r such that $r\in Q\setminus(f(C_q)\cup f(C_p))$. Let C_r be a component of $f^{-1}(Q)$ which is defined by (1). We can assume that $f(C_p)\cap f(C_q)\neq\emptyset$ (or $f(C_p)\cap f(C_r)\neq\emptyset$ or $f(C_q)\cap f(C_r)\neq\emptyset$; in any case the proof is the same) by (2). Then $f(C_r)\cap(f(C_p)\cup f(C_q))=\emptyset$, because in the opposite case we obtain a contradiction of (3) simply by substituting C_p, C_q, C_r for C_1, C_2, C_3, respectively, if $f(C_r)\cap f(C_p)\neq\emptyset$. Therefore there is a point s such that $s\in Q\setminus(f(C_p)\cup f(C_q)\cup f(C_r))$. Take C_s, which is determined by (1). If $f(C_s)\cap(f(C_p)\cup f(C_q))\neq\emptyset$, then we can assume that $f(C_s)\cap f(C_p)\neq\emptyset$ by (2). Thus substituting C_p, C_q and C_r for C_1, C_2 and C_3, respectively, we obtain a contradiction of (3). If $f(C_s)\cap(f(C_p)\cup f(C_q))=\emptyset$, we have $f(C_s)\cap f(C_r)\neq\emptyset$ by (2). So $f(C_p)\cap f(C_q)\neq\emptyset$ and $f(C_r)\cap f(C_s)\neq\emptyset$, and

$$(f(C_p)\cup f(C_q))\cap(f(C_r)\cup f(C_s))=\emptyset\,.$$

Therefore there is a point x such that $x\in Q\setminus(f(C_p)\cup f(C_q)\cup f(C_r)\cup f(C_s))$. Consider a component C_x determined by (1). It follows from (2) that either $f(C_x)\cap(f(C_p)\cup f(C_q))\neq\emptyset$ or $f(C_x)\cap(f(C_r)\cup f(C_s))\neq\emptyset$. In both cases we obtain a contradiction as above. Hence (4) holds.

Let C be an arbitrary component of $f^{-1}(Q)$ such that $f(C)\setminus f(C_p)\neq\emptyset$ and $f(C)\setminus f(C_q)\neq\emptyset$. Then, by (1), we have $f(C_p)\setminus f(C)\neq\emptyset$ and $f(C_q)\setminus f(C)\neq\emptyset$. If $f(C_p)\cap f(C_q)\subset f(C)$, then the sets $f(C)\cap f(C_p)$, $f(C)\cap f(C_q)$, $f(C_p)\setminus(f(C)\cup f(C_q))$ and $f(C_q)\setminus(f(C)\cup f(C_p))$ are non-empty. This contradicts to (3). If $(f(C_p)\cap f(C_q))\setminus f(C)\neq\emptyset$, then putting $P_1=f(C)\cap f(C_p)$, $P_2=f(C)\cap f(C_q)$ and $P_3=f(C_p)\cap f(C_q)$, we infer that the set $P_1\cup P_2\cup P_3$ is a triod — a contradiction, because X is atriodic.

Therefore, by (4), the components $C'=C_p$ and $C''=C_q$ satisfy conditions (i) and (ii) of the definition of an atriodic mapping. Hence the mapping f is atriodic. Moreover, since the property of being an atriodic continuum is hereditary, we infer that the mapping f is hereditarily atriodic. The proof of Theorem (3.6) is complete.

(3.7) THEOREM. *If the projection mapping $p\colon X\times I\to X$ is hereditarily atriodic, then the continuum X is atriodic.*

Proof. Suppose, on the contrary, that X is not atriodic. Then there are three subcontinua A, B and C of X such that the set $A\cup B\cup C$ is a continuum, $A\cap B\cap C=A\cap B=A\cap C=B\cap C$ and there are three points a, b and c of X such that $a\in A\setminus(B\cup C)$, $b\in B\setminus(A\cup C)$ and $c\in C\setminus(A\cup B)$. Denote the set $A\cup B\cup C$ by T. Define the continuum M contained in $X\times I$ as follows:

$$M=((A\cup C)\times\{0\})\cup((A\cup B)\times\{\tfrac12\})\cup((B\cup C)\times\{1\})\cup(\{a\}\times[0,\tfrac12])\cup(\{b\}\times[\tfrac12,1])\,.$$

The mapping $p|M$ is not atriodic. In fact, let V and U be open sets in T such that $a\in V\subset \overline{V}\subset T\setminus(B\cup C)$ and $b\in U\subset\overline{U}\subset T\setminus(A\cup C)$. Take the component D of the set $A\setminus V$ such that $A\cap B\cap C\subset D$ and take the component E of the set $B\setminus U$ such that $A\cap B\cap C\subset E$. Then $D\cap\overline{V}\neq\emptyset$ and $E\cap\overline{U}\neq\emptyset$ by Theorem 1 in [11], § 47, III, p. 172. We infer that there are points d and e such that $d\in D\setminus(B\cup C)$ and $e\in E\setminus(A\cup C)$. Put $K=C\cup D\cup E$. The set $p^{-1}(K)$ has exactly three components, namely: $(C\cup D)\times\{0\}$, $(D\cup E)\times\{\frac{1}{2}\}$ and $(C\cup E)\times\{1\}$; none of them is mapped under p onto the continuum K and the image under p of none of them is contained in the image under p of an other one. Therefore the mapping $p|M$ is not atriodic, i.e., p is not hereditary atriodic, a contradiction.

(3.8) COROLLARY. *Let a continuum X be hereditarily unicoherent. The following conditions are equivalent:*

(i) *X is atriodic,*

(ii) *any mapping of a continuum onto X is hereditarily atriodic,*

(iii) *any atriodic mapping of a continuum onto X is hereditarily atriodic,*

(iv) *the projection mapping $p\colon X\times I\to X$ is hereditarily atriodic.*

Indeed, (i) implies (ii) by Theorem (3.6). Obviously (ii) implies (iii). Since the projection mapping p is open (thus weakly confluent), we infer that p is atriodic by Proposition (3.5). Therefore (iii) implies (iv); (iv) implies (i) by Theorem (3.7).

(3.9) THEOREM. *If the projection mapping $p\colon X\times I\to X$ is hereditarily weakly confluent, then the continuum X is atriodic and hereditarily unicoherent.*

Proof. By Proposition (3.5) and Theorem (3.7) we infer that the continuum X is atriodic. Suppose, on the contrary, that X is not hereditarily unicoherent. Then there are subcontinua A and B of X such that $A\cap B=R\cup S$, where R and S are closed, nonempty, separated (disjoint) sets. Let r be a point of the set R, and let s be a point of the set S. Consider the continuum $Q\subset X\times I$ defined as follows:

$$Q=(A\times\{0\})\cup(B\times\{1\})\cup(\{r\}\times I)\,.$$

The partial mapping $p|Q$ is not weakly confluent. In fact, let V be an open set in X such that $R\subset V\subset\overline{V}\subset X\setminus S$. Take the component C of the set $A\setminus V$ such that $s\in C$, and take the component D of the set $B\setminus V$ such that $s\in D$. Then

$$\emptyset\neq C\cap\overline{A\setminus(A\setminus V)}=C\cap\overline{A\cap V}\subset A\setminus B$$

and

$$\emptyset\neq D\cap\overline{B\setminus(B\setminus V)}=D\cap\overline{B\cap V}\subset B\setminus A$$

by Theorem 1 in [11], § 47, III, p. 172. We infer that there are points c and d such that $c\in C\setminus D$ and $d\in D\setminus C$. Consider the continuum $K=C\cup D$. The set $Q\cap p^{-1}(K)$ has exactly two components, namely $C\times\{0\}$ and $D\times\{1\}$. We have $p(C\times\{0\})=C\subset K\setminus\{d\}$ and $p(D\times\{1\})=D\subset K\setminus\{c\}$; thus $p|Q$ is not weakly confluent, i.e., p is not hereditarily weakly confluent, a contradiction.

(3.10) COROLLARY. *If any mapping of a continuum onto X is hereditarily weakly confluent, then X is hereditarily unicoherent and atriodic.*

One can ask the following

(3.11) QUESTION. *Let a continuum X be hereditarily unicoherent and atriodic. Does it follow that any mapping of a continuum onto X is hereditarily weakly confluent (weakly confluent)?*

I am not able to answer to this question but I have some partial results.

Recall that a continuum X is said to be *snake-like* (*arc-like*) if for each $\varepsilon>0$ there exists an arc I and a map f from X to I such that the diameter of $f^{-1}(t)$ is less than ε for each $t\in I$ (see [1]). It is easy to observe that any subcontinuum of a snake-like continuum is snake-like. Theorem 4 of [17], p. 237 says that any mapping of a continuum onto a snake-like continuum is weakly confluent. Therefore, we have

(3.12) PROPOSITION. *Any mapping of a continuum onto a snake-like continuum is hereditarily weakly confluent.*

Hereditarily weakly confluent mappings characterize hereditarily decomposable snake-like continua; namely

(3.13) COROLLARY. *Let a continuum X be hereditarily decomposable. The following conditions are equivalent:*

(i) *X is snake-like,*

(ii) *X is hereditarily unicoherent and atriodic,*

(iii) *any mapping of a continuum onto X is hereditarily weakly confluent,*

(iv) *any weakly confluent mapping of a continuum onto X is hereditarily weakly confluent,*

(v) *the projection mapping $p\colon X\times I\to X$ is hereditarily weakly confluent.*

Indeed, (i) and (ii) are equivalent by Theorem 2 of [16], p. 55 (cf. [1]). Conditions (i) and (ii) imply (iii) by Proposition (3.12). Obviously (iii) implies (iv). Since the projection mapping p is open, and thus weakly confluent, we infer that (iv) implies (v). Condition (v) implies (ii) by Theorem (3.9).

The problems considered above are associated with the following problem asked by A. Lelek: characterize continua which have the property that any mapping of a continuum onto X is weakly confluent (see [12], Problem 1). The class of those continua does not coincide with any class of continua considered here. We have the following

(3.14) EXAMPLE. There exists a hereditarily decomposable, atriodic and unicoherent continuum M such that any mapping of a continuum onto M is weakly confluent.

Let (r,φ) denote a point of the Euclidean plane having r and φ as its polar coordinates. Define a continuum M as follows:

$$S=\{(r,\varphi)\colon r=1,\ 0\leqslant\varphi\leqslant 2\pi\}\,,$$
$$M=S\cup\{(r,\varphi)\colon 1<r\leqslant 2,\ \varphi=1/(r-1)\}\,.$$

The continuum M is a spiral winding up the circle S. Obviously, M is hereditarily decomposable, atriodic and unicoherent. Let a mapping f map an arbitrary continuum N onto M and let N' be a minimal subcontinuum of N such that $f(N') = M$. Put $b = (2,1)$. Since $b \in M$, there is a point a such that $a \in N'$ and $f(a) = b$. Consider a sequence $Q_1, Q_2, \ldots$ of proper subcontinua of N' such that $\overline{\bigcup_{n=1}^{\infty} Q_n} = N'$ and $a \in Q_n \subset Q_{n+1}$ for each $n = 1, 2, \ldots$ (cf. [11], § 47, III, Theorem 5, p. 173). Since N' is a minimal subcontinuum of N such that $f(N') = M$, we have $f(Q_n) \subset M \setminus S$ for each $n = 1, 2, \ldots$ Since $\overline{\bigcup_{n=1}^{\infty} Q_n} = N'$ and $f(N') = M$, we infer that any subcontinuum K of $M \setminus S$ is contained in some $f(Q_n)$. Moreover, since $f(Q_n)$ is an arc, the mapping $f|Q_n$ is weakly confluent by Proposition (3.12). Therefore, for each subcontinuum K of $M \setminus S$ there is a continuum Q in N' such that $f(Q) = K$. Further, let R be an arbitrary subcontinuum of M. Then there is a sequence of subcontinua R_n of $M \setminus S$ such that $\operatorname{Lim}_{n\to\infty} R_n = R$. Therefore there are subcontinua C_n of N' such that $f(C_n) = R_n$. We choose a convergent subsequence $\{C_{n_m}\}$ of the sequence $\{C_n\}$ (cf. [11], § 42, I, Theorem 1, p. 45 and § 42, II). Put $C = \operatorname{Lim}_{m\to\infty} C_{n_m}$. The set C is a subcontinuum of N' (cf. [11], § 47, II, Theorem 4, p. 170) and

$$f(C) = \operatorname{Lim}_{m\to\infty} f(C_{n_m}) = \operatorname{Lim}_{m\to\infty} R_{n_m} = \operatorname{Lim}_{n\to\infty} R_n = R$$

by the continuity of f. We conclude that f is weakly confluent, and thus any mapping of a continuum onto M is weakly confluent.

(3.15) Example. There is a continuum M having the following properties: M is irreducible; M is hereditarily decomposable and hereditarily unicoherent (i.e., M is a *λ-dendroid*; cf. [5], Theorem 1, p. 16); M is not atriodic; any mapping of a continuum onto M is weakly confluent.

Let (x, y, z) denote a point of the Euclidean 3-space having x, y and z as its rectangular coordinates. Let $\{r_n\}$ be a sequence of rationals of the closed unit interval $[0, 1]$. Let A_n denote the union of the straight line intervals joining consecutively point $(0, 1, 1/n)$, $(0, 1/n, 1/n)$, $(r_n, 0, 1/n)$, $(0, -1/n, 1/n)$ and $(0, -1, 1/n)$; and let

$$B_n = \left\{(0, 1, z)\colon \frac{1}{2n} \leqslant z \leqslant \frac{1}{2n-1}\right\}$$

and

$$C_n = \left\{(0, -1, z)\colon \frac{1}{2n+1} \leqslant z \leqslant \frac{1}{2n}\right\}.$$

We define:

$$T = \{(0, y, 0)\colon -1 \leqslant y \leqslant 1\} \cup \{(x, 0, 0)\colon 0 \leqslant x \leqslant 1\},$$

$$M = T \cup \bigcup_{n=1}^{\infty} (A_n \cup B_n \cup C_n).$$

The continuum M is a simple triod which is approximated in a special way by the broken line. Obviously, the continuum M is irreducible, hereditarily decomposable, hereditarily unicoherent and not atriodic. In the same way as in Example (3.14) we infer that if the mapping f maps the continuum N onto M, then

(1) for each continuum K contained in $M \setminus T$ there is a continuum Q in N such that $f(Q) = K$.

Let R be an arbitrary subcontinuum of M that is not contained in $M \setminus T$. Consider two cases.

1'. $R \cap (M \setminus T) \neq \emptyset$. Then $T \subset R$ and there is a sequence $\{R_n\}$ of continua such that $R_n \subset M \setminus T$ and $\operatorname{Lim}_{n\to\infty} R_n = R$ (for example $R_n = R \cap \{(x, y, z)\colon z \geqslant 1/n\}$). By (1), there are continua C_n in N such that $f(C_n) = R_n$. Take, as in Example (3.14), a convergent subsequence $\{C_{n_m}\}$ of the sequence $\{C_n\}$ and put $C = \operatorname{Lim}_{m\to\infty} C_{n_m}$. Then $f(C) = R$ by the continuity of f.

2'. $R \subset T$. If $R \cap \{(x, 0, 0)\colon 0 \leqslant x \leqslant 1\} = \emptyset$, then we take

$$R_m = \{(x, y, z)\colon (x, y, 0) \in R,\ z = 1/n\};$$

and if $R \cap \{(x, 0, 0)\colon 0 \leqslant x \leqslant 1\} \neq \emptyset$, then we take a maximal number r_0 such that the point $(r_0, 0, 0)$ belongs to R and we take a convergent subsequence $\{r_{n_m}\}$ of the sequence $\{r_n\}$ such that $\lim_{m\to\infty} r_{n_m} = r_0$ and then we put

$$R_m = A_{n_m} \cap \big(\{(x, y, z)\colon (0, y, 0) \in R\} \cup \{(x, y, z)\colon \text{there is an } x_0 \text{ such that } x_0 \leqslant x \text{ and } (x_0, 0, 0) \in R\}\big).$$

In both cases the sequence $\{R_m\}$ is convergent to R and $R_m \subset M \setminus T$ for each $n = 1, 2, \ldots$ By (1) we conclude as before that there is a continuum C in N such that $f(C) = R$.

Therefore f is weakly confluent, and thus any mapping of a continuum onto M is weakly confluent.

Remarks. Recall that an irreducible continuum X is said to be of *type λ* (see [11], § 48, III, p. 197, the footnote) if there is a monotone mapping φ of X onto the closed unit interval $[0, 1]$ such that for each $t \in [0, 1]$ the set $\varphi^{-1}(t)$ has a void interior. Any set P of the form $P = \bigcup_{u<t<v} \varphi^{-1}(t)$ is called a *portion* of X. Z. Waraszkiewicz in [19], Theoreme 1, p. 182 proved the following

(3.16) Proposition. *If a mapping f maps a continuum M onto an irreducible continuum X of type λ, then for each portion P of X there is a continuum K in M such that $f(K) = \overline{P}$.*

Denote by $\mathscr{K}$ the collection of all irreducible continua X of type λ such that any subcontinuum X' of X is the topological limit of some sequence of portions of X. By Proposition (3.16) we conclude that any continuum X of the collection $\mathscr{K}$ has the property that any mapping of a continuum onto X is weakly confluent.

In particular, the continua described in Examples (3.14) and (3.15) belong to the collection $\mathscr{K}$.

By similar arguments to those used in the proofs of Theorems (3.7) and (3.9) it is easy to obtain the following

(3.17) PROPOSITION. *If any mapping of a continuum onto a continuum X is weakly confluent, then X is unicoherent and X is not a triod.*

Every unicoherent continuum which is not a triod is an irreducible continuum (see [18], Theorem 3.2, p. 456). We have

(3.18) QUESTION. *Does it follow that if an irreducible Suslinian (with no uncountable collection of nondegenerate subcontinua) continuum X has the property that any mapping of a continuum onto X is weakly confluent, then $X \in \mathscr{K}$?*

(3.19) QUESTION. *Is any snake-like Suslinian continuum in the class $\mathscr{K}$?* (Any snake-like continuum is irreducible, because it is unicoherent and atriodic).

Note also the following

(3.20) COROLLARY. *Let X be a locally connected continuum. The following conditions are equivalent:*

(i) *X is an arc,*
(ii) *X is a snake-like continuum,*
(iii) *any mapping of a continuum onto X is hereditarily weakly confluent,*
(iv) *any weakly confluent mapping of a continuum onto X is hereditarily weakly confluent,*
(v) *the projection mapping $p\colon X\times I \to X$ is hereditarily weakly confluent,*
(vi) *any mapping of a continuum onto X is weakly confluent,*
(vii) *any mapping of X onto itself is weakly confluent.*

In fact, obviously (i) implies (ii); (ii) implies (iii) by Proposition (3.12). It is obvious that (iii) implies (iv) and (vi), and (iv) implies (v). Condition (v) implies that X is hereditarily unicoherent and atriodic by Theorem (3.9). Since X is locally connected, it is easy to see that X is an arc (cf. [11], § 51, V, p. 291), i.e., (v) implies (i). Obviously (vi) implies (vii), and condition (vii) implies (i) by Corollary 3.5 in [15].

§ 4. Hereditarily confluent mappings on some special spaces.

If M is a continuum, an *essential sum decomposition* of M (see [9], p. 221) is a finite collection $\mathscr{D}$ of subcontinua of M such that

(i) $M = \bigcup \{D\colon D \in \mathscr{D}\}$,
(ii) if $D \in \mathscr{D}$, then D contains a point not in the union of the other members of $\mathscr{D}$.

A mapping f of a continuum X onto Y is called *irreducible* provided no proper subcontinuum of X is mapped onto the whole Y under f (see [21], p. 162).

We first prove the following

(4.1) THEOREM. *If an irreducible hereditarily confluent mapping f maps a continuum X onto Y, then for each essential sum decomposition $\mathscr{D}$ of Y and for each D of $\mathscr{D}$ the set $f^{-1}(D)$ is connected.*

Proof. We claim that

(1) if A and B are proper subcontinua of Y such that $A \cup B = Y$, then sets $f^{-1}(A)$ and $f^{-1}(B)$ are connected.

Indeed, sets $A \cap B$, $A \setminus B$ and $B \setminus A$ are nonempty, and thus there are points a, b and c such that $a \in A\setminus B$, $b \in B \setminus A$ and $c \in A \cap B$. Let c' be an arbitrary point of the set $f^{-1}(c)$. Therefore the components A' and B' of the sets $f^{-1}(A)$ and $f^{-1}(B)$, respectively, such that $c' \in A' \cap B'$ have the property $f(A' \cup B') = f(A') \cup f(B') = Y$ by the confluence of f. This implies that $A' \cup B' = X$ by the minimality of X with respect to the property $f(X) = Y$. It suffices to prove that $f^{-1}(A) = A'$ and $f^{-1}(B) = B'$. Let C be an arbitrary component of the set $f^{-1}(A)$. Since f is confluent, we have $f(C) = A$; thus $f^{-1}(a) \cap C \neq \emptyset$. But $f^{-1}(a) = f^{-1}(a) \cap A'$, because $X = A' \cup B'$ and $f(B') = B \subset Y \setminus \{a\}$. We infer $A' \cap C \neq \emptyset$. Since A' and C are components of $f^{-1}(A)$ and $A' \cap C \neq \emptyset$, we have $A' = C$, i.e., $f^{-1}(A) = A'$. In the same way we obtain $f^{-1}(B) = B'$.

Now let $\mathscr{D}$ be an essential sum decomposition of Y and let D_0 be an element of $\mathscr{D}$. If the set $E_0 = \bigcup \{D\colon D \in \mathscr{D}$ and $D \neq D_0\}$ is connected, then $Y = D_0 \cup E_0$ and D_0 and E_0 are proper subcontinua of Y. Thus, by (1), we infer that $f^{-1}(D_0)$ is connected. Assume that E_0 is not connected. Then E_0 has a finite number of components, say $E_1, E_2, \ldots, E_n$, by the finiteness of $\mathscr{D}$. We have $E_i \cap E_j = \emptyset$ and $D_0 \cap E_i \neq \emptyset$ for $i \neq j$ and $i, j = 1, 2, \ldots, n$. Put $A = E_1 \cup D_0$ and $B = D_0 \cup \bigcup_{i=2}^{n} E_i$. It follows from (1) that the sets $f^{-1}(A)$ and $f^{-1}(B)$ are connected. Let K be a minimal subcontinuum in $f^{-1}(A)$ such that $f(K) = A$. Applying (1) to the confluent mapping $f|K$, we infer that the set $K \cap f^{-1}(D_0)$ is connected, because E_1 and D_0 are proper subcontinua of A and $E_1 \cup D_0 = A$. Moreover, the set $K \cap \cap f^{-1}(B)$ is nonempty, because $f^{-1}(D_0) \subset f^{-1}(B)$ and $K \cap f^{-1}(D_0) \neq \emptyset$. Thus $K \cup f^{-1}(B) = X$ by the irreducibility of f. Since $f^{-1}(A) = K \cup f^{-1}(D_0)$ and the sets $f^{-1}(A)$ and $K \cap f^{-1}(D_0)$ are continua, we infer that $f^{-1}(D_0)$ is a continuum by Theorem 2 in [11], § 47, I, p. 168. The proof of Theorem (4.1) is complete.

(4.2) COROLLARY. *If a hereditarily confluent mapping f maps a hereditarily unicoherent continuum X onto Y, then for each subcontinuum Q of Y there is a continuum K in X such that $f(K) = Q$ and for any subcontinuum R which has the nonempty interior in Q, we find that the set $f^{-1}(Q) \cap K$ is connected.*

Proof. Let Q be an arbitrary subcontinuum of Y. Since f is hereditarily confluent, there is a subcontinuum K in X such that $f|K$ is an irreducible hereditarily confluent mapping from K onto Q. Let R be an arbitrary subcontinuum of Q such that the interior of R is nonempty. If the set $Q \setminus R$ is connected, then the closure of $Q \setminus R$ is a continuum (see [11], § 46, II, Corollary 3, p. 132) and if $Q\setminus R \neq \emptyset$, then $\mathscr{D} = \{R, \overline{Q\setminus R}\}$ is an essential sum decomposition of Q. Therefore the set $(f|K)^{-1}(R) = K \cap f^{-1}(R)$ is connected by Theorem (4.1). If the set $Q \setminus R$ is not connected, then $Q \setminus R = M \cup N$, where M and N are separated sets. This implies that the sets $R\cup M$ and $R\cup N$ are continua (see [11], § 47, I, Theorem 3, p. 168). Then $\mathscr{D}=\{R\cup M,\ R\cup N\}$ is an essential sum decomposition of Q. Therefore the sets $K\cap f^{-1}(R\cup M)$ and $K\cap f^{-1}(R\cup N)$ are connected by Theorem (4.1); and since

$$K\cap f^{-1}(R)=\big(K\cap f^{-1}(R\cup M)\big)\cap\big(K\cap f^{-1}(R\cup N)\big)$$

and X is hereditarily unicoherent, we conclude that the set $K\cap f^{-1}(R)$ is connected.

Now we prove

(4.3) LEMMA. *Let a continuum Y be locally connected and let y be an arbitrary point of Y. Then there is a sequence $\{Q_n\}$ of subcontinua of Y such that*

(i) *$Q_{n+1}\subset Q_n$ for each $n=1,2,\ldots$,*

(ii) *$\bigcap_{n=1}^{\infty} Q_n=\{y\}$,*

(iii) *any Q_n is an element of some essential sum decomposition $\mathscr{D}_n$ of Y.*

Proof. We shall proceed by induction. If $n=1$, then we take $Q_1=Y$. Assume that $Q_1,\ldots,Q_s$ are subcontinua of Y such that $y\in \operatorname{Int}Q_s\subset Q_s\subset Q_{s-1}\ldots\subset Q_1$, $\operatorname{diam}Q_i\leqslant\frac{2}{3}\operatorname{diam}Q_{i-1}$ for $i=2,3,\ldots$ and for $i=1,2,\ldots,s$ the continuum Q_i is an element of some essential sum decomposition $\mathscr{D}_i$ of Y. Since $y\in\operatorname{Int}Q_s$, we can assume that the set $\{x:\varrho(x,y)\leqslant\varepsilon\}$ is contained in Q_s for some positive number ε, where ϱ denotes the metric in Y. Since Y is locally connected, we infer by Theorem 2 of [11], § 50, II, p. 256 that there are continua $K_1,\ldots,K_m$ of diameter less than $\frac{1}{3}\varepsilon$ and such that $Y=\bigcup_{i=1}^{m}K_i$. We can assume that $\mathscr{D}'_{s+1}=\{K_1,\ldots,K_m\}$ is an essential sum decomposition of Y. Put $Q_{s+1}=\bigcup\{K\colon y\in K \text{ and } K\in\mathscr{D}'_{s+1}\}$. Then $y\in\operatorname{Int}Q_{s+1}\subset Q_{s+1}\subset Q_s$ and $\operatorname{diam}Q_{s+1}\leqslant\frac{2}{3}\operatorname{diam}Q_s$. Moreover, Q_{s+1} is an element of an essential sum decomposition $\mathscr{D}_{s+1}$ defined as follows:

$$\mathscr{D}_{s+1}=\{K\colon K=Q_{s+1}\ \text{or}\ K\in\mathscr{D}'_{s+1}\ \text{and}\ y\in Y\setminus K\}.$$

Therefore the sequence $\{Q_n\}$ satisfies (i) and (iii) and $\operatorname{diam}Q_{i+1}\leqslant\frac{2}{3}\operatorname{diam}Q_i$ for each $i=1,2,\ldots$ by the construction, i.e., it also satisfies (ii).

(4.4) THEOREM. *If a hereditarily confluent mapping f maps a continuum X onto Y and Y' is a locally connected subcontinuum of Y, then there is a subcontinuum X' of X such that $f(X')=Y'$ and the partial mapping is monotone.*

Proof. Since f is hereditarily confluent, we infer that there is a subcontinuum X' of X such that $f(X')=Y'$ and the partial mapping $f|X'$ is an irreducible hereditarily confluent mapping from X' onto Y'. It suffices to prove that the set $f^{-1}(y)\cap X'$ is connected for each $y\in Y'$. Let y be an arbitrary point of Y'. It follows from Lemma (4.3) that there is a sequence $\{Q_n\}$ of subcontinua of Y' such that (i) $Q_{n+1}\subset Q_n$ for each $n=1,2,\ldots$, (ii) $\bigcap_{n=1}^{\infty}Q_n=\{y\}$, and (iii) any Q_n is an element of some essential sum decomposition $\mathscr{D}_n$ of Y'. By Theorem (4.1) and (iii) we infer that the sets $f^{-1}(Q_n)\cap X'$ are continua for $n=1,2,\ldots$ Thus by (i) and (ii) and by Theorem 5 in [11], § 47, II, p. 170 we conclude that the set $f^{-1}(y)\cap X'$ is a continuum. This completes the proof.

Furthermore,

(4.5) THEOREM. *Any hereditarily confluent mapping of an arcwise connected continuum is monotone.*

Proof. Let a hereditarily confluent mapping f map an arcwise connected continuum X onto Y, and let p be an arbitrary point of Y. Suppose, on the contrary, that the set $f^{-1}(p)$ is not connected. Take two components C_1 and C_2 of the set $f^{-1}(p)$. Since X is arcwise connected and $C_1\cap C_2=\emptyset$, we infer that there is an arc A in X such that A is irreducible between C_1 and C_2. Since the image of the end-points of A under f is the point p, and $f(A)\setminus\{p\}\neq\emptyset$, we can choose a non-degenerate subarc ab of A having a and b as its end-points and such that $f^{-1}(p)\cap ab=\{a,b\}$. The mapping f is hereditarily confluent, and thus $f|ab$ is hereditarily confluent (cf. Proposition (2.4)). Therefore the set $f(ab)$ is an arc (cf. [4], Corollary 20, p. 32). Denote the end-points of $f(ab)$ by c and d. Obviously we have $c\neq d$. Consider two cases.

1'. $c=p$ (if $d=p$, the proof is quite similar). Let a point d' be the first one in the arc ab which goes into d under f $(a<d'<b)$. Let e be an arbitrary point of $f(ab)\setminus\{d,p\}$ and let a point e' be the first one in the arc from d' to b in ab which goes into e under f $(a<d'<e'<b)$. Take the arc ae' in ab. Then $f(ae')=f(ab)$, and the one-point set $\{e'\}$ is a component of the set $f^{-1}(\overline{f(ab)\setminus f(d'e')})\cap ae'$, where $d'e'$ is an arc in ab which has d' and e' as its endpoints. Thus, since

$$\{f(e')\}\subset\overline{f(ab)\setminus f(d'e')}\neq\{f(e')\}$$

and the set $\overline{f(ab)\setminus f(d'e')}$ is a continuum, the partial mapping $f|ae'$ is not confluent, a contradiction.

2'. $c\neq p$ and $d\neq p$. Let a point c' be the first one in the arc ab which goes into c or d under f $(a<c'<b)$. Then $f(c')\subset\{c,d\}$ and we can assume that $f(c')=c$ (if $f(c')=d$, the proof is quite similar). This implies that the one-point set $\{a\}$ is a component of the set $f^{-1}(\overline{f(ab)\setminus f(ac')})\cap ab$, where ac' is an arc in ab which has a and c' as its end-points. Since the set $\overline{f(ab)\setminus f(ac')}$ is a continuum in $f(ab)$ and $\{f(a)\}\subset\overline{f(ab)\setminus f(ac')}\neq\{f(a)\}$, we conclude that the partial mapping $f|ab$ is not confluent, a contradiction. The proof of Theorem (4.5) is complete.

(4.6) COROLLARY. *Any hereditarily confluent mapping of a hereditarily arcwise connected continuum is hereditarily monotone.*

It is easy to observe that there are hereditarily confluent mappings which are not monotone. We have

(4.7) QUESTION. *For which class of continua is any hereditarily confluent mapping monotone?*

In particular,

(4.8) QUESTION. *Is any hereditarily confluent mapping onto arcwise connected continua monotone?*

§ 5. **Invariance properties.** Recall that a continuum X is called *discoherent* provided for each closed sets A and B such that $X = A \cup B$ and $A \neq X \neq B$, the set $A \cap B$ is not connected. We have (see [11], § 46, X, Theorem 1, p. 163) the following

(5.1) PROPOSITION. *A continuum X is discoherent if and only if for each continuum $K \subset X$ the set $X \setminus K$ is connected.*

Obviously any indecomposable continuum is discoherent. We will prove that a continuum which is not hereditarily unicoherent contains a continuum which is discoherent and decomposable (cf. [11], § 56, VII, Theorem 1, p. 421 and VIII, Theorem 3, p. 425). We have

(5.2) LEMMA. *Let a continuum X be irreducible between a point a and each point of a set A. If for each component C of a closed set M we have $C \cap A \neq \emptyset$, then the set $X \setminus M$ is connected.*

Proof. If the point a belongs to the set M, then the component C of M such that $a \in C$, is equal to X by the irreducibility of X and by $C \cap A \neq \emptyset$. Then $X \setminus M = \emptyset$, i.e., the set $X \setminus M$ is connected. Suppose $a \in X \setminus M$. Let L be a component of the set $X \setminus M$ such that $a \in L$. Then (see [11], § 47, III, Theorem 2, p. 172) $\bar{L} \cap M \neq \emptyset$. Thus, there exists a component C of the set M such that $\bar{L} \cap C \neq \emptyset$. Therefore the set $\bar{L} \cup C$ is a continuum, and $a \in \bar{L} \cup C$ and $A \cap (\bar{L} \cup C) \neq \emptyset$. By the irreducibility of X we have $\bar{L} \cup C = X$. But then

$$L \subset X \setminus M = (X \setminus M) \cap (\bar{L} \cup C) \subset (X \setminus M) \cap (\bar{L} \cup M) = (X \setminus M) \cap \bar{L} \subset \bar{L},$$

and since L is connected, we infer that the set $X \setminus M$ is connected (see [11], § 46, II, Corollary 3, p. 132).

(5.3) LEMMA. *Let a continuum X be irreducible between each point of a set A and each point of a set B. If for each component C of a closed set M we have $C \cap A \neq \emptyset$ and for each component C' of a closed set N we have $C' \cap B \neq \emptyset$, then the set $X \setminus (M \cup N)$ is connected.*

Proof. If $M \cap N \neq \emptyset$, then there are components C and C' of the sets M and N, respectively, such that $A \cap C \neq \emptyset$, $B \cap C' \neq \emptyset$ and $C \cap C' \neq \emptyset$. Then the set $C \cup C'$ is a continuum containing some point of A and some point of B. By the irreducibility of X, we have $C \cup C' = X$, i.e., $X \setminus (M \cup N) = \emptyset$, and thus the set $X \setminus (M \cup N)$ is connected. Assume that $M \cap N = \emptyset$. Then there is a component L of $X \setminus (M \cup N)$ such that $\bar{L} \cap M \neq \emptyset$ and $\bar{L} \cap N \neq \emptyset$.

Indeed, if L is a component of the set $X \setminus (M \cup N)$, then $\bar{L} \cap (M \cup N) \neq \emptyset$ (see [11], § 47, III, Theorem 2, p. 172). Suppose, on the contrary, that for each component L of the set $X \setminus (M \cup N)$ we have either $\bar{L} \cap M = \emptyset$ or $\bar{L} \cap N \neq \emptyset$. Put $R = \bigcup \{L\colon L$ is a component of $X \setminus (M \cup N)$ and $\bar{L} \cap M \neq \emptyset\}$, $S = \bigcup \{L\colon L$ is a component of $X \setminus (M \cup N)$ and $\bar{L} \cap N \neq \emptyset\}$. Then the sets $R \cup M$ and $S \cup N$ are separated and $X = (R \cup M) \cup (S \cup N)$, contrary to the connectedness of X.

Let L be a component of $X \setminus (M \cup N)$ such that $\bar{L} \cap M \neq \emptyset$ and $\bar{L} \cap N \neq \emptyset$. Take components C and C' of the sets M and B, respectively, such that $C \cap \bar{L} \neq \emptyset$ and $C' \cap \bar{L} \neq \emptyset$. By the irreducibility of X we obtain the equality $X = C \cup \bar{L} \cup C'$. Therefore

$$L \subset X \setminus (M \cup N) = (X \setminus (M \cup N)) \cap (C \cup \bar{L} \cup C') = (X \setminus (M \cup N)) \cap \bar{L} \subset \bar{L},$$

and since the set L is connected, we conclude that the set $X \setminus (M \cup N)$ is connected (see [11], § 46, II, Corollary 3, p. 132).

Lemmas (5.2) and (5.3) are generalizations of Theorems 3 and 4 in [11], § 48, II, p. 193). We now prove

(5.4) THEOREM. *If a continuum Q is the union of two continua I and I' such that $I \cap I' = A \cup B$, where A and B are nonempty and separated, and the continua I and I' are both irreducible between each point of A and each point of B, then Q is decomposable and discoherent.*

Proof. Obviously the continuum Q is decomposable. By Proposition (5.1) it suffices to prove that for each continuum $K \subset Q$ the set $Q \setminus K$ is connected. Let K be an arbitrary subcontinuum of Q. Consider the sets:

$$M = \bigcup \{C\colon C \text{ is a component of } K \cap I \text{ and } C \cap A \neq \emptyset\},$$
$$N = \bigcup \{C\colon C \text{ is a component of } K \cap I \text{ and } C \cap B \neq \emptyset\},$$
$$M' = \bigcup \{C\colon C \text{ is a component of } K \cap I' \text{ and } C \cap A \neq \emptyset\},$$
$$N' = \bigcup \{C\colon C \text{ is a component of } K \cap I' \text{ and } C \cap B \neq \emptyset\}.$$

It is easy to verify that

(1) the sets M, M', N and N' are closed,

(2) $M = \emptyset$ ($N = \emptyset$) if and only if $M' = \emptyset$ ($N' = \emptyset$, respectively). Moreover,

(3) if C is a component of $K \cap I$ ($K \cap I'$) such that $C \cap A = \emptyset$ and $C \cap B = \emptyset$, then $C \cap I' = \emptyset$ ($C \cap I = \emptyset$, respectively).

Indeed, if $C \cap I' \neq \emptyset$, then $\emptyset \neq (C \cap I') \cap (C \cap I) = C \cap I' \cap I = C \cap (A \cup B) = (C \cap A) \cup (C \cap B)$ by the assumptions and by the connectedness of C, because $C = (C \cap I') \cup (C \cap I)$. Therefore, either $C \cap A$ or $C \cap B$ is nonempty, a contradiction.

(4) if either M or N is nonempty and C is a component of $K \cap I$ (or $K \cap I'$), then either $C \cap A \neq \emptyset$ or $C \cap B \neq \emptyset$.

In fact, let C be a component of $K \cap I$ such that $C \cap A = \emptyset$ and $C \cap B = \emptyset$. By (3) we have $C \cap I' = \emptyset$. Therefore C is a component of the set $K \setminus I'$ because $C \subset K \setminus I' \subset K \cap I$. Since either $M \neq \emptyset$ or $N \neq \emptyset$, we conclude that the set $K \cup I'$ is a continuum. Furthermore, C is a component of the set $K \setminus I' = (K \cup I') \setminus I'$, and by Theorem 2 in [11], § 47, III, p. 172 we obtain $\emptyset \neq \bar{C} \cap \overline{(K \cup I') \setminus (K \setminus I')} = C \cap I'$, a contradiction.

Consider three cases.

1'. $M = N = \emptyset$. Then, since $K = (K \cap I) \cup (K \cap I')$, either $K \cap I$ or $K \cap I'$ is empty. Suppose $K \cap I = \emptyset$ (if $K \cap I' = \emptyset$; then the proof is analogous). Then

$Q\setminus K = (I\setminus K) \cup (I'\setminus K) = I \cup (I'\setminus K)$ and $A \cup B \subset I'\setminus K$. It follows from Theorem 3 in [11], § 48, II, p. 193, that the set $I'\setminus K$ is a union of two connected sets, C_1 and C_2, such that $C_1 \cap A \neq \emptyset$ and $C_2 \cap B \neq \emptyset$. Since $A \cup B \subset I$, we infer that $Q\setminus K = I \cup C_1 \cup C_2$ is connected.

2'. $M \neq \emptyset$ and $N = \emptyset$ (if $M = \emptyset$ and $N \neq \emptyset$, then the proof is analogous). By (3) and (4) we obtain $K = M \cup M'$. Since $(M \cup M') \cap I = M$ and $(M \cup M') \cap I' = M'$, we have $Q\setminus K = (I\setminus M) \cup (I'\setminus M')$. The sets $I\setminus M$ and $I'\setminus M'$ are connected by Lemma (5.2). Moreover, $B \subset (I\setminus M) \cap (I'\setminus M')$, and thus the set $Q\setminus K$ is connected.

3'. $M \neq \emptyset$ and $N \neq \emptyset$. Then, by (3) and (4), the equality $K = M \cup M' \cup N \cup N'$ holds. Therefore $(M \cup M') \cap (N \cup N') \neq \emptyset$, because K is connected. Thus one of the intersections $M \cap N$, $M' \cap N'$, $M \cap N'$ and $M' \cap N$ is nonempty. If $M \cap N \neq \emptyset$ (analogous by if $M' \cap N' \neq \emptyset$), then for some components C_1 and C_2 of M and N, respectively, we have $C_1 \cap C_2 \neq \emptyset$; thus $C_1 \cup C_2$ is a continuum in I having nonempty intersections with the sets A and B, i.e., $C_1 \cup C_2 = I$ by the irreducibility of I, hence $I \subset K$. If $M \cap N' \neq \emptyset$ (if $M' \cap N \neq \emptyset$, analogously), then, since $M \cap N' \subset A \cap B$, we have either $M \cap N' \cap A \neq \emptyset$ or $M \cap N' \cap B \neq \emptyset$. If $M \cap N' \cap A \neq \emptyset$ (if $M \cap N' \cap B \neq \emptyset$, similarly), then there is a component C of N' such that $C \cap A \neq \emptyset$. Moreover, by the definition of N', $C \cap B \neq \emptyset$. By the irreducibility of I' we infer that the inclusion $I' \subset K$ holds.

Therefore either $I \subset K$ or $I' \subset K$ in any case. Suppose $I \subset K$ (if $I' \subset K$, the proof is quite similar). Then

$$Q\setminus K = I'\setminus((K \cap I') \cup A \cup B) = I'\setminus((M' \cup A) \cup (N' \cup B)).$$

The set $I'\setminus((M' \cup A) \cup (N' \cup B))$ is connected by Lemma (5.3) if we substitute I' for X, $M' \cup A$ for M and $N' \cup B$ for N. Thus set $Q\setminus K$ is connected. The proof of Theorem (5.4) is complete.

(5.5) Corollary. *If a continuum is not hereditarily unicoherent, then it contains a decomposable discoherent continuum.*

Indeed, suppose that a continuum X is not hereditarily unicoherent. Then there are two continua E and F in X such that $E \cap F = P \cup R$ and the sets P and R are closed, disjoint and nonempty. Let I be an irreducible continuum between the sets P and R in E, and let I' be an irreducible continuum between the sets $I \cap P$ and $I \cap R$ in F. Put $A = I \cap I' \cap P$, $B = I \cap I' \cap R$ and $Q = I \cup I'$. By Theorem 2 in [11], § 48, IX, p. 222 we conclude that the continua I and I' are irreducible between each point of A and each point of B. Thus, Theorem (5.4) implies the conclusion of the corollary.

We now prove

(5.6) Theorem. *A hereditarily weakly confluent image of a hereditarily decomposable, hereditarily unicoherent continuum is hereditarily unicoherent.*

Proof. Let a hereditarily weakly confluent mapping f map a hereditarily decomposable, hereditarily unicoherent continuum X onto a continuum Y. Suppose, on the contrary, that the continuum Y is not hereditarily unicoherent. It follows from Corollary (5.5) that Y contains a decomposable, discoherent continuum Q. By the weak confluence of f there is a subcontinuum K of X such that $f(K) = Q$ and K is minimal with respect to this property. Since the continuum X is hereditarily decomposable, there are proper subcontinua A and B of K such that $K = A \cup B$. Then the sets $f(A)$ and $f(B)$ are proper subcontinua of Q with $Q = f(A) \cup f(B)$. Since the continuum Q is discoherent, we infer that $f(A) \cap f(B) = P \cup R$ where the sets P and R are closed, disjoint and nonempty. The continuum X is hereditarily unicoherent, hence the set $A \cap B$ is a continuum, and thus the set $f(A \cap B)$ is a continuum. Therefore we have either $f(A \cap B) \subset P$ or $f(A \cap B) \subset R$. Assume $f(A \cap B) \subset R$. By the normality of Y there is an open set U such that $R \subset U \subset \overline{U} \subset Y\setminus P$. Let $p \in P$. Take a component L_1 of the set $f(A)\setminus U$ such that $p \in L_1$ and take a component L_2 of the set $f(B)\setminus U$ such that $p \in L_2$. According to Theorem 2 in [11], § 47, III, p. 172 we have $L_1 \cap \overline{U} \neq \emptyset$ and $L_2 \cap \overline{U} \neq \emptyset$. Let $a \in L_1 \cap \overline{U}$ and $b \in L_2 \cap \overline{U}$. Since $\overline{U} \subset Y\setminus P$ and $L_i \cap U = \emptyset$ for $1, 2$, we obtain $a \in f(A)\setminus f(B)$ and $b \in f(B)\setminus f(A)$. Consider the continuum $Q' = L_1 \cup L_2$. Since $A \cap B \subset K\setminus f^{-1}(Q')$ we infer that each component of the set $f^{-1}(Q') \cap K$ is contained either in A or in B. But $a \in f(A)\setminus Q'$, $b \in f(B)\setminus Q'$ and $\{a, b\} \subset Q'$; thus no component of the set $f^{-1}(Q') \cap K$ is mapped under f onto the whole Q'. Therefore the mapping $f|K$ is not weakly confluent, a contradiction.

(5.7) Theorem. *A hereditarily confluent image of a hereditarily unicoherent continuum is hereditarily unicoherent.*

Proof. Let a hereditarily confluent mapping f map a hereditarily unicoherent continuum X onto Y and let Q be an arbitrary subcontinuum of Y. It suffices to show that Q is unicoherent. Let A and B be proper subcontinua of Q such that $A \cup B = Q$ and let K be a minimal subcontinuum of X with respect to the property $f(K) = Q$ (such K exists by the confluence of f). Then the partial mapping $f|K$ is an irreducible hereditarily confluent mapping of K onto Q. By Theorem (4.1) we conclude that the sets $f^{-1}(A) \cap K$ and $f^{-1}(B) \cap K$ are continua. Therefore the set $f^{-1}(A) \cap f^{-1}(B) \cap K$ is a continuum by the hereditary unicoherence of X. Then the equalities

$$A \cap B = A \cap B \cap Q = (A \cap B) \cap f(K) = f(f^{-1}(A \cap B) \cap K)$$
$$= f(f^{-1}(A) \cap f^{-1}(B) \cap K)$$

imply that the set $A \cap B$ is a continuum, i.e., Q is unicoherent.

One can ask the following

(5.8) Question. *Is a hereditarily weakly confluent image of a hereditarily unicoherent continuum also hereditarily unicoherent?*

Recall that a hereditarily decomposable, hereditarily unicoherent continuum is called a *λ-dendroid* (see [5], Theorem 1, p. 16). We have

(5.9) Theorem. *If a mapping f maps a hereditarily decomposable continuum X onto a hereditarily unicoherent continuum Y, then Y is hereditarily decomposable.*

Proof. Let Q be an arbitrary subcontinuum of Y and let K be a minimal subcontinuum of X with respect to the property $Q\subset f(K)$. Since the continuum X is hereditarily decomposable, there are proper subcontinua A and B of K with $K = A\cup B$. The sets $f(A)\cap Q$ and $f(B)\cap Q$ are continua by the hereditary unicoherence of Y. Moreover,

$$(f(A)\cap Q)\cup(f(B)\cap Q) = Q \quad \text{and} \quad f(A)\cap Q\neq Q\neq f(B)\cap Q.$$

Thus the continuum Q is decomposable, i.e., the continuum Y is hereditarily decomposable.

Theorems (5.6) and (5.9) imply a partial answer to Question (5.8).

(5.10) COROLLARY. *A hereditarily weakly confluent image of a λ-dendroid is a λ-dendroid.*

Remarks. The last corollary may be obtain in an other way. Namely, we first prove that a hereditarily weakly confluent image of an arc is an arc (an elementary proof); secondly, in the same way as in [3] (cf. [14]), we prove that there is no hereditarily weakly confluent mapping of a λ-dendroid onto a circumference. One can prove that if a hereditarily decomposable continuum X is not hereditarily unicoherent, then there are a subcontinuum M of X and an irreducible monotone mapping φ of M onto a circumference S. The propositions mentioned above imply Corollary (5.10) in the same way as in the proof of Theorem XIV in [3], p. 217.

Recall that a *dendroid* is an arcwise connected and hereditarily unicoherent continuum (see [2], p. 239). Since every dendroid is hereditarily decomposable (see [2], (47), p. 239), and since arcwise connectedness is an invariant under an arbitrary continuous mapping (see [21], p. 39) by Corollary (5.10), we have the following

(5.11) COROLLARY. *Every hereditarily weakly confluent image of a dendroid is a dendroid.*

Moreover, since a *dendrite* is a locally connected dendroid, and since local connectedness is an invariant under an arbitrary continuous mapping, we conclude from Corollary (5.11) that

(5.12) COROLLARY. *Every hereditarily weakly confluent image of a dendrite is a dendrite.*

We will now study some invariant properties of atriodic continua.

Atriodic continua need not be hereditarily unicoherent (even unicoherent; for example: a circumference), but they are hereditarily bicoherent (see [21], p. 153, the definition of function $r(X)$); namely we have

(5.13) THEOREM. *The common part of each two subcontinua of an atriodic continuum is the union of two continua.*

Proof. Let A and B be arbitrary subcontinua of an atriodic continuum X. Suppose, on the contrary, that the set $A\cap B$ has more than two components. Then

(see [11], § 46, II, Theorem 6, p. 133) we have $A\cap B = C_1\cup C_2\cup C_3$ where C_1, C_2 and C_3 are closed, nonempty, mutually disjoint sets. By the normality of X there are open sets G_1, G_2 and G_3 such that $C_i\subset G_i$ and $\bar{G}_i\cap\bar{G}_j=\emptyset$ for $i\neq j$ and $i,j=1,2,3$. Let $c_i\in C_i$. Consider a component K_i of the set $G_i\cap B$ such that $c_i\in K_i$. Then (see [11], § 47, III, Theorem 2, p. 172)

$$\bar{K}_i\cap(B\setminus G_i)\neq\emptyset. \tag{1}$$

Put $L_i=\bar{K}_i\cup A$ for $i=1,2,3$. Since the sets $\bar{K}_i$ and A are continua and $c_i\in K_i\cap A$, we infer that the sets L_i are continua. Moreover, since $\bar{K}_i\cap\bar{K}_j\subset\bar{G}_i\cap\bar{G}_j=\emptyset$ for $i\neq j$, we conclude that

$$L_i\cap L_j=(\bar{K}_i\cup A)\cap(\bar{K}_j\cup A)=(\bar{K}_i\cap\bar{K}_j)\cup A=A \quad \text{for} \quad i\neq j. \tag{2}$$

This implies by

$$\bar{K}_i\cap(B\setminus G_i)\cap A\subset\bar{G}_i\cap\bigcup_{j\neq i}G_j=\emptyset$$

that $\bar{K}_i\cap(B\setminus G_i)\subset\bar{K}_i\setminus A=L_i\setminus A$. Therefore we conclude from (1) that the sets $L_i\setminus A$ are nonempty. Thus the set $T=L_1\cup L_2\cup L_3$ is a triod, a contradiction.

(5.14) LEMMA. *Let L_1, L_2 and L_3 be subcontinua of an atriodic continuum X such that $L_1\cap L_2\neq\emptyset$, $L_2\cap L_3\neq\emptyset$, $L_1\cap L_3=\emptyset$ and $L_1\setminus L_2\neq\emptyset$. If K_1 and K_2 are continua such that $K_1\cup K_2\subset L_1\cup L_2\cup L_3$ and $L_1\cup L_2\subset K_1\cap K_2$, then we have either $K_1\subset K_2$ or $K_2\subset K_1$.*

Proof. We prove that

(1) if R is a continuum such that $R\subset L_1\cup L_2\cup L_3$ and $L_2\subset R$, then the set $R\cap(L_2\cup L_3)$ is a continuum.

In fact, since $R=(R\cap L_1)\cup L_2\cup(R\cap L_3)$ and $L_1\cap L_3=\emptyset$, both components of the set $R\cap L_3$ (cf. Theorem (5.13)) intersect the set L_2. Therefore the set $L_2\cup(R\cap L_3)=R\cap(L_2\cup L_3)$ is a continuum.

Let K_1 and K_2 be subcontinua of X such that $K_1\cup K_2\subset L_1\cup L_2\cup L_3$ and $L_1\cup L_2\subset K_1\cap K_2$. Consider two cases.

1′. The set $K_1\cap K_2$ is connected. Then put $A_1=K_1\cap K_2$, $A_2=(L_2\cup L_3)\cap K_1$ and $A_3=(L_2\cup L_3)\cap K_2$. The set A_1 is a continuum by the assumptions, and the sets A_2 and A_3 are continua by (1) (put K_1 and K_2 for R). Moreover, the set $A_0=A_i\cap A_j=K_1\cap K_2\cap(L_2\cup L_3)$ for $i\neq j$ is a continuum according to (1) if we put $K_1\cap K_2$ for R. Since the continuum X is atriodic and $\emptyset\neq L_1\setminus L_2\subset A_1\setminus A_0$, we conclude that either $A_2\setminus A_0$ or the set $A_3\setminus A_0$ is empty, because in the opposite case the set $A_1\cup A_2\cup A_3$ is a triod. If $A_2\setminus A_0=\emptyset$, then $K_1\setminus K_2=\emptyset$, i.e., $K_1\subset K_2$, because $K_1\setminus K_2\subset A_2\setminus A_0$. If $A_3\setminus A_0=\emptyset$, then $K_2\setminus K_1=\emptyset$, i.e., $K_2\subset K_1$, because $K_2\setminus K_1\subset A_3\setminus A_0$.

2′. The set $K_1\cap K_2$ is the union of two nonempty disjoint continua C and D (cf. Theorem (5.13)). Then either C or D contains the set $L_1\cup L_2$. Assume $L_1\cup L_2\subset C$. By the normality of X there is an open set U such that $D\subset U\subset\bar{U}\subset X\setminus C$.

Denote the component of the set $K_1\setminus U$ containing C by K_1', and denote the component of the set $K_2\setminus U$ containing C by K_2'. Obviously $C\subset K_1'\cap K_2'\subset(K_1\cap K_2)\setminus D = C$, i.e., $K_1'\cap K_2' = C$, and thus the set $K_1'\cap K_2'$ is connected. It follows from 1′ that either $K_1'\subset K_2'$ or $K_2'\subset K_1'$ holds. Theorem 2 in [11], § 47, III, p. 172 implies that $K_1'\cap \overline{U}\neq\emptyset$ and $K_2'\cap\overline{U}\neq\emptyset$. Therefore $\emptyset\neq K_1'\cap K_2'\cap\overline{U} = C\cap\overline{U}$, a contradiction, because $\overline{U}\subset X\setminus C$.

(5.15) Lemma. *Let continua A', B' and C' be irreducible between a continuum W and points a, b and c, respectively, and let $W = (W\cup A')\cap(W\cup B') = (W\cup A')\cap(W\cup C') = (W\cup B')\cap(W\cup C')$. If a continuum $T = W\cup A'\cup B'\cup C'$ is hereditarily decomposable and if a continuum $S\subset T$ contains the points a, b and c, then there are continua A_1, B_1 and C_1 such that*

(i) $A_1\cap B_1\cap C_1\neq\emptyset$,

(ii) $A_1\cup B_1\cup C_1 = S$,

(iii) $a\in A_1\setminus(B_1\cup C_1)$, $b\in B_1\setminus(A_1\cup C_1)$ *and* $c\in C_1\setminus(A_1\cup B_1)$.

Proof. It follows from the assumptions that

(1) $$\{a, b, c\}\subset S\subset T.$$

Therefore $S = S\cap T = (S\cap W)\cup(S\cap A')\cup(S\cap B')\cup(S\cap C')$. Moreover

(2) $$S\cap W\neq\emptyset.$$

In fact, if $S\cap W = \emptyset$, then, since S is a continuum, the intersection of two sets of $S\cap A'$, $S\cap B'$ and $S\cap C'$ is nonempty. Assume $(S\cap A')\cap(S\cap B')\neq\emptyset$. Then $\emptyset\neq(S\cap A')\cap(S\cap B') = S\cap A'\cap B'\subset S\cap W$ by the inclusion $A'\cap B'\subset W$; a contradiction.

Let a continuum $D\subset S$ be irreducible between a and W (cf. (1) and (2)) and let P be a composant of a point a in D (for the definition of a composant see [11], § 48, VI, p. 208). Since the continuum D is irreducible between the point a and each point of the set $W\cap D$ (see [11], § 48, VIII, Theorem 2, p. 220), we have $P\cap W = \emptyset$; thus $P\subset S\setminus W\subset(A'\setminus W)\cup(B'\setminus W)\cup(C'\setminus W)$. Sets $A'\setminus W$, $B'\setminus W$ and $C'\setminus W$ are separated and $a\in P\cap(A'\setminus W)$, we infer that $P\subset A'$. Therefore $D = \overline{P}\subset A'$ (see [11], § 48, VI, Theorem 2, p. 209). Since the continuum A' is irreducible between the point a and the set W, we conclude that $D = A'$, and thus $A'\subset S$. Since the role of A', B' and C' is symmetric, we obtain the inclusion

(3) $$A'\cup B'\cup C'\subset S.$$

Let R be a component of the set $S\cap W$. By Theorem 1 in [11], § 47, III, p. 172 we have

(4) $$R\cap\overline{S\setminus W}\neq\emptyset.$$

It follows from (1) and (2) that $S\setminus W\subset A'\cup B'\cup C'$. Therefore, (4) implies that the set $(R\cap A')\cup(R\cap B')\cup(R\cap C')$ is nonempty, i.e., each component of the set $S\cap W$ has a nonempty intersection with either A', or B', or C'. We define $A'' = \bigcup\{R\colon R \text{ is a component of } S\cap W \text{ and } R\cap A'\neq\emptyset\}$ and similarly we define the sets B'' and C''. Then

(5) $$S\cap W = A''\cup B''\cup C''.$$

It is easy to check that the sets A'', B'' and C'' are closed. Therefore the sets $A'\cup A''$, $B'\cup B''$ and $C'\cup C''$ are continua. Moreover, conditions (1), (3) and (5) imply the equality

(6) $$S = (A'\cup A'')\cup(B'\cup B'')\cup(C'\cup C'').$$

Since the set S is a continuum, we see that there are two continua of this decomposition, both intersecting the third one. Assume

(7) $$(A'\cup A'')\cap(C'\cup C'')\neq\emptyset \quad\text{and}\quad (B'\cup B'')\cap(C'\cup C'')\neq\emptyset.$$

The continuum T is hereditarily decomposable, thus there are continua C_1 and C_2 with $C' = C_1\cup C_2$ and $C_1\neq C'\neq C_2$. The continuum C' is irreducible between the point c and each point of the set $C'\cap W$; we may assume

(8) $$C'\cap W\subset C_2\setminus C_1 \quad\text{and}\quad c\in C_1\setminus C_2.$$

Put $A_1 = (A'\cup A'')\cup(C_2\cup C'')$ and $B_1 = (B'\cup B'')\cup(C_2\cup C'')$. Since

$$(A'\cup A'')\cap(C_2\cup C'') = (A'\cup A'')\cap(C'\cup C'')$$

and

$$(B'\cup B'')\cap(C_2\cup C'') = (B'\cup B'')\cap(C'\cup C'')$$

and since sets the $A'\cup A''$, $B'\cup B''$ and $C_2\cup C''$ are continua, we conclude that the sets A_1 and B_1 are continua by (7). Moreover, $C_1\cap C_2\neq\emptyset$ and $C_1\cap C_2\subset A_1\cap B_1\cap C_1$; thus

$$A_1\cap B_1\cap C_1\neq\emptyset.$$

Equality (6) implies the equality

$$S = A_1\cup B_1\cup C_1.$$

By (8) and by the definitions of A_1, B_1 and C_1, we conclude from the assumptions relative to the sets A', B' and C' that

$$a\in A_1\setminus(B_1\cup C_1),\quad b\in B_1\setminus(A_1\cup C_1)\quad\text{and}\quad c\in C_1\setminus(A_1\cup B_1).$$

Therefore, the continua A_1, B_1 and C_1 satisfy the required conditions.

We now prove

(5.16) Theorem. *Each hereditarily weakly confluent image of a hereditarily decomposable atriodic continuum is hereditarily decomposable and atriodic.*

Proof. Let a hereditarily weakly confluent mapping f map a hereditarily decomposable, atriodic continuum X onto Y. Then the continuum Y is hereditarily decomposable by the weak confluence of f (cf. [3], XIII, p. 217). Suppose, on the contrary, that the continuum Y is not atriodic. Then there continua A, B and C in Y such that $A\cap B\cap C = A\cap B = A\cap C = B\cap C\neq\emptyset$ and the set $A\cap$

$\cap B \cap C$ is a proper subcontinuum of each continuum A, B and C. Let $a \in A \setminus (B \cup C)$, $b \in B \setminus (A \cup C)$ and $c \in C \setminus (A \cup B)$. Take a continuum $A' \subset A$ irreducible between the point a and the set $W = A \cap B \cap C$, a continuum $B' \subset B$ irreducible between the point b and the set W, and a continuum $C' \subset C$ irreducible between the point c and the set W. The set $T = W \cup A' \cup B' \cup C'$ is a continuum; thus there is a subcontinuum Q of X such that $f(Q) = T$ by the weak confluence of f. Take a subcontinuum Q' of Q which is minimal with respect to the property $\{a, b, c\} \subset f(Q')$. Put $S = f(Q')$.

The continua T, W, A', B', C' and S satisfy the assumptions of Lemma (5.15); thus there are continua A_1, B_1 and C_1 such that (i), (ii) and (iii) of Lemma (5.15) hold.

Since the continuum X is hereditarily decomposable, we infer that there are continua L and L' contained in Q' such that $Q' = L \cup L'$ and $L \neq Q' \neq L'$. By the minimality of Q' with respect to the property $\{a, b, c\} \subset f(Q')$ we may assume that $f^{-1}(a) \cap L' = \emptyset$ and $f^{-1}(b) \cap L = \emptyset$. Thus

(1) $$\emptyset \neq f^{-1}(a) \cap Q' \subset L \setminus L' \quad \text{and} \quad \emptyset \neq f^{-1}(b) \cap Q' \subset L' \setminus L .$$

Since $f^{-1}(c) \cap Q' \neq \emptyset$, we may assume

(2) $$f^{-1}(c) \cap L' \neq \emptyset .$$

Take a continuum $L_0 \subset L$ irreducible between the set $f^{-1}(a)$ and the set L'. Since $\{a, b, c\} \subset f(L_0 \cup L')$, we conclude that $L_0 \cup L' = Q'$ by the minimality of Q' wi.h respect to the property $\{a, b, c\} \subset f(Q')$. Therefore the continuum L_0 is irreducible between each point of the set $f^{-1}(a) \cap Q'$ and each point of the set $L_0 \cup L'$. Moreover,

(3) if a subcontinuum K of Q' is such that $K \cap f^{-1}(a) \neq \emptyset$ and $K \cap L' \neq \emptyset$, then $L_0 \subset K$.

Since a continuum L_0 is decomposable and irreducible between the sets $f^{-1}(a)$ and L', we infer that there are continua L_1 and L_2 such that

(4) $$L_0 = L_1 \cup L_2 , \quad L_1 \setminus L_2 \neq \emptyset, \quad L_1 \cap L' = \emptyset \quad \text{and} \quad L_2 \cap L' \neq \emptyset .$$

Putting $L' = L_3$, we obtain

(4′) $$L_1 \cap L_2 \neq \emptyset , \quad L_2 \cap L_3 \neq \emptyset , \quad L_1 \cap L_3 = \emptyset \quad \text{and} \quad L_1 \setminus L_2 \neq \emptyset .$$

The sets $A_1 \cup B_1$ and $A_1 \cup C_1$ are continua which are contained in $f(Q') = S$ (cf. (i) and (ii) of Lemma (5.15)); thus, by the hereditary weak confluence of f, there are components K_1 and K_2 of sets $Q' \cap f^{-1}(A_1 \cup B_1)$ and $Q' \cap f^{-1}(A_1 \cup C_1)$, respectively, such that

(5) $$f(K_1) = A_1 \cup B_1 \quad \text{and} \quad f(K_2) = A_1 \cup C_1 .$$

Therefore $K_1 \cap f^{-1}(a) \neq \emptyset$ and $K_1 \cap f^{-1}(b) \neq \emptyset$. Hence, by (1), we obtain $K_1 \cap f^{-1}(a) \neq \emptyset$ and $K_1 \cap L' \neq \emptyset$. It follows from (3) that $L_0 \subset K_1$, thus by (4) we conclude that

(6) $$L_1 \cup L_2 \subset K_1 .$$

Conditions (5) and (6) imply that the set $f^{-1}(c) \cap L_0$ is empty. But $f^{-1}(c) \cap K_2 \neq \emptyset$ and $f^{-1}(a) \cap K_2 \neq \emptyset$; thus $K_2 \cap L' \neq \emptyset$ and $K_2 \cap f^{-1}(a) \neq \emptyset$. Thus, by (3), we infer that $L_0 \subset K_2$. Therefore condition (4) implies

(7) $$L_1 \cup L_2 \subset K_2 .$$

Conditions (4′), (6) and (7) imply that the continua L_1, L_2, L_3, K_1 and K_2 satisfy the assumptions of Lemma (5.14), thus we have either $K_1 \subset K_2$ or $K_2 \subset K_1$.

If $K_1 \subset K_2$, then $f(K_1) \subset f(K_2)$. Thus, by (5), we have $A_1 \cup B_1 \subset A_1 \cup C_1$. Therefore $B_1 \setminus (A_1 \cup C_1) \subset B_1 \setminus (A_1 \cup B_1) = \emptyset$, i.e., $B_1 \setminus (A_1 \cup C_1) = \emptyset$, contrary to (iii) of Lemma (5.15).

If $K_2 \subset K_1$, then $f(K_2) \subset f(K_1)$. Thus, by (5), we have $A_1 \cup C_1 \subset A_1 \cup B_1$. Therefore $C_1 \setminus (A_1 \cup B_1) \subset C_1 \setminus (A_1 \cup C_1) = \emptyset$, i.e., $C_1 \setminus (A_1 \cup B_1) = \emptyset$, contrary to (iii) of Lemma (5.15).

The proof of Theorem (5.15) is complete.

Corollary (5.9) and Theorem (5.16) imply

(5.17) COROLLARY. *Each hereditarily weakly confluent image of an atriodic λ-dendroid is an atriodic λ-dendroid.*

Theorem (5.16) is a partial solution of the following problem.

(5.18) QUESTION. *Is a hereditarily weakly confluent image of an atriodic continuum also atriodic?*

Applying Theorem (5.13), it is easy to obtain

(5.19) PROPOSITION. *Each confluent image of an atriodic continuum is atriodic.*

In fact, let f be confluent and let f map an atriodic continuum X onto Y. Suppose, on the contrary, that Y is not atriodic. Then there are continua A, B, and C in Y such that $A \cap B \cap C = A \cap B = A \cap C = B \cap C \neq \emptyset$ and the set $A \cap B \cap C$ is a proper subcontinuum of each of the sets A, B and C. Let $x' \in f^{-1}(A \cap B \cap C)$. Take the components A', B' and C' of $f^{-1}(A)$, $f^{-1}(B)$ and $f^{-1}(C)$, respectively, such that x' belongs to each of them. By the confluence of f we have

(1) $$f(A') = A, \quad f(B') = B \quad \text{and} \quad f(C') = C .$$

Theorem (5.13) implies that any of sets $A' \cap B'$, $A' \cap C'$ and $B' \cap C'$ has at most two components. Denote the component of $A' \cap B'$ containing x' by W_1, the component of $A' \cap C'$ containing x' by W_2 and the component of $B' \cap C'$ containing x' by W_3. Since

$$f(W_1) \cup f(W_2) \cup f(W_3) \subset \big((f(A') \cap f(B')) \cup (f(A') \cap f(C')) \cup (f(B') \cap f(C')\big)$$
$$= ((A \cap B) \cup (A \cap C) \cup (B \cap C)) = A \cap B \cap C$$

by (1), we conclude that $W_1 = W_2 = W_3$, because $A \cap B = A \cap C = B \cap C = A \cap B \cap C$ and $x' \in W_1 \cap W_2 \cap W_3$.

Take an open neighbourhood U of the set W_1 such that

$$((A' \cap B') \cup (A' \cap C') \cup (B' \cap C')) \cap U = W_1$$

(it is possible by Theorem (5.13)). Let A'', B'' and C'' be components of the sets $A' \cap \bar{U}$, $B' \cap \bar{U}$ and $C' \cap \bar{U}$, respectively, and such that $W_1 \subset A'' \cap B'' \cap C''$. Then

$$W_1 = A'' \cap B'' \cap C'' = A'' \cap B'' = A'' \cap C'' = B'' \cap C''.$$

Therefore, by (1), we infer that the set $A'' \cup B'' \cup C''$ is a triod, a contradiction.

We have

(5.20) EXAMPLE. There is a weakly confluent mapping of an atriodic λ-dendroid onto a simple triod.

Let (x, y) denote a point of Euclidean plane having x and y as its rectangular coordinates. Let C_0 be the Cantor set (cf. Example (2.8)) situated on the line $y = 0$ and let C_1 be the same set located on the line $y = 1$. Join each point of C_0 with the corresponding point of C_1 by a vertical segment and add the contiguous intervals to C_0 with lengths $1/3, 1/3^3, \ldots$ and add the contiguous intervals to C_1 with lengths $1/3^2, 1/3^4, \ldots$ (cf. [11], § 48, I, Example 4, p. 191). The continuum X obtained in this way is an atriodic λ-dendroid. Put $g(x, y) = (h(x), y)$ if $(x, 0) \in C_0$, where $h\colon C \to [0, 1]$ is the "step-function" recalled here in Example (2.8). Take an extension g^* of g such that $g^*|X \cap \{(x, y)\colon y = 0\}$ and $g^*|X \cap \{(x, y)\colon y = 1\}$ are monotone. Thus g^* maps X onto the unit square I^2. Further, take a mapping f^* defined as follows:

$$f^*(x, y) = \begin{cases} \left(0, \dfrac{2y}{2-x}\right) & \text{if} \quad 0 \leqslant y \leqslant \dfrac{2-x}{4}, \\ \left(4y-(2-x), \dfrac{1}{2}\right) & \text{if} \quad \dfrac{2-x}{4} \leqslant y \leqslant \dfrac{1}{2}, \\ \left(-4y+(x+2), \dfrac{1}{2}\right) & \text{if} \quad \dfrac{1}{2} \leqslant y \leqslant \dfrac{2+x}{4}, \\ \left(0, \dfrac{2(y-1)}{2-x}+1\right) & \text{if} \quad \dfrac{2+x}{4} \leqslant y \leqslant 1. \end{cases}$$

Put $f(x, y) = f^*(g^*(x, y))$ for each $(x, y) \in X$. The mapping f is weakly confluent and maps X onto a simple triod.

Corollary (5.12) and Theorem (5.16) imply

(5.21) COROLLARY. *A hereditarily weakly confluent image of an arc is an arc.*

Remark. Corollary (5.21) follows also from Corollary (5.12) and from Corollary (3.3) in [15] (cf. [7], Corollary 2.3).

Recall that a point p of a continuum X is called a *ramification point* (in the classical sense) if it is the common end-point of three (or more) arcs in X whose only common point is p. Denote the set of all ramification points of X by $R(X)$. A dendroid X having exactly one ramification point is called a *fan* (see [4], p. 6). The ramification point of a fan is called its *top*.

We have

(5.22) THEOREM. *Let a hereditarily arcwise connected continuum X have a finite set $R(X)$. If a hereditarily weakly confluent mapping f maps X onto Y, then Y is hereditarily arcwise connected, and $R(Y) \subset f(R(X))$.*

Proof. Obviously Y is hereditarily arcwise connected, because the mapping f is weakly confluent and arcwise connectedness is an invariant under an arbitrary continuous mapping (see [21], p. 39). Suppose, on the contrary, that $R(Y) \subset f(R(X)) \neq \emptyset$. Since $R(X)$ is finite, we infer that the set $f(R(X))$ is finite. Therefore there is a triod Q in Y such that $Q \subset Y \setminus f(R(X))$. So $f^{-1}(Q) \subset X \setminus R(X)$, whence every component of $f^{-1}(Q)$ is either an arc or a circle. Let A be the component of $f^{-1}(Q)$ such that $f(A) = Q$ which does exist by the weak confluence of f. Since f is hereditarily weakly confluent, we infer that $f|A$ is weakly confluent; consequently Q is either an arc or a circle by Corollary 2.3 in [7], a contradiction.

Corollary (5.11) and Theorem (5.22) imply

(5.23) COROLLARY. *A hereditarily weakly confluent image of a fan is a fan (or an arc) and the top of the model is mapped on the top of the image.*

There is a hereditarily weakly confluent mapping f which maps a smooth fan (for the definition of smoothness, see [4], p. 7) onto a fan which is not smooth (for example a mapping f described in Example 5.7 of [14], p. 263).

The Cantor fan may be mapped onto a dendrite with two (or more) ramification points by a weakly confluent mapping (an easy modification of Example I in [7]).

Finally note that unicoherence is not preserved by hereditarily weakly confluent mappings, but we have

(5.24) QUESTION. *Is the hereditarily confluent image of unicoherent continuum also unicoherent?*

Moreover, we have

(5.25) QUESTION. *Is the hereditarily confluent (hereditarily weakly confluent) image of an arc-like (tree-like) continuum also arc-like (tree-like)?* (for the definition of tree-likeness see [1], p. 653).

Added in proof. Question (3.11) has a negative answer and Questions (4.8) and (5.24) have positive answers (see my paper *Continuous mappings on continua*, Dissertationes Math., to appear, Example (6.14) and Theorems (6.7) and (7.1), respectively). The assumption of the hereditary unicoherence in Theorem (3.6) can be omitted (see ibidem, Theorem (6.12)).

References

[1] R. H. Bing, *Snake-like continua*, Duke Math. J. 18 (1951), pp. 653–663.
[2] J. J. Charatonik, *On ramification points in the classical sense*, Fund. Math. 51 (1962), pp. 229–252.
[3] — *Confluent mappings and unicoherence of continua*, Fund. Math. 56 (1964), pp. 213–220.
[4] — *On fans*, Dissertationes Math. 54, Warszawa.

[5] J. J. Charatonik, *On decompositions of λ-dendroids*, Fund. Math. 67 (1970), pp. 15–30.
[6] H. Cook, *Continua which admit only the identity mapping onto non-degenerate subcontinua*, Fund. Math. 60 (1967), pp. 241–249.
[7] C. A. Eberhart, J. B. Fugate and G. R. Gordh, Jr., *Branchpoint covering theorems for confluent and weakly confluent maps*, Proc. Amer. Math. Soc. 55 (1976), pp. 409–415.
[8] A. Emeryk and Z. Horbanowicz, *On atomic mappings*, Colloq. Math. 27 (1973), pp. 49–55.
[9] J. B. Fugate and L. Mohler, *Quasi-monotone and confluent images of irreducible continua*, Colloq. Math. 28 (1973), pp. 221–224.
[10] K. Kuratowski, *Topology*, vol. I, New York–London–Warszawa 1966.
[11] — *Topology*, vol. II, New York–London–Warszawa 1968.
[12] A. Lelek, *Report on weakly confluent mappings*, Proc. Virginia Polytechnic Institute State Univ., Topology Conference 1973.
[13] — and D. R. Read, *Compositions of confluent mappings and some other classes of functions*, Colloq. Math. 29 (1974), pp. 101–112.
[14] T. Maćkowiak, *Semi-confluent mappings and their invariants*, Fund. Math. 79 (1973), pp. 251–264.
[15] — *Locally weakly confluent mappings on hereditarily locally connected continua*, Fund. Math. 83 (1975), pp. 225–240.
[16] L. Mohler, *A characterization of hereditarily decomposable snake-like continua*, Colloq. Math. 28 (1973), pp. 51–56.
[17] D. R. Read, *Confluent and related mappings*, Colloq. Math. 29 (1974), pp. 233–239.
[18] R. H. Sorgenfrey, *Concerning triodic continua*, Amer. J. Math. 66 (1944), pp. 439–460.
[19] Z. Waraszkiewicz, *Sur un problème de M. H. Hahn*, Fund. Math. 22 (1934), pp. 180–205.
[20] L. E. Ward, Jr., *Fixed point theorems for pseudo monotone mappings*, Proc. Amer. Math. Soc. 13 (1962), pp. 13–16.
[21] G. T. Whyburn, *Analytic Topology*, New York 1942.

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