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The axiom of choice and linearly ordered sets

by

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Abstract. Let ZF be Zermelo-Fraenkel set theory without the axiom of choice and let ZF^0 be the modification which allows urelements or atoms. In this paper we show that there are many forms of the axiom of choice and the axiom of multiple choice involving linearly ordered sets which are equivalent to the axiom of choice in ZF but not in ZF^0. The independence proofs use permutation models of Fraenkel-Mostowski.

§ 1. Introduction. The variants of the axiom of choice which we consider in this paper are listed below.

The statements

A: *Antichain Principle.* Every partially ordered set contains a maximal antichain. (I.e. a maximal subset of mutually incomparable elements.)

AC: *Axiom of Choice.* For every family x of non-empty sets, there is a function f such that for each $u \in x$, $f(u) \in u$.

AC^{LO}: Axiom of choice for a linearly ordered family of non-empty sets.

AC_{LO}: Axiom of choice for a family of non-empty sets each of which can be linearly ordered.
$(\forall x)[(\forall u \in x)(\exists R_u)\ (R_u$ linearly orders $u) \to$ AC holds for $x]$.

AC_{DLO}: Axiom of choice for a family of non-empty sets, each of which has a defined linear ordering.
$(\forall x)[(\exists R)\ (\forall u \in x)\ (R_u$ linearly orders $u) \to$ AC holds for $x]$.

AC^{LO}_{LO}: Axiom of choice for a linearly ordered family of non-empty sets, each of which can be linearly ordered.

AC^{LO}_{DLO}: Axiom of choice for a linearly ordered family of non-empty sets, each of which has a defined linear ordering.

LW: Every linearly ordered set can be well ordered.

MC: *Axiom of Multiple Choice.* For every family x of non-empty sets, there is a function f such that for each $u \in x$, $f(u)$ is a non-empty, finite subset of u.

MC^{LO}: Axiom of multiple choice for a linearly ordered family of non-empty sets.

MC_{LO}: Axiom of multiple choice for a family of non-empty sets, each of which can be linearly ordered.

MC_{DLO}: Axiom of multiple choice for a family of non-empty sets, each of which has a defined linear ordering.

MC_{LO}^{LO}: Axiom of multiple choice for a linearly ordered family of non-empty sets, each of which can be linearly ordered.

MC_{DLO}^{LO}: Axiom of multiple choice for a linearly ordered family of non-empty sets, each of which has a defined linear ordering.

PW: The power set of each well ordered set can be well-ordered.

In § 2 we show that all the implications and equivalences shown in the diagram below (Fig. 1) hold in ZF^0, ZF with atoms. However, it is known [4], that PW→AC in ZF. Thus, it follows that all the statements listed above are equivalent in ZF.

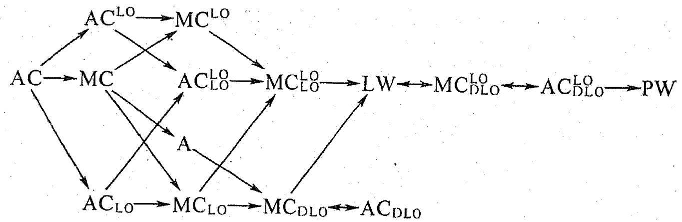

Fig. 1

Then, in § 3 we construct Fraenkel–Mostowski (F-M) models and prove the non-implications. We show, with very few exceptions that no arrow in Figure 1 is reversible in ZF^0 and if there is no arrow between two statements then it cannot be shown in ZF^0 that one implies the other. There are still some unsolved problems due to the fact that we have not been able to construct an F-M model in which AC_{LO} is true(1). However, all other questions have been resolved. (See Fig. 3 in § 3.)

§ 2. **The implications.** The following implications are clear.

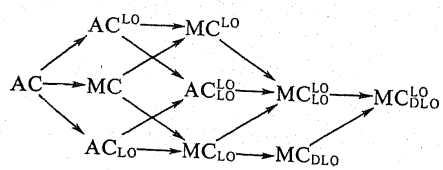

Fig. 2

The proof that MC→A is given in [1], p. 80 and the proof that LW→PW is given in [5], p. 77.

(1) Added in proof, John Tross has discovered a model due to R. J. Gauntt in which AC_{LO} is true and MC^{LO} is false.

LEMMA 2.1. $A \to AC_{DLO}$.

Proof. Let x be a set of non-empty sets. (There is no loss of generality if we assume the sets in x are pairwise disjoint.) Suppose R is a relation which linearly orders each set in x. Define a relation S on $\bigcup x$ such that for each $a, b \in \bigcup x$, aSb iff $(\exists u \in x)(a, b \in u \,\&\, aRb)$. S is a partial ordering on $\bigcup x$ and a maximal S-anti-chain is a choice set for x.

It is clear that $AC_{DLO} \to MC_{DLO}$ and $AC_{DLO}^{LO} \to MC_{DLO}^{LO}$.

LEMMA 2.2. $MC_{DLO} \to AC_{DLO}$.

Proof. Let x be a set of non-empty pairwise disjoint sets, each of which is linearly ordered by R. Let F be a multiple choice function on x. Then $\{a\colon (\exists u \in x)$ (a is the R-first element of $F(u)$)$\}$ is a choice set in x.

LEMMA 2.3. $LW \to AC_{DLO}^{LO}$.

Proof. Suppose $\langle x, R\rangle$ is a linearly ordered set of non-empty pairwise disjoint sets such that each set in x is linearly ordered by S. Then, $\bigcup x$ can be linearly ordered so LW implies that $\bigcup x$ can be well-ordered. Using this well-ordering a choice function on x can be defined.

The proof that $MC_{DLO}^{LO} \to AC_{DLO}^{LO}$ is similar to the proof of Lemma 2.2.

LEMMA 2.4. $AC_{DLO}^{LO} \to LW$.

Proof. Let $\langle x, R\rangle$ be a linearly ordered set. Define

$$y = \{u \times \{S\}\colon u \subseteq x \,\&\, S \text{ well-orders } u\}.$$

The set y can be linearly ordered by a relation T as follows:

$$(u_1 \times \{S_1\})T(u_2 \times \{S_2\}) \quad \text{iff} \quad \overline{\langle u_1, S_1\rangle} < \overline{\langle u_2, S_2\rangle}$$

or $[f\colon \langle u_1, S_1\rangle \cong \langle u_2, S_2\rangle$ & if a is the S_1-first element in $\{b \in u_1\colon b \neq f(b)\}$ then $aRf(a)]$ where $\overline{\langle u, S\rangle}$ is the ordinal number of $\langle u, S\rangle$ and $f\colon \langle u_1, S_1\rangle \cong \langle u_2, S_2\rangle$ means f is the unique isomorphism from $\langle u_1, S_1\rangle$ onto $\langle u_2, S_2\rangle$.

The set

$$z = \{(x-u) \times \{S\}\colon u \subset x \,\&\, S \text{ well-orders } u\}$$

is a linearly ordered set (T induces a linear ordering on z) of non-empty sets, each of which is linearly ordered by R. Thus, AC_{DLO}^{LO} implies there is a choice function F, on z. Then well-order x so that the αth element of x, $a_\alpha =$ the first coordinate of $F\big((x - \bigcup_{\beta<\alpha} \{a_\beta\}) \times \{\leqslant\}\big)$ where $\leqslant$ is the well-ordering of $\bigcup_{\beta<\alpha} a_\beta$ induced by α.

The proof that PW→AC in ZF is due to H. Rubin and can be found in [5], p. 77. Thus, all the statements listed in Figure 1 are equivalent in ZF. We shall show in § 3 that this is not the case in ZF^0.

As a closing note for this section we show that AC^{LO} is equivalent in ZF^0 to the following maximal principle.

M: Every transitive and connected ordered set contains a maximal linearly ordered subset.

LEMMA 2.5. $M \to AC^{LO}$.

Proof. Let $\langle x, R\rangle$ be a linearly ordered set of pairwise disjoint sets. Define S on $\bigcup x$ so that for $a, b \in \bigcup x$, where $a \in u \in x$ and $b \in v \in x$, aSb iff uRv. S is a transitive and connected relation on $\bigcup x$. An S-linearly ordered maximal subset of $\bigcup x$ is a choice set for x.

LEMMA 2.6. $AC^{LO} \to M$.

Proof. Suppose $\langle x, R\rangle$ is a transitive and connected set. Define a relation S on x so that

$$uSv \quad \text{iff} \quad u = v \quad \text{or} \quad (uRv \,\&\, vRu).$$

S is an equivalence relation. For each $u \in x$, let $[u] = \{v \in x\colon uSv\}$. Let $y = \{[u]\colon u \in x\}$. Since R is transitive, we can define a relation T on y so that $[u]T[v]$ iff uRv. (If $u' \in [u]$ and $v' \in [v]$, $u'Rv'$ iff uRv.) The relation T linearly orders y and a choice set for y is a maximal linearly ordered subset of x.

§ 3. **The non-implications.** Given a model M' of ZF^0+AC which has U as the set of urelements, a permutation model M of ZF^0 is determined by a group G of permutations of U and a filter Γ of subgroups of G which satisfies

(1) $$(\forall a \in U)(\exists H \in \Gamma)(\forall \varphi \in H)\varphi(a) = a$$

and

(2) $$(\forall \varphi \in G)(\forall H \in \Gamma)\varphi H \varphi^{-1} \in \Gamma.$$

Each permutation of U extends uniquely to a permutation of M' by ε-induction and for any $\varphi \in G$, we identify φ with its extension. The following notation from [2] will be adopted: For $x \in M'$,

$$\mathrm{fix}_G(x) = \{\varphi \in G\colon \varphi(y) = y \text{ for all } y \in x\}.$$

When no confusion will arise, we will write $\mathrm{fix}(x)$ for $\mathrm{fix}_G(x)$. Also if H is a subgroup of G, $x \in M'$ and $(\forall \varphi \in H)(\varphi(x) = x)$ we say *H fixes x*. If it is also the case that $(\forall \varphi \in H)(\forall y \in x)(\varphi(y) = y)$ we say that *H fixes x pointwise*.

The permutation model M determined by U, G and Γ consists of all those $x \in M'$ such that for every y in the transitive closure of x, there is some $H \in \Gamma$ such that H fixes y. We refer the reader to [2], p. 46 for the proof that M is a model of ZF^0.

Each of the non-implications is proved by using one of the six permutation models $M_1, M_2, \ldots, M_6$ described below. In each case we describe M_i by giving a set U_i of urelements, a group G_i of permutations of U_i and a filter Γ_i of subgroups of G_i.

U_1 is a countable set of urelements and $<$ is a dense linear ordering of U_1 without first or last element. G_1 is the group of all automorphisms of $\langle U_1, <\rangle$ and Γ_1 is the filter generated by

$$\{\mathrm{fix}(E)\colon E \text{ is a finite subset of } U_1\}.$$

(This is the linear ordered model of Mostowski [3].)

U_2 is countable and $<$ is a partial ordering of U_2 such that $\langle U_2, <\rangle$ is a countable, universal, homogeneous partially ordered set. G_2 is the group of all automorphisms of $\langle U_2, <\rangle$ and Γ_2 is the filter generated by

$$\{\mathrm{fix}(E)\colon E \text{ is a finite subset of } U_2\}.$$

We refer the reader to [2], p. 101 for definitions.

U_3 is countable, G is the group of all permutations of U_3 and Γ_3 is the filter generated by

$$\{\mathrm{fix}(E)\colon E \text{ is a finite subset of } U_3\}.$$

$U_4 = \bigcup_{i\in\omega} \{a_i, b_i\}$ where a_i and b_i are atoms for all $i \in \omega$ and $i \neq j$ implies $\{a_i, b_i\} \cap \{a_j, b_j\} = \varnothing$,

$$G_4 = \{\varphi\colon (\exists A \subseteq \omega)\big(A \text{ finite and } (\forall i \in A)(\varphi(a_i) = b_i \text{ and } \varphi(b_i) = a_i) \text{ and } (\forall i \notin A)(\varphi(a_i) = a_i \text{ and } \varphi(b_i) = b_i)\big)\}$$

and Γ_4 is the filter generated by

$$\{\mathrm{fix}(E)\colon E \text{ is a finite subset of } U_4\}.$$

U_5 is a set of urelements of cardinality $\aleph_1$, G_5 is the group of all permutations of U_5 and Γ_5 is the filter generated by

$$\{\mathrm{fix}(E)\colon E \text{ is a countable subset of } U_5\}.$$

$U_6 = \bigcup_{i\in\omega} C_i$ where each C_i is a countable set of urelements and $i \neq j$ implies $C_i \cap C_j = \varnothing$,

$$G_6 = \{\varphi\colon (\forall i \in \omega)(\varphi(C_i) = C_i) \text{ and } (\exists B)\big(B \text{ is a finite subset of } U_6 \text{ and } (\forall a \notin B)(\varphi(a) = a)\big)\},$$

Γ_6 is the filter generated by $\{\mathrm{fix}(C_i)\colon i \in \omega\}$.

We note that if one prefers a countable set of atoms M_5 could be replaced by M_5' where U_5' is a countable set of atoms and $<$ is a dense linear ordering of U_5' without first or last element,

$$G_5' = \{\varphi\colon (\exists E \subseteq U_5')\big(E \text{ is bounded and } (\forall a \notin E)(\varphi(a) = a)\big)\}.$$

and Γ_5' is the filter generated by $\{\mathrm{fix}(E)\colon E$ is a bounded subset of $U_5'\}$. All the theorems which we prove concerning the model M_5 remain true if M_5 is replaced by M_5'. We summarize the results of this section (and of Sections 1 and 2) by the following table. An $\to$ in a box indicates that the row label implies the column label and a number i in a box indicates that the row label is true in M_i and the column label is false. An empty box indicates an open problem.

(See the footnote on p. 112 for additional results on AC_{LO}.

We now look at the models one at a time beginning with M_1.

	AC	AC^{LO}	MC	AC_{LO}	MC^{LO}	AC_{LO}^{LO}	A	MC_{LO}	MC_{LO}^{LO}	MC_{DLO}	LW	PW
AC	→	→	→	→	→	→	→	→	→	→	→	→
AC^{LO}	5	→	5	5	→	→	5	5	→	5	→	→
MC	4	4	→	4	→	4	→	→	→	→	→	→
AC_{LO}				→		→		→	→	→	→	→
MC^{LO}	4	4	5	4	→	4	5	5	→	5	→	→
AC_{LO}^{LO}	3	3	3	3	3	→	5	5	→	5	→	→
A	3	3	3	3	3	4	→	6	6	→	→	→
MC_{LO}	3	3	3	3	3	4	2	→	→	→	→	→
MC_{LO}^{LO}	3	3	3	3	3	4	2	5	→	5	→	→
MC_{DLO}	3	3	3	3	3	4	2	6	6	→	→	→
LW	3	3	3	3	3	4	5	5	6	5	→	→
PW	1	1	1	1	1	1	1	1	1	1	1	→

Fig. 3

By considering Figure 1 we see that to prove all of the claims made in Figure 3 about model M_1, the following theorem suffices:

THEOREM 3.1. *In M_1, PW is true and LW is false.*

The proof can be found in [2], p. 134 ff.

Before proceeding with the remaining models, it is convenient to prove the following two lemmas.

LEMMA 3.1. *Suppose M is a permutation model determined by a set U of urelements, a group G of permutations of U and a filter Γ of subgroups of G. Suppose that W is a set in M and H is a function defined on W such that:*

1) *$H \in M$,*
2) *$G' \in \Gamma$ fixes H and*
3) *for each $z \in W$, there is a $y \in H(z)$ such that*

$$(\forall \varphi \in G')\big(\varphi(z) = z \to \varphi(y) = y\big).$$

Then there is a function $F \in M$ defined on W such that

$$(\forall z \in W)\big(F(z) \in H(z)\big).$$

Proof. Assume the hypotheses. Define the equivalence relation $\sim$ on W by

$$u \sim v \leftrightarrow (\exists \varphi \in G')(\varphi(u) = v).$$

Let C be the set of equivalence classes and choose $u_c \in c$ for each $c \in C$. Then choose $t_c \in H(u_c)$ such that

$$(\forall \varphi \in G')\big(\varphi(u_c) = u_c \to \varphi(t_c) = t_c\big).$$

(Neither of the functions u or t need by in M). We then claim that

$$F = \{\varphi(\langle u_c, t_c\rangle) \colon c \in C \ \&\ \varphi \in G'\}$$

is a function with the required properties. We verify that F is a function. The other properties are easily verified. Suppose that $\langle \varphi(u_c), \varphi(t_c)\rangle$ and $\langle \varphi'(u_{c'}), \varphi'(t_{c'})\rangle$ are in F, where $c, c' \in C$ and $\varphi, \varphi' \in G'$ and suppose that $\varphi(u_c) = \varphi'(u_{c'})$. Then

$$(*) \qquad \varphi'^{-1}\varphi(u_c) = u_{c'}.$$

Hence $u_c \sim u_{c'}$ so $u_c = u_{c'}$ and therefore $c = c'$. Then by (*) we have $\varphi'^{-1}\varphi(u_c) = u_c$. So by the choice of t_c, $\varphi'^{-1}\varphi(t_c) = t_c$. This gives $\varphi(t_c) = \varphi'(t_c) = \varphi'(t_{c'})$ and therefore F is a function.

We prove Lemma 3.2 in model M_3. However, with very little modification the same proof holds in M_2, and M_5. Mostowski ([3], p. 236 ff) gives a proof for M_1.

LEMMA 3.2. *If $x \in M_3$ and E_1 and E_2 are supports of x, then $E_1 \cap E_2$ is a support of x.*

Proof. Assume the hypotheses and suppose $\varphi \in \operatorname{fix}(E_1 \cap E_2)$. The first step is to find a permutation $\varphi' \in \operatorname{fix}(E_1 \cap E_2)$ such that $\{u \in U_3 \colon \varphi'(u) \neq u\}$ is finite and $\varphi'(u) = \varphi(u)$ for all $u \in E_1$. (So that $\varphi(x) = \varphi'(x)$.) φ' is obtained as follows: Write φ as a product of disjoint cycles and let $C_1, C_2, \ldots, C_n$ be those cycles which have an element in common with E_1. φ' will be a product of a finite number of finite disjoint cycles $C_1', C_2', \ldots, C_m'$ where

$$C_i' = C_i \quad \text{if } C_i \text{ is finite},$$

and

$$C_i' = (u_0, u_1, \ldots, u_{k-1}, u_k) \quad \text{if } C_i \text{ is infinite},$$

where $C_i = (\ldots, u_0, u_1, \ldots, u_{k-1}, u_k, \ldots)$ and all elements common to C_i and E_1 occur among $u_0, u_1, \ldots, u_k$.

The next step is to show that there are permutations $\sigma \in \operatorname{fix}(E_1)$, $\eta \in \operatorname{fix}(E_2)$ and $\varphi'' \in \operatorname{fix}(E_1 \cup E_2)$ such that $\varphi' = \sigma^{-1}\eta^{-1}\varphi''\eta\sigma$ so that we can conclude

$$\varphi(x) = \varphi'(x) = \sigma^{-1}\eta^{-1}\varphi''\eta\sigma(x) = x$$

and the proof will be complete. σ, η and φ'' are constructed as follows: Let $C = \{u \colon \varphi'(u) \neq u\}$ and let A and B be two disjoint subsets of U_3 such that each is disjoint with $E_1 \cup E_2 \cup C$, A has the same cardinality as $(E_1 \cup C) - E_2$ and B has the same cardinality as $E_2 - E_1$. (We note here that $(E_1 \cup C) - E_2$ and $E_2 - E_1$ are disjoint.) Let f be a 1-1 correspondence between $(E_1 \cup C)-E_2$ and A and let g be a 1-1 correspondence between E_2-E_1 and B. Then σ, η and φ'' are defined by

$$\eta(u) = \begin{cases} f(u) & \text{if } u \in (E_1 \cup C)-E_2\,, \\ f^{-1}(u) & \text{if } u \in A\,, \\ u & \text{otherwise}\,, \end{cases}$$

$$\sigma(u) = \begin{cases} g(u) & \text{if } u \in E_2-E_1\,, \\ g^{-1}(u) & \text{if } u \in B\,, \\ u & \text{otherwise}\,, \end{cases}$$

$$\varphi''(u) = \eta\sigma\varphi'\sigma^{-1}\eta^{-1}(u)\,.$$

It can be easily shown that σ, η and φ'' have the required properties. This completes the proof of Lemma 3.2.

Now, considering Figures 1 and 3 again, we see that in order to verify the claims concerning M_2, it suffices to show:

THEOREM 3.2. *In* M_2, MC_{LO} *is true and* A *is false.*

Proof. It is proved in [1], p. 82 that A is false in M_2. Hence it remains to prove MC_{LO} in M_2.

Suppose W is a set of pairwise disjoint sets in M_2 each of which can be linearly ordered in M_2. It is shown in [1] that LW holds in M_2 so we also have that each element of W can be well-ordered in M_2. Suppose that E is a support of W (i.e., E is a finite subset of U_2 such that for every $\varphi \in \text{fix}(E)$, $\varphi(W) = W$.) For each $z \in W$, let

$$H(z) = \{t\colon\ t \text{ is a non-empty finite subset of } z\}\,.$$

Fix(E) fixes H, therefore in view of Lemma 3.1, it suffices to find for each $z \in W$ an element $y \in H(z)$ such that

$$(**)\qquad (\forall\varphi \in \text{fix}(E))\big(\varphi(z) = z \rightarrow \varphi(y) = y\big)\,.$$

Choose $t \in z$ and let $y = \{\varphi(t)\colon\ \varphi \in \text{fix}(E) \text{ and } \varphi(z) = z\}$. It is clear that y satisfies (**). We complete the proof by showing that y is finite.

Each element of M_2 has a minimal support. (This follows from the fact that the intersection of two supports is a support, Lemma 3.2.) Let E' be the minimal support of z and E'' the minimal support of t. We claim $E'' \subseteq E'$. For if not, there is some $a \in E''-E'$. Then the set

$$B = \{\varphi(t)\colon\ \varphi \in \text{fix}\big((E'' \cup E')-\{a\}\big)\}$$

is a subset of z and further the set

$$\{\langle\varphi(t), \varphi(a)\rangle\colon\ \varphi \in \text{fix}\big((E'' \cup E')-\{a\}\big)\}$$

is a one to one function in M_2 from B to U_2. One can easily show using Lemma 9.5 in [2], p. 137 that $\{\varphi(a)\colon\ \varphi \in \text{fix}(E'' \cup E')-\{a\})\}$ cannot be well-ordered in M_2. Therefore B cannot be well-ordered in M_2, hence z cannot be well-ordered in M_2. A contradiction which proves the claim.

Since E' is a minimal support of z, any $\varphi \in \text{fix}(E)$ which fixes z must fix E'. (Although not necessarily pointwise). Therefore

$$y = \{\varphi(t)\colon\ \varphi(z) = z \text{ and } \varphi \in \text{fix}(E)\} \subseteq \{\varphi(t)\colon\ \varphi(E') = E' \text{ and } \varphi \in \text{fix}(E)\}$$

and in view of the fact that $E'' \subseteq E'$,

$$y \subseteq \{\varphi(t)\colon\ \varphi(E'') \subseteq E' \text{ and } \varphi \in \text{fix}(E)\}\,,$$

and this set is clearly finite since E'' is the least support of t.

Although it is not necessary for Figure 3, we can also show that AC_{LO}^{LO} is true in M_3 while AC_{LO} and MC^{LO} are false. (The proof is similar to the proof of Theorem 3.3.) From these latter results Figure 1 and Theorem 3.2 the truth and falsity of all the statements is determined in M_2. See Figure 4.

For the claims involving M_3 we need the following theorem:

THEOREM 3.3. *In* M_3, A, AC_{LO}^{LO} *and* MC_{LO} *are true and* AC_{LO} *and* MC^{LO} *are false.*

Proof. The proof that A is true in M_3 was given by Halpern in his Ph. D. Thesis 1962. For the proof see for example [2], p. 134 ff.

We now prove AC_{LO}^{LO} in M_3. By Lemma 3.2 each element of M_3 has a minimal support. It also follows that if $x \in M_3$ and E is a minimal support of x, then

$$(*)\qquad \varphi(x) = x \rightarrow \varphi(E) = E$$

for any $\varphi \in G_3$. Further we claim that if

1) $x \in M_3$,
2) x can be linearly ordered in M_3,
3) E is minimal support of x,
4) E' is a minimal support of $y \in x$.

Then $E' \subseteq E$. The proof of the claim is by contradiction. Suppose $u \in E'-E$, then the set of pairs

$$\{\langle\varphi(y), \varphi(u)\rangle\colon\ \varphi \in \text{fix}\big((E' \cup E)-\{u\}\big)\}$$

is a one to one function in M_3. (By (*).) Further its domain is a subset of x (since $u \notin E$) and its range is $U_3-\big((E' \cup E)-\{u\}\big)$ which cannot be linearly ordered in M_3.

Now the proof that AC_{LO}^{LO} holds is as follows: Suppose that W is a linearly ordered set of linearly orderable sets in M_3. Suppose $x \in W$ and $y \in x$. Suppose further that E, E' and E'' are minimal supports of W, x and y respectively. By the claim, $E'' \subseteq E' \subseteq E$ hence fix(E) fixes a well-ordering of $\bigcup W$ and therefore a choice function on W.

The proof of MC_{LO} in M_3 also makes use of the claim. Suppose W is a set of linearly orderable sets in M_3. Let E be a support of W. For each $z \in W$, let

$$H(z) = \{y\colon\ y \subseteq z \text{ and } y \neq \varnothing \text{ and } y \text{ finite}\}\,.$$

E is also a support of H and therefore to prove MC_{LO} using Lemma 3.1 it suffices to find for each $z \in W$ a $y \in H(z)$ such that

$$(\forall\varphi \in \text{fix}(E))\big(\varphi(z) = z \rightarrow \varphi(y) = y\big)\,.$$

So choose $z \in W$, let E' be a support of z and let $x_0 \in z$. By the claim if E'' is a minimal support of x_0, then $E'' \subseteq E'$. Now let

$$y = \{\varphi(x_0)\colon \varphi \in \mathrm{fix}(E) \text{ and } \varphi(z) = z\}.$$

By (*), $\varphi(z) = z \to \varphi(E') = E'$, hence

$$y \subseteq \{\varphi(x_0)\colon \varphi \in \mathrm{fix}(E) \text{ and } \varphi(E') = E'\}$$
$$\subseteq \{\varphi(x_0)\colon \varphi \in \mathrm{fix}(E) \text{ and } \varphi(E'') \subseteq E'\}$$

which is finite since E'' is a support of x_0. So y is finite and further if $\psi \in \mathrm{fix}(E)$ and $\psi(z) = z$, then $\psi(y) = y$. Therefore y satisfies the required properties and applying Lemma 3.1 gives a function F with domain W such that $F(z)$ is a non-empty finite subset of z for each $z \in W$.

The set of non-empty finite subsets of U_3 provides an example of a set each of whose elements can be linearly ordered but which has no choice function. Therefore $\mathrm{AC}_{\mathrm{LO}}$ is false in M_3.

To show that $\mathrm{MC}^{\mathrm{LO}}$ is false in M_3, we let $z_i = \{E\colon E \subseteq U_3$ and E has cardinality $i\}$ and let $W = \{z_i\colon i \in \omega\}$. W is well-ordered in M_3 and we claim there is no function f in M_3 such that for all $i \in \omega$, $f(z_i)$ is a finite subset of z_i. The existence of such an f leads to a contradiction when one considers $f(z_i)$ where i is chosen to be larger than the cardinal number of a support of f. This completes the proof of Theorem 3.3.

THEOREM 3.4. *In M_4 MC is true and $\mathrm{AC}^{\mathrm{LO}}_{\mathrm{LO}}$ is false.*

Proof. The proof in [2], p. 134 ff shows that MC is true in M_4.

The set $\{\{a_i, b_i\}\colon i \in \omega\}$ provides an example of a linearly ordered set of linearly-orderable sets which has no choice function in M_4.

THEOREM 3.5. *In M_5 $\mathrm{AC}^{\mathrm{LO}}$ is true and $\mathrm{AC}_{\mathrm{DLO}}$ is false.*

Proof. We begin by proving $\mathrm{AC}^{\mathrm{LO}}$ in M_5. A set $E \subseteq U_5$ is said to be a support of $x \in M_5$ if E is countable and $(\forall \varphi \in \mathrm{fix}(E))(\varphi(x) = x)$.

First we shall show that LW holds in M_5. Suppose X is linearly ordered in M_5 and that E is a support of a linear ordering of X. We claim that E is a support for a well-ordering of X. If not then $\mathrm{fix}(E)$ does not fix X pointwise. That is, for some $y \in X$ and $\varphi \in \mathrm{fix}(E)$, $\varphi(y) \neq y$.

Our plan is to show that for some $\psi \in \mathrm{fix}(E)$, $\psi(y) \neq y$ but $\psi^2(y) = y$. (This will contradict the assumption that E is a support of a linear ordering of X.) Suppose that E' is a support of y where $E' \supseteq E$. Let C be a subset of U_5 of the same cardinality as $E' - E \neq \emptyset$ such that $C \cap E' = \emptyset$. Let ψ be the permutation of U_5 that interchanges C and $E' - E$. I.e., let f be a 1-1 function from $E' - E$ onto C and define

$$\psi(a) = \begin{cases} f(a) & \text{if } a \in E' - E, \\ f^{-1}(a) & \text{if } a \in C, \\ a & \text{otherwise}. \end{cases}$$

Then ψ^2 is the identity so $\psi^2(y) = y$, $\psi \in \mathrm{fix}(E)$ and $\psi(y) \neq y$, for if $\psi(y) = y$, then $\psi(E')$ is a support of $\psi(y)$ and hence a support of y so by Lemma 3.2, $\psi(E') \cap E' = E$ is a support of y, a contradiction. So LW holds in M_5.

Now let X be a linearly ordered set of non-empty sets (in M_5). Then X is well-ordered in M_5, so we let E be a support of a well-ordering of X. Then $\mathrm{fix}(E)$ fixes X pointwise. Let E' be a countable subset of U_5 such that $E' \supseteq E$ and $E' - E$ is countably infinite.

Claim. For every $y \in X$, there is some $z \in y$ with support E'. (This will give a choice function in M_5, for X.) Choose $y \in X$ and $z \in y$. If E' is a support of z then we are done. Otherwise, suppose E'' is a support of z. Choose a permutation $\psi \in \mathrm{fix}(E)$ such that $\psi(E'' - E) \subseteq E' - E$. Then $\psi(z) \in y$ and $\psi(z)$ has support $\psi(E'') \subseteq E'$, hence has support E'. This proves $\mathrm{AL}^{\mathrm{LO}}$ in M_5.

To show that $\mathrm{AC}_{\mathrm{DLO}}$ is false in M_5, we let

$$W = \{A \times \{R\}\colon A \subseteq U_5,\ \bar{\bar{A}} = \aleph_0, \text{ and } R \text{ is a linear ordering of type } \eta \text{ (the order type of the rationals)}\}.$$

W satisfies the hypothesis of $\mathrm{AC}_{\mathrm{DLO}}$. Suppose F is a choice function on W and E is a support of F. Let E' be a countably infinite subset of U_5 disjoint from E and suppose R is a linear ordering of E' of type η. Then $E' \times \{R\} \in W$ so suppose $F(E' \times \{R\}) = \langle a, R\rangle$. Let $\varphi \in \mathrm{fix}(E)$ such that $\varphi(a) \neq a$ but $\varphi(E') = E'$ and $\varphi(R) = R$. (I.e. shifts the elements of E' preserving the linear ordering R). Then we have the contradiction that $\varphi(F) = F$, $\varphi(E' \times \{R\}) = E' \times \{R\}$, but $\varphi(\langle a, R\rangle) \neq \langle a, R\rangle$. The proof of Theorem 3.5 is therefore complete.

THEOREM 3.6. *In M_6 A is true and $\mathrm{MC}^{\mathrm{LO}}_{\mathrm{LO}}$ is false.*

Proof. To show that A holds in M_6, let $\langle P, <\rangle$ be a partially ordered set in M_6. Suppose $\langle P, <\rangle$ is fixed by $G' \in \Gamma_6$. Let B be an antichain maximal among those fixed by G' and suppose B is not maximal. Then for some $x \in P - B$, $B \cup \{x\}$ is an antichain. Further, $B' = B \cup \{\varphi(x)\colon \varphi \in G'\}$ fails to be an antichain since B' is fixed by G' and properly includes B. There are two ways B' can fail to be an antichain. Either $\varphi(x) > y$ (or $y > \varphi(x)$) for some $\varphi \in G'$ and $y \in B$ or for some $\varphi, \psi \in G'$, $\psi(x) > \varphi(x)$.

The first alternative is impossible since $\varphi(x) > y$ implies $x > \varphi^{-1}(y)$ which contradicts the assumption that $B \cup \{x\}$ is an antichain. ($y > \varphi(x)$ is treated similarly.) The other alternative is also impossible. To show this we use the property of the model M_6 that each permutation in G_6 permutes at most a finite number of elements. Therefore, for every $\eta \in G_6$, there is some $n \in \omega$ such that $\eta^n = e$ (the identity permutation on U_6). $\psi(x) > \varphi(x) \to x > \psi^{-1}\varphi(x)$. So if we let $\eta = \psi^{-1}\varphi$ and if we suppose $\eta^n = e$ we get

$$x > \eta(x) > \eta^2(x) > \ldots > \eta^n(x) = x$$

a contradiction. Therefore B is a maximal antichain and A is true in M_6.

$W = \{C_i\colon i \in \omega\}$ is an example to show that MC_{LO}^{LO} is false in M_6. This completes the proof of Theorem 3.6 and therefore the proof of the non-implications given in Figure 3.

To summarize the results of this section we include Figure 4. It shows for each of the models M_1-M_6, which of our statements are true (T) and which are false (F).

	M_1	M_2	M_3	M_4	M_5	M_6
AC	F	F	F	F	F	F
AC^{LO}	F	F	F	F	T	F
MC	F	F	F	T	F	F
AC_{LO}	F	F	F	F	F	F
MC^{LO}	F	F	F	T	T	F
AC_{LO}^{LO}	F	T	T	F	T	F
A	F	F	T	T	F	T
MC_{LO}	F	T	T	T	F	F
MC_{LO}^{LO}	F	T	T	T	T	F
MC_{DLO}	F	T	T	T	F	T
LW	F	T	T	T	T	T
PW	T	T	T	T	T	T

Fig. 4

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The hereditary classes of mappings

by

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Abstract. If $\mathscr{C}$ is an arbitrary class of mappings, then a mapping $f\colon X \to Y$ is hereditarily $\mathscr{C}$ if for each continuum $K \subset X$ the partial mapping $f|K$ is in $\mathscr{C}$. In the paper we study some properties of hereditarily monotone, hereditarily confluent, hereditarily weakly confluent and hereditarily atriodic mappings (for the definition see § 3). In particular, it is proved that a continuum X is hereditarily unicoherent if and only if any monotone mapping of X is hereditarily monotone. We give also other mapping characterizations of some classes of continua. Namely, we prove that a continuum X is hereditarily indecomposable (atriodic) if and only if any confluent (atriodic) mapping of a continuum onto X is hereditarily confluent (hereditarily atriodic). Using these results, we characterize hereditarily decomposable snake-like continua and an arc by hereditarily weakly confluent mappings. These results are connected with the problem posed in [12], and imply some partial solutions of this problem.

Further, it is proved that any (irreducible) hereditarily confluent mapping of an arcwise connected continuum (onto a locally connected continuum, respectively) is monotone. We discuss also some invariance properties of the above mappings. In particular, we show that if a continuum X is hereditarily decomposable, then the hereditary unicoherence of X as well as the atriodicity of X is an invariant under hereditarily weakly confluent mappings.

§ 1. **Introduction.** The topological spaces under consideration are assumed to be metric and compact, and the mappings — to be continuous and surjective. A continuum means a compact connected space.

Pseudo-monotone mappings have been introduced in [20], p. 13, by L. E. Ward, Jr. Namely, we call a mapping $f\colon X \to Y$ *pseudo-monotone* if, for each pair of closed connected sets $A \subset X$ and $B \subset f(A)$, some component of $A \cap f^{-1}(B)$ is mapped by f onto B. Simple examples show that the pseudo-monotoneity of f neither implies nor is implied by its monotoneity. We describe below a monotone mapping which is not pseudo-monotone. This example will be used in further considerations.

(1.1) Example. There exists a monotone mapping f of a circle S onto itself such that f is not pseudo-monotone.

Let (r, φ) denote a point of the Euclidean plane having r and φ as its polar coordinates. Take the unit circle $S = \{(r, \varphi)\colon r = 1 \text{ and } 0 \leqslant \varphi \leqslant 2\pi\}$. We define

$$f(r, \varphi) = \begin{cases} (r, 2\varphi) & \text{if } 0 \leqslant \varphi \leqslant \pi, \\ (r, 0) & \text{if } \pi \leqslant \varphi \leqslant 2\pi. \end{cases}$$

Observe that a mapping $f\colon S \to S$ is monotone but it is not pseudo-monotone.