On the actions of SO(3) on lens space II

by

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Abstract. In this paper, an effective (smooth) action of SO(3) on a generalized lens space $L_{2n+1}(m; q_1, \ldots, q_n)$, m odd (>3), is studied via the lifting action of SO(3) on the universal covering space S^{2n+1} over the lens space.

The main result is that there is an isotropy subgroup isomorphic to a finite cyclic group contrary to the case of SO(3) actions on spheres for which there is no such subgroup. If the cohomology dimension of the singular set over Z is $2n-1$ (or less than $2n-1$), then there is a point in $L_{2n+1}(m)$ at which the isotropy subgroup is Z_m (or a subgroup of Z_m). Furthermore, if γ is the dimension of the maximal orbit then the cohomology dimension of the fixed point set over Z must be less than, $2n-\gamma$. If it is equal to $2n-\gamma-1$ over Z_2, then γ must be 3 and some isotropy subgroup is Z_m.

1. The results concerning actions of SO(3) on the sphere S^n ([6], [7]) indicate that there is no finite cyclic isotropy subgroup of SO(3). However, the authors showed in [5] that if SO(3) acts effectively on a 5-dimensional lens space $L_5(m)$, m odd, with 3-dimensional orbits, then there exists a nontrivial cyclic isotropy subgroup of SO(3). We used the lifting action of SO(3) on the universal covering space S^5.

This paper is a continuation of our paper [5]. Using the lifting action again, we study an effective action of SO(3) in an arbitrary dimensional lens space. We show that an isotropy subgroup is isomorphic to a subgroup of the fundamental group of the lens space and it is nontrivial for certain cases, (see (2.2), (3.2)). We also prove a theorem about the dimension of the fixed point set, (see (3.1)).

It is well known that the conjugacy classes of closed subgroups of the group SO(3) are the maximal torus S, the normalizer N of S in SO(3), the cyclic groups Z_n of order n, the dihedral group D_n of order $2n$, and the groups H_T, H_C, H_I of all the rotational symmetries of the tetrahedron, the cube, the icosahedron, respectively. The group H_T, H_C and H_I are maximal finite subgroups of SO(3) and the groups H_T, H_C, H_I, D_{2n}, N, SO(3) all contain $D_2 \cong Z_2 \oplus Z_2$ as a subgroup.

A lens space $L_{2n+1}(m, q_1, \ldots, q_n)$ is defined as the orbit space of a Z_m-action on the sphere $S^{2n+1} = \{z = (z_0, \ldots, z_n) \in C^{n+1} |\ |z| = 1\}$ induced by a homeomorphism $\alpha\colon S^{2n+1} \to S^{2n+1}$ defined by

$$\alpha(z) = (z_0 e^{2\pi i/m}, z_1 e^{2\pi i q_1/m}, \ldots, z_n e^{2\pi i q_n/m}),$$

where $m \geqslant 2$ and $q_1, \ldots, q_n$ are integers such that $(m, q_i) = 1$ for each i. A lens space will be denoted by $L(m)$ or $L_{2n+1}(m)$ where no confusion arises. The projection mapping $p\colon S^{2n+1} \to L(m)$ is the universal covering mapping, and the group of covering transformations is $\pi_1(L(m)) \cong Z_m$.

For an action (G, X) of a topological group G on a space X, we call $G(x) = \{gx \mid g \in G\}$ the *orbit* of $x \in X$ and $G_x = \{g \in G \mid gx = x\}$ the *isotropy subgroup* of G at $x \in X$. The orbit space of an action (G, X) is denoted by X/G.

Let $(\mathrm{SO}(3), \varnothing, L_{2n+1}(m))$, m odd, be an effective smooth action of $\mathrm{SO}(3)$ on a lens space $L_{2n+1}(m)$. Denote by U the union of all the principal orbits, by D the union of all the exceptional orbits, by B the union of all the singular orbits, and by F the set of all the fixed points of $(\mathrm{SO}(3), \varnothing, L_{2n+1}(m))$. Then U, D, and B are all $\mathrm{SO}(3)$-invariant and they are mutually disjoint. It is known that if B_k is the union of all the k-dimensional orbits, then $\dim_Z B_k \leqslant 2n+1-r+k-1$, where r is the dimension of the principal orbit. Hence $\dim_Z B \leqslant 2n-1$ and $\dim_Z F \leqslant 2n-r$ (see [1]).

LEMMA (1.1). *Any effective action* $(\mathrm{SO}(3), \varnothing, L_{2n+1}(m))$, *$m$ odd, can be lifted equivariantly to an effective action of* $(\mathrm{SO}(3), \tilde{\varnothing}, S^{2n+1})$, *i.e., the diagram*

$$\begin{array}{ccc} \mathrm{SO}(3) \times S^{2n+1} & \xrightarrow{\tilde{\varnothing}} & S^{2n+1} \\ {\scriptstyle \mathrm{id} \times p} \downarrow & & \downarrow {\scriptstyle p} \\ \mathrm{SO}(3) \times L(m) & \xrightarrow{\varnothing} & L(m) \end{array}$$

commutes, where $p\colon S^{2n+1} \to L_{2n+1}(m)$ *is the covering mapping.*

Proof. See [4], (4.3) or [5], Lemma 1.

We denoted by $\tilde{A}$ the inverse image of a subset A of $L(m)$ under the covering mapping p.

LEMMA (1.2). *In the lifted action* $(\mathrm{SO}(3), S^{2n+1})$, $\widetilde{U \cup D}$, $\tilde{B}$, *and* $\tilde{F}$ *are the union of all the highest dimensional orbits, the union of all the singular orbits, and the fixed point set in* S^{2n+1}, *respectively.*

Proof. Since $p(\mathrm{SO}(3)(y)) = \mathrm{SO}(3)(p(y))$ for each $y \in S^{2n+1}$ and $\mathrm{SO}(3)$ is connected and p is a local homeomorphism, the lemma follows immediately.

As an immediate corollary we have:

COROLLARY (1.3). $\dim_Z \tilde{B} = \dim_Z B$ *and* $\dim_Z \tilde{F} = \dim_Z F$.

If $(\mathrm{SO}(3), S^{2n+1})$ is the lifted action of $(\mathrm{SO}(3), L_{2n+1}(m))$, it can be seen easily that there is an induced action of $Z_m = \pi_1(L(m))$ on the orbit space $S^{2n+1}/\mathrm{SO}(3)$. It is given by the formula $g(y\alpha) = (gy)\alpha$ for all $g \in \mathrm{SO}(3)$, $y \in S^{2n+1}$, and $\alpha \in \pi_1(L_{2n+1}(m))$. In [4], Conner and Raymond defined a mapping $\eta_y\colon \mathrm{SO}(3)_{p(y)} \to \pi_1(L(m))$ for each $y \in S^{2n+1}$ and proved the following lemma:

LEMMA (1.4) (Conner and Raymond [4]). *For each $y \in S^{2n+1}$, there is an exact sequence*

$$e \to \mathrm{SO}(3)_y \xrightarrow{i} \mathrm{SO}(3)_x \xrightarrow{\eta_y} \pi_1(L(m))_{\bar{p}(y)} \to 0\,,$$

where i is the inclusion and $x = p(y)$ and $\bar{p}\colon S^{2n+1} \to S^{2n+1}/\mathrm{SO}(3)$ *is the projection.*

First we prove the following theorem:

THEOREM (1.5). *Let* $(\mathrm{SO}(3), L_{2n+1}(m))$, *$m$ odd, be an effective action of* $\mathrm{SO}(3)$ *on* $L_{2n+1}(m)$. *If the principal isotropy subgroup* $\mathrm{SO}(3)_x$ *is a finite cyclic group, then* $\mathrm{SO}(3)_x$ *is isomorphic to a subgroup of* $\pi_1(L(m)) \cong Z_m$.

Proof. Let S_u be the set of all principal orbits of the lifted action $(\mathrm{SO}(3), S^{2n+1})$. Then S_u is an open dense subset of S^{2n+1}. Therefore, $p(S_u) \subset L(m)$ is an open dense subset of $L_{2n+1}(m)$. This implies that $p(S_u) \cap U \neq \varnothing$. Thus there are points $x \in L(m)$ and $y \in p^{-1}(x)$ such that both $\mathrm{SO}(3)_y$ and $\mathrm{SO}(3)_x$ are principal isotropy subgroups. Then by Lemma (1.4), we have an exact sequence

$$e \to \mathrm{SO}(3)_y \to \mathrm{SO}(3)_x \to (Z_m)_{\bar{p}(y)} \to 0$$

where $p(y) = x$ and $\mathrm{SO}(3)_y$ and $\mathrm{SO}(3)_x$ are principal isotropy subgroups. Since $\mathrm{SO}(3)_x$ is a finite cyclic group, $\mathrm{SO}(3)_y$ is a finite cyclic principal isotropy subgroup. Then by Theorem 1 of [6], $\mathrm{SO}(3)_y$ must be a trivial group. Hence $\mathrm{SO}(3)_x \cong (Z_m)_{\bar{p}(y)}$.

Note that if $(Z_m, S^{2n+1}/\mathrm{SO}(3))$ has a fixed point in $S^{2n+1}/\mathrm{SO}(3)$ that corresponds to a principal orbit of $(\mathrm{SO}(3), S^{2n+1})$, then $\mathrm{SO}(3)_x \cong Z_m$, i.e., it is a nontrivial cyclic group. ($\mathrm{SO}(3)_x$ may not be the principal isotropy subgroup.)

2. We know that $\dim_Z B \leqslant 2n-1$. Suppose that the dimension of B over Z is the highest possible dimension, i.e., $\dim_Z B = 2n-1$. Then the principal orbit of $(\mathrm{SO}(3), L_{2n+1}(m))$ has dimensions 3 for if $\dim_Z \mathrm{SO}(3)(x) = 2$, $x \in U$, then $B = F$ since there is no 1-dimensional orbit. This implies that $\dim_Z B = \dim_Z F \leqslant 2n-2$. This contradicts the assumption about the dimension of B. There is a 2-dimensional orbit and $n \geqslant 2$ since $\dim_Z B_2 = 2n-1$, where B_2 is the set of all the 2-dimensional orbits. Therefore $B_2 \neq \varnothing$ and $\dim_Z B = 2n-1 \geqslant 2$, i.e., $n \geqslant 2$. Moreover, a 2-dimensional orbit is either a sphere S^2 or a projective space p^2. If $\mathrm{SO}(3)(z)$ is 2-dimensional, then there is a $g \in \mathrm{SO}(3)$ such that $\mathrm{SO}(3)_{gz} \supset \mathrm{SO}(3)_x$, $x \in U$. Hence the principal isotropy subgroup is either cyclic or dihedral. Exactly the same things can be said about the lifted action $(\mathrm{SO}(3), S^{2n+1})$ of the action $(\mathrm{SO}(3), L_{2n+1}(m))$.

LEMMA (2.1). *Let* $(\mathrm{SO}(3), L_{2n+1}(m))$, *$m$ odd, be an effective action of* $\mathrm{SO}(3)$ *on* $L_{2n+1}(m)$ *with* $\dim_Z B = 2n-1$. *Then the lifted action* $(\mathrm{SO}(3), S^{2n+1})$ *has 2- and 3-dimensional orbits,* $n \geqslant 2$ *and* $\dim_Z \tilde{B} = 2n-1$. *Furthermore, 2-dimensional orbits are either* S^2 *or* P^2 *and the principal isotropy subgroup is either trivial or dihedral. The principal isotropy subgroup can be a dihedral group only when the induced action* $(Z_m, S^{2n+1}/\mathrm{SO}(3))$ *has an isotropy subgroup isomorphic to* Z_3 *that corresponds to principal orbits of* $(\mathrm{SO}(3), S^{2n+1})$.

Proof. All that remains is to prove the last part of the lemma about the principal isotropy subgroup. We know that the principal isotropy subgroup is either cyclic or dihedral. However by [7] it is actually either a dihedral group of order 4, D_2, or the trivial group. Suppose the principal isotropy subgroup is a dihedral group D_2, then by (1.4) $\mathrm{SO}(3)_x/D_2$ is isomorphic to $(Z_m)_{\bar{p}(y)}$, $p(y) = x$, which is a cyclic group of odd order. But H_T is the only subgroup of $\mathrm{SO}(3)$ that contains D_2 as a normal subgroup and the quotient group has an odd order. Since $H_T/D_2 \cong Z_3$, this is impossible unless the action $(Z_m, S^{2n+1}/SO(3))$ has an isotropy subgroup isomorphic to Z_3. In this case $SO(3)(x)$ may not be a principal orbit.

In fact, we have the following:

THEOREM (2.2). *Let* $(SO(3), L_{2n+1}(m))$, *m odd* (>3), *be an effective action of* $SO(3)$ *on* $L_{2n+1}(m)$ *with* $\dim_Z B = 2n-1$. *Then* $n \geqslant 2$, *and there is a point* $x \in L_{2n+1}(m)$ *at which the isotropy subgroup is* Z_m. *The fixed point set* F *is the orbit space of a free* Z_m*-action on a cohomology* $(2(n-3)+1)$*-sphere over* Z_2.

Proof. The fixed point set $\tilde{F}$ of the lifted action $(SO(3), S^{2n+1})$ is either a cohomology $(2(n-2))$-sphere over Z_2 and the principal isotropy subgroup is D_2, or a cohomology $(2(n-3)+1)$-sphere over Z_2 and the principal isotropy subgroup is trivial (by a theorem of Montgomery and Yang [7]). We know that $Z_m = \pi_1(L(m))$ acts freely on S^{2n+1} and $\tilde{F}$ is an invariant subset of the Z_m-action. Furthermore, the same theorem in [7] says that $S^{2n+1}/SO(3)$ is a compact Hausdorff space which is cohomologically trivial over Z. Hence $(Z_m, S^{2n+1}/SO(3))$ has a fixed point, and we may take a fixed point $\bar{p}(y)$ in the interior of $S^{2n+1}/SO(3)$, $y \in S^{2n+1}$, that corresponds to a principal orbit $SO(3)(y)$ in S^{2n+1}. If the principal isotropy subgroup $SO(3)_y$ is D_2 then $SO(3)_{p(y)}/D_2 \cong (Z_m)_{\bar{p}(y)} = Z_m$, $m>3$. Hence by Lemma (2.1), D_2 cannot be the principal isotropy subgroup. Therefore the principal isotropy subgroup of the action $(SO(3), S^{2n+1})$ is trivial and the fixed point set $\tilde{F}$ is a cohomology $(2(n-3)+1)$-sphere over Z_2. Hence $F = \tilde{F}/Z_m$. Furthermore, (1.4) implies $SO(3)_x \cong Z_m$, $p(y) = x \in L_{2n+1}(m)$.

Now suppose that $\dim_Z B<2n-1$.

THEOREM (2.3). *Let* $(SO(3), L_{2n+1}(m))$, *m odd, be an effective smooth action of* $SO(3)$ *on* $L_{2n+1}(m)$. *Suppose there is a 3-dimensional orbit and* $\dim_Z B<2n-1$. *Then the principal isotropy subgroup* $SO(3)_x$ *is subgroup of* Z_m.

Proof. There is a 3-dimensional orbit and $\dim_Z \tilde{B}<2n-1$ in the lifted action $(SO(3), S^{2n+1})$. By Theorem 2 of [6] the principal isotropy subgroup of the action $(SO(3), S^{2n+1})$ is trivial. Therefore by Lemma (1.4) $SO(3)_x \cong (Z_m)_{\bar{p}(y)}$, $y \in p^{-1}(x)$, $x \in U$.

3. In this section, we consider the dimension of the fixed point set F. It is known that $\dim_Z F \leqslant 2n-r$, where r is the dimension of the principal orbit. We first show that $\dim_Z F$ cannot be equal to $2n-r$.

THEOREM (3.1). *Let* $(SO(3), L_{2n+1}(m))$, *m odd, be an effective smooth action of* $SO(3)$ *on* $L_{2n+1}(m)$. *Then* $\dim_Z F<2n-r$.

Proof. Suppose $r=3$ and $\dim_Z F = 2n-3$. Then the lifted action $(SO(3), S^{2n+1})$ has a 3-dimensional orbit and $\dim_Z \tilde{F} = 2n-3$, the highest possible dimension of $\tilde{F}$. Then by Bredon ([1], p. 197), $H^*(SO(3)/SO(3)_y) \cong H^*(S^3)$, where $SO(3)_y$ is the principal isotropy subgroup, and there are exactly two orbit types in the action $(SO(3), S^{2n+1})$. Therefore $\tilde{F} = \tilde{B}$. Then $\dim_Z \tilde{B} = 2n-3<2n-1$. Hence by Theorem 2 of [6] the principal isotropy subgroup is trivial. This is a contradiction since the only subgroup H of $SO(3)$ such that $H^*(SO(3)/H) \cong H^*(S^3)$ is H_I, the group of all the rotational symmetries of the icosahedron. (Note that in this case $S^{2n+1}/SO(3)$ is a cohomology $(2n-2)$-cell over Z, and $(Z_m, S^{2n+1}/SO(3))$ has a fixed point. This will also lead to a contradiction.)

Now suppose $r = 2$ and $\dim_Z F = 2n-2$. Then $\dim_Z \tilde{F} = 2n-2$. Then $\dim_Z \tilde{F} = 2n-2$, the highest possible dimension of $\tilde{F}$ in the action $(SO(3), S^{2n+1})$. By Bredon again, $H^*(SO(3)/SO(3)_y) \cong H^*(S^2)$, where $SO(3)_y$ is the principal isotropy subgroup. Therefore, $SO(3)_y$ is the maximal torus S of $SO(3)$. Then the exact sequence of (1.4) says that $SO(3)_x$ is either S or N or $SO(3)$ itself. Since we may assume $x \in U \cup D$, $SO(3)_x \neq SO(3)$. Therefore $SO(3)_x/S = 0$ or Z_2 which is isomorphic to $(Z_m)_{\bar{p}(y)}$. Since $S^{2n+1}/SO(3)$ is compact cohomology $(2n-1)$-cell over Z, the action $(Z_m, S^{2n+1}/SO(3))$ has a fixed point $\bar{p}(y) \in S^{2n+1}/SO(3)$ such that $SO(3)(y)$ is a principal orbit. By taking this fixed point, we have $(Z_m)_{\bar{p}(y)} \cong Z_m$, a contradiction.

In the next theorem, we consider the case when $\dim_{Z_2} F = 2n-r-1$.

THEOREM (3.2). *Let* $(SO(3), L_{2n+1}(m))$, *m odd* (>3), *be an effective smooth action of* $SO(3)$ *on* $L_{2n+1}(m)$. *If* $\dim_{Z_2} F = 2n-r-1$, *then* $r = 3$ *and there is a point* $x \in L_{2n+1}(m)$ *at which the isotropy subgroup is* Z_m.

Proof. Let $(SO(3), S^{2n+1})$ be the lifted action of $(SO(3), L_{2n+1}(m))$. Then by Bredon [3], $S^{2n+1}/SO(3)$ is a compact cohomology $(2n+1-r)$-cell with $\partial(S^{2n+1}/SO(3)) = \tilde{B}/SO(3)$, and there are no exceptional orbits. Therefore, $(Z_m, S^{2n+1}/SO(3))$ has a fixed point in $S^{2n+1}/SO(3)$ that corresponds to a principal orbit of $(SO(3), S^{2n+1})$. So we have an exact sequence

$$e \to SO(3)_y \to SO(3)_x \to Z_m \to 0\,, \quad y \in p^{-1}(x)\,,$$

$SO(3)_y$ is the principal isotropy subgroup. If $r = 2$, then $SO(3)_y = S$ or N. Therefore $SO(3)_x$ must be S or N. Then $SO(3)_x/SO(3)_y \cong 0$ or Z_2. This contradicts $m \neq 2$. Therefore the principal orbits are 3-dimensional and $\dim_{Z_2} \tilde{F} = 2n-4$. We claim that $\dim_Z \tilde{B} \neq 2n-1$. If $\dim_Z \tilde{B} = 2n-1$, then by [7], $\tilde{F}$ is either a cohomology $(2n-4)$-sphere over Z_2 and the principal isotropy subgroup is D_2, or a cohomology $(2n-5)$-sphere over Z_2 and the principal isotropy subgroup is trivial. For the dimensional reason $\tilde{F}$ must be a cohomology $(2n-4)$-sphere over Z_2 with the principal isotropy subgroup D_2. However if the principal isotropy subgroup is D_2, then $SO(3)_x$ must be one of the groups H_I, H_C, H_T, D_k. Since $SO(3)_x/D_2 \cong Z_m$, $m =$ odd (>3), none of these can be $SO(3)_x$. Therefore $\dim_Z \tilde{B}<2n-1$. Then, by Montgomery and Samelson [6], the principal isotropy subgroup $SO(3)_y$ is trivial. Hence we have $SO(3)_x \cong Z_m$, $x = p(y) \in L_{2n+1}(m)$.

COROLLARY (3.3). *If* $SO(3)$ *acts effectively on* $L_5(m)$, *m odd* (>3), *then the action has no fixed points.*

Proof. By Theorem 3 of [5] if the fixed point set $F(SO(3), L_5(m))$ is nonempty, then it is a circle. However by Theorem (3.1) $\dim_Z F<4-r$. Therefore $r = 2$ is the only choice. This contradicts Theorem (3.2) since if $\dim_{Z_2} F = 1 = 4-r-1$, then $r = 3$.

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The axiom of choice and linearly ordered sets

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Abstract. Let ZF be Zermelo-Fraenkel set theory without the axiom of choice and let ZF^0 be the modification which allows urelements or atoms. In this paper we show that there are many forms of the axiom of choice and the axiom of multiple choice involving linearly ordered sets which are equivalent to the axiom of choice in ZF but not in ZF^0. The independence proofs use permutation models of Fraenkel–Mostowski.

§ 1. **Introduction.** The variants of the axiom of choice which we consider in this paper are listed below.

The statements

A: *Antichain Principle.* Every partially ordered set contains a maximal antichain. (I.e. a maximal subset of mutually incomparable elements.)

AC: *Axiom of Choice.* For every family x of non-empty sets, there is a function f such that for each $u \in x$, $f(u) \in u$.

AC^{LO}: Axiom of choice for a linearly ordered family of non-empty sets.

AC_{LO}: Axiom of choice for a family of non-empty sets each of which can be linearly ordered.
$(\forall x)[(\forall u \in x)(\exists R_u)$ (R_u linearly orders u) $\to$ AC holds for x].

AC_{DLO}: Axiom of choice for a family of non-empty sets, each of which has a defined linear ordering.
$(\forall x)[(\exists R)$ $(\forall u \in x)$ (R_u linearly orders u) $\to$ AC holds for x].

AC^{LO}_{LO}: Axiom of choice for a linearly ordered family of non-empty sets, each of which can be linearly ordered.

AC^{LO}_{DLO}: Axiom of choice for a linearly ordered family of non-empty sets, each of which has a defined linear ordering.

LW: Every linearly ordered set can be well ordered.

MC: *Axiom of Multiple Choice.* For every family x of non-empty sets, there is a function f such that for each $u \in x$, $f(u)$ is a non-empty, finite subset of u.

MC^{LO}: Axiom of multiple choice for a linearly ordered family of non-empty sets.