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Symmetric words in nilpotent groups of class $\leqslant 3$

by

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Abstract. Let G be a group. A word $w = w(x_1, \ldots, x_n)$ is said to be *symmetric* if $w(a_1, \ldots, a_n) = w(a_{\pi 1}, \ldots, a_{\pi n})$ for all $a_1, \ldots, a_n \in G$ and all permutations π from the symmetric group S_n on n-letters. In this note we describe symmetric words in nilpotent groups of class $\leqslant 3$.

1. Introduction and notation. Let G be a group, and let $F_G(x_1, \ldots, x_r)$ be the group freely generated by $x_1, \ldots, x_r$ in the smallest variety $\mathrm{var}(G)$ of groups containg G. Let A be the group of automorphisms of $F_G(x_1, \ldots, x_r)$ induced by the mappings

$$x_i \to x_{\mu i}, \quad 1 \leqslant i \leqslant r,$$

μ belonging to the symmetric group S_r on r letters. Let $S^{(r)}(G)$ be the set of all fixed points of A, i.e.,

$$S^{(r)}(G) = \{w\colon \xi w = w \text{ for all } \xi \in A\}.$$

The elements of $S^{(r)}(G)$ are called *symmetric words* (of r variables) in G.

Clearly, $S^{(r)}(G)$ is a group. The aim of this note is to describe symmetric words in nilpotent groups of class $\leqslant 3$. We prove that in this case $S^{(r)}(G)$ is Abelian.

2. Symmetric words. In an Abelian group every word w of r variables is of the form

$$w = \prod_{1 \leqslant i \leqslant r} x_i^{a(i)}.$$

We thus have

THEOREM 1. *If G is an Abelian group, then $w \in S^{(r)}(G)$ if and only if*

$$w = \prod_{1 \leqslant i \leqslant r} x_i^a.$$

In [3] all elements of $S^{(r)}(G)$ for a nilpotent G of class 2 are described. Namely

THEOREM 2. *If G is a nilpotent group of class 2, then $w \in S^{(r)}(C)$ if and only if*

$$w = \prod_{1 \leqslant i \leqslant r} x_i^a \prod_{1 \leqslant j < i \leqslant r} [x_i, x_j]^b,$$

where the integers a, b satisfy the condition

$$a^2 \equiv 2b(\exp G').$$

($\exp G'$ *means the exponent of the group of commutators of* G.)

To determine the symmetric words in a nilpotent group of class 3 we need some well-known identities:

$$(*)\qquad \begin{aligned} &[x, y, z][y, z, x][z, x, y] = 1,\\ &[x, y, z] = [y, x, z]^{-1},\\ &[x^n, y, z] = [x, y^n, z] = [x, y, z^n] = [x, y, z]^n,\\ &[y^n, x^m] = [y, x]^{mn}[y, x, x]^{\frac{1}{2}mn(m-1)}[y, x, y]^{\frac{1}{2}mn(n-1)} \end{aligned}$$

valid in an arbitrary nilpotent group of class 3 for all integers n, m. Let $C(m, n, p, q)$ be the class of all nilpotent groups of class $\leqslant 3$ which satisfy the identities

$$x^m = 1, \quad [y, x]^n = 1, \quad [y, x, z]^p = 1, \quad [y, x, x]^q = 1.$$

In [2] B. Jónsson has proved the following

THEOREM (B. Jónsson). *There is a 1-1 correspondence between all the varieties of nilpotent groups of class $\leqslant 3$ and the quadruples (m, n, p, q) satisfying the conditions*

$$n\cdot\gcd(2, m)\,|\,m,\ \ p\,|\,n,\ \ q\,|\,p,\ \ q\cdot\gcd(m, 6)\,|\,m,\ \ p\,|\,3q.$$

The following lemma readily follows from [2].

LEMMA. *Let G be a nilpotent group of class 3 with $\mathrm{var}(G) = C(m, n, p, q)$. Then for all natural numbers r the identity in G*

$$\prod_{1\leqslant i\leqslant r} x_i^{a(i)} \prod_{1\leqslant j<i\leqslant r} [x_i, x_j]^{b(i,j)}[x_i, x_j, x_j]^{c(i,j)}[x_i, x_j, x_i]^{c'(i,j)} \prod_{\substack{1\leqslant j<i\leqslant r\\ j<k\leqslant r}} [x_i, x_j, x_k]^{d(i,j,k)}$$
$$= \prod_{1\leqslant i\leqslant r} x_i^{\bar a(i)} \prod_{1\leqslant j<i\leqslant r} [x_i, x_j]^{\bar b(i,j)}[x_i, x_j, x_j]^{\bar c(i,j)}[x_i, x_j, x_i]^{\bar c'(i,j)} \prod_{\substack{1\leqslant j<i\leqslant r\\ j<k\leqslant r}} [x_i, x_j, x_k]^{\bar d(i,j,k)}$$

is equivalent to the conditions

$$\begin{aligned} a(i) &\equiv \bar a(i)\ (m), && 1\leqslant i\leqslant r,\\ b(i,j) &\equiv \bar b(i,j)\ (n), && 1\leqslant j<i\leqslant r,\\ c(i,j) &\equiv \bar c(i,j)\ (q), && 1\leqslant j<i\leqslant r,\\ c'(i,j) &\equiv \bar c'(i,j)\ (q), && 1\leqslant j<i\leqslant r,\\ d(i,j,k) &\equiv \bar d(i,j,k)\ (q), && 1\leqslant j<i\leqslant r,\ j<k\leqslant r,\\ d(i,j,k)-d(k,j,i) &\equiv \bar d(i,j,k)-\bar d(k,j,i)\ (p), && 1\leqslant j<i<k\leqslant r. \end{aligned}$$

We start with

THEOREM 3. *If G is a nilpotent group of class 3 with $\mathrm{var}(G) = C(m, n, p, q)$, then $w \in S^{(2)}(G)$ if and only if*

$$w = x^a y^a [y, x]^b [y, x, x]^c [y, x, y]^{c'},$$

and

$$a^2 \equiv 2b\ (n), \qquad c+c' \equiv \tfrac{1}{2}a^2(a-1)\ (q).$$

Proof. Let

$$w = x^{a(1)}y^{a(2)}[y, x]^b[y, x, x]^c[y, x, y]^{c'}$$

be a word in G. Using (*), we calculate

$$\begin{aligned} w(y, x) &= y^{a(1)}x^{a(2)}[x, y]^b[x, y, y]^c[x, y, y]^{c'}\\ &= x^{a(2)}y^{a(1)}[y, x]^{-b+a(1)a(2)}[y, x, x]^{-c'+\frac{1}{2}a(1)a(2)(a(2)-1)}\times\\ &\quad\times[y, x, y]^{-c+\frac{1}{2}a(1)a(2)(a(1)-1)}. \end{aligned}$$

Hence, by the lemma, we infer that w is symmetric if and only if

$$\begin{aligned} a(1) &\equiv a(2)\ (\equiv a)\ (m),\\ a^2 &\equiv 2b\ (n),\\ c+c' &\equiv \tfrac{1}{2}a^2(a-1) \end{aligned}$$

as required.

THEOREM 4. *If G is a nilpotent group of class 3 and $\mathrm{var}(G) = C(m, n, p, q)$, then for every $r\geqslant 3$, $w \in S^{(r)}$ if and only if w is of the form*

$$(1)\qquad w = \prod_{1\leqslant i\leqslant r} x_i^a \prod_{1\leqslant j<i\leqslant r} [x_i, x_j]^b[x_i, x_j, x_j]^c[x_i, x_j, x_i]^{c'} \prod_{\substack{1\leqslant j<i\leqslant r\\ j<k\leqslant r}} [x_i, x_j, x_k]^d$$

and the three congruences

$$\begin{aligned} a^2 &\equiv 2b\ (n),\\ c+c' &\equiv \tfrac{1}{2}a^2(a-1)\ (q),\\ a^3 &\equiv 3d\ (p). \end{aligned}$$

are fulfilled.

Proof. Let w be a word in G. Then (cf. e.g. [2])

$$w = \prod_{1\leqslant i\leqslant r} x_i^{a(i)} \prod_{1\leqslant j<i\leqslant r} [x_i, x_j]^{b(i,j)}[x_i, x_j, x_j]^{c(i,j)}[x_i, x_j, x_i]^{c'(i,j)} \prod_{\substack{1\leqslant j<i\leqslant r\\ j<k\leqslant r}} [x_i, x_j, x_k]^{d(i,j,k)}.$$

Let α, β, γ be integers satisfing $1\leqslant\alpha<\beta<\gamma\leqslant r$. We define

$$u(x_\alpha, x_\beta, x_\gamma) = w(1, \ldots, 1, x_\alpha, 1, \ldots, 1, x_\beta, 1, \ldots, 1, x_\gamma, 1, \ldots, 1).$$

If the word w is symmetric, then the words $u(x, y, 1), u(x, 1, y), u(1, x, y)$ are also symmetric and, of course, the equalities

$$u(x, y, 1) = u(x, 1, y) = u(1, x, y)$$

hold. This together with the lemma and Theorem 3 yields

$$\begin{aligned} a(\alpha) &\equiv a(\beta) \equiv a(\gamma)\ (\equiv a)\ (m),\\ b(\beta, \alpha) &\equiv b(\gamma, \alpha) \equiv b(\gamma, \beta)\ (\equiv b)\ (n),\\ c(\beta, \alpha) &\equiv c(\gamma, \alpha) \equiv c(\gamma, \beta)\ (\equiv c)\ (q),\\ c'(\beta, \alpha) &\equiv c'(\gamma, \alpha) \equiv c'(\gamma, \beta)\ (\equiv c')\ (q). \end{aligned}$$

Since α, β, γ have been arbitrary, we can assume that the word w is of the form (1) with $[x_i, x_j, x_k]^{d(i,j,k)}$ instead of $[x_i, x_j, x_k]^d$.

Clearly w is symmetric if and only if

$$w(x_1, \dots, x_r) = w(x_2, x_1, x_3, \dots, x_r) = w(x_2, \dots, x_r, x_1)$$

because the cycles $(1, 2)$ and $(1, 2, \dots, r)$ generates the group S_r.

We have

$$\begin{aligned} w(x_2, x_1, x_3, \dots, x_r) = {} & x_2^a x_1^a \prod_{3\leqslant i\leqslant r} x_i^a [x_1, x_2]^b \\ & \prod_{2\leqslant j<i\leqslant r} [x_i, x_j]^b [x_1, x_2, x_1]^c [x_1, x_2, x_2]^{c'} [x_i, x_j, x_j]^c [x_i, x_j, x_i]^{c'} \\ & \prod_{3\leqslant i\leqslant r} [x_1, x_2, x_i]^{d(2,1,i)} [x_i, x_2, x_1]^{d(i,1,2)} \\ & \prod_{\substack{3\leqslant i,j\leqslant r\\ i\neq j}} [x_i, x_1, x_j]^{d(i,2,j)} [x_i, x_2, x_j]^{d(i,1,j)} \\ & \prod_{\substack{3\leqslant j<i\leqslant r\\ j<k\leqslant r}} [x_i, x_j, x_k]^{d(i,j,k)} \end{aligned}$$

$$\begin{aligned} = {} & \prod_{1\leqslant i\leqslant r} x_i^a [x_2, x_1]^{a^2-b} \\ & \prod_{2\leqslant j<i\leqslant r} [x_i, x_j]^b [x_2, x_1, x_1]^{-c'+\frac{1}{2}a^2(a-1)} [x_2, x_1, x_2]^{-c+\frac{1}{2}a^2(a-1)} [x_i, x_j, x_j]^c [x_i, x_j, x_i]^{c'} \\ & \prod_{3\leqslant i\leqslant r} [x_2, x_1, x]^{-d(2,1,i)-d(i,1,2)+a^3} [x_i, x_1, x_2]^{d(i,1,2)} \\ & \prod_{\substack{3\leqslant i,j\leqslant r\\ i\neq j}} [x_i, x_1, x_j]^{d(i,2,j)} [x_i, x_2, x_j]^{d(i,1,j)} \\ & \prod_{\substack{3\leqslant j<i\leqslant r\\ j<k\leqslant r}} [x_i, x_j, x_k]^{d(i,j,k)}, \end{aligned}$$

and likewise

$$\begin{aligned} w(x_2, x_3, \dots, x_r, x_1) = {} & x_2^a \dots x_r^a x_1^a \prod_{2\leqslant i\leqslant r} [x_1, x_i]^b \prod_{2\leqslant j<i\leqslant r} [x_i, x_j]^b \\ & \prod_{2\leqslant i\leqslant r} [x_1, x_i, x_i]^c [x_1, x_i, x_1]^{c'} \\ & \prod_{2\leqslant j<i\leqslant r} [x_i, x_j, x_j]^c [x_i, x_j, x_i]^{c'} \\ & \prod_{1\leqslant j<i<r} [x_{i+1}, x_{j+1}, x_1]^{d(i,j,n)} [x_1, x_{j+1}, x_{i+1}]^{d(n,j,i)} \\ & \prod_{\substack{1\leqslant j<i<r\\ j<k<r}} [x_{i+1}, x_{j+1}, x_{k+1}]^{d(i,j,k)} \end{aligned}$$

$$\begin{aligned} = {} & \prod_{1\leqslant i\leqslant r} x_i^a \prod_{2\leqslant i\leqslant r} [x_i, x_1]^{-b+a^2} \prod_{2\leqslant j<i\leqslant r} [x_i, x_j]^b \\ & \prod_{2\leqslant i\leqslant r} [x_i, x_1, x_1]^{-c'+\frac{1}{2}a^2(a-1)} [x_i, x_1, x_i]^{-c+\frac{1}{2}a^2(a-1)} \\ & \prod_{2\leqslant j<i\leqslant r} [x_i, x_j, x_j]^c [x_i, x_j, x_i]^{c'} \\ & \prod_{1\leqslant j<i<r} [x_{i+1}, x_1, x_{j+1}]^{d(i,j,n)} [x_{j+1}, x_1, x_{i+1}]^{-d(n,j,i)-d(i,j,n)+a^3} \\ & \prod_{\substack{1\leqslant j<i<r\\ j<k<r}} [x_{i+1}, x_{j+1}, x_{k+1}]^{d(i,j,k)}. \end{aligned}$$

This together with the lemma gives

$$a^2 \equiv 2b \ (n),$$

$$c+c' \equiv \tfrac{1}{2}a^2(a-1) \ (q),$$

(2) $$a^3 \equiv 2d(2,1,i)+d(i,1,2) \ (p), \quad 3\leqslant i\leqslant r,$$

(3) $$d(i,2,j) \equiv d(i,1,j) \ (q), \quad 3\leqslant i,j\leqslant r, \ i\neq j,$$

(4) $$a^3 \equiv 2d(i,j,r)+d(r,j,i)+d(j+1,1,i+1)-d(i+1,1,j+1) \ (p), \quad 1\leqslant j<i<r,$$

(5) $$d(i,j,r) \equiv d(i+1,1,j+1) \ (q), \quad 1\leqslant j<i<r,$$

(6) $$a^3 \equiv d(r,j,i)+d(i,j,r)+d(j+1,1,i+1) \ (q), \quad 1\leqslant j<i<r,$$

(7) $$d(i+1,j+1,k+1) \equiv d(i,j,k) \ (q), \quad 1\leqslant j<i<r, \ j<k<r.$$

We are now going to show that for all $r\geqslant 3$ these congruences imply the equality of all $d(i,j,k)$ modulo q. This is done by induction on r.

For $r = 3$ this is obvious. Suppose it holds for $r-1$ $(r\geqslant 4)$. Then consider the integers $d(i,j,r)$ where $1\leqslant j<i<r$.

If $j>1$, then by (7) we have $d(i-1,j-1,r-1)\equiv d(i,j,r)$ (q).

If $j = 1$, $i<r-1$, then by (5) we have $d(i,1,r) \equiv d(i+1,1,2)$ (q).

If $j = 1$, $i = r-1$, then using (3) and (7) we obtain $d(r-1,1,r) \equiv d(r-2,1,r-1)$ (q).

It is enough to apply (2) and (4) for $r = 3$ to get

$$d(3,1,2) \equiv d(2,1,3) \ (p),$$

$$a^3 \equiv 3d(3,1,2) \ (p).$$

Suppose now that for $r-1$ $(r\geqslant 4)$ the congruences (2)-(7) imply

$$d(k,j,i) \equiv d(i,j,k) \ (p), \quad 1\leqslant j<i<k\leqslant r-1,$$

$$a^3 \equiv 3d(k,j,i) \ (p), \quad 1\leqslant j<i<k\leqslant r-1.$$

If $i\leqslant r-2$, then it follows from (4) that

$$a^3 \equiv 2d(i,j,r)+d(r,j,i) \ (p),$$

since, in view of the induction hypothesis, we have

$$d(j+1,1,i+1) \equiv d(i+1,1,j+1) \ (p).$$

But $p|3q$, and therefore $3d(i,j,r) \equiv 3d(3,1,2) \equiv a^3$ (p). This gives

$$d(i,j,r) \equiv d(r,j,i) \ (p), \quad 1 \leqslant j < i \leqslant r-2. \tag{8}$$

If $i = r-1$, $j \leqslant r-3$, then combining (8) and (4) we get

$$a^3 \equiv 2d(r-1,j,r)+d(r,j,r-1) \ (p).$$

Consequently

$$d(r-1,j,r) \equiv d(r,j,r-1) \ (p), \quad 1 \leqslant j \leqslant r-3. \tag{9}$$

The congruence $d(r,r-2,r-1) \equiv d(r-1,r-2,r)$ follows now from the one above and (4).

We have thus proved that the integers $d(i,j,k)$ are equal modulo q, and that

$$3d(i,j,k) \equiv a^3 \ (p), \quad 1 \leqslant j < i < k \leqslant r,$$
$$d(i,j,k) \equiv d(k,j,i) \ (p), \quad 1 \leqslant j < i < k \leqslant r.$$

It follows from the lemma that the identities

$$[x_i, x_j, x_k]^q [x_k, x_j, x_i]^q \equiv 1, \quad 1 \leqslant j < i < k \leqslant r$$

hold in G. This and the fact that $p|3q$ complete the result.

3. Homomorphisms ∂_r^{r+1}. If G is a group, then the mapping $\partial_r^{r+1}\colon S^{(r+1)} \to S^{(r)}$ defined by

$$(\partial_r^{r+1} w)(x_1, \ldots, x_{r+1}) = w(x_1, \ldots, x_r, 1)$$

is a homomorphism. From Theorems 1 and 2 follows

COROLLARY. *If G is a nilpotent group of class $\leqslant 3$, then for all $r \geqslant k$ the mapping ∂_r^{r+1} is an isomorphism.*

Let us consider the following two examples:

EXAMPLE 1. Let Q be the eight-element group of quaternions. Since $m(Q) = 4$, $n(Q) = 2$, $p(Q) = q(Q) = 0$, we get from Theorem 2

$$S^{(1)} = \{1, x, x^2, x^3\}, \quad S^{(2)} = \{1, x^2y^2, [y, x], x^2y^2[y, x]\}$$

and therefore $\partial_1^2(S^{(2)}(Q)) \neq S^{(1)}(Q)$.

EXAMPLE 2. By Jónsson's theorem there exists a nilpotent group G of class 3 for which $m = 18$, $n = p = 9$, $q = 3$. As is easy to verify, the word $w = xy[y, x]$ is symmetric in G, while there is in G no symmetric word of 3 variables which would be of the form $xyz\ldots$, because the congruence $3d \equiv 1$ (9) does not have a solution, thus $w \notin \partial_2^3(S^{(3)}(G))$.

These examples show that the assumption $r \geqslant k$ is indispensable. Theorem 4 and the corollary deduced from it depend very heavily on Jónsson's theorem. Since it seems that there is nothing like Jónsson's theorem for a nilpotent group of class $\geqslant 4$, the following problem requires another method.

PROBLEM. Let G be a nilpotent group of class k. Is the mapping ∂_r^{r+1} an isomorphism for every $r \geqslant k$?

In the case of a free nilpotent group, this question is answered in the affirmative in [4].

4. The groups $S^{(r)}(G)$.

THEOREM 5. *If G is a nilpotent group of class $\leqslant 3$, then for every r the groups $S^{(r)}(G)$ are Abelian.*

Proof. Let G be a nilpotent group of class 2, and $w_1, w_2 \in S^{(2)}(G)$. In view of Theorem 2 we have

$$w_1 w_2 = x^{a_1}y^{a_1}[y, x]^{b_1}x^{a_2}y^{a_2}[y, x]^{b_2} = x^{a_1+a_2}y^{a_1+a_2}[y, x]^{b_1+b_2+a_1a_2}.$$

Since $b_1+b_2+a_1a_2$ is invariant under transposition a_1, a_2 and b_1, b_2, we see that the group $S^{(2)}(G)$ is Abelian.

For symmetric words w_1, w_2 of three variables in a nilpotent group G of class 3 we get

$$\begin{aligned} w_1 w_2 \equiv\ & x^{a_1}y^{a_1}z^{a_1}[y, x]^{b_1}[z, x]^{b_1}[z, y]^{b_1}[y, x, x]^{c_1}[y, x, y]^{c_1'}[z, x, x]^{c_1}[z, x, z]^{c_1'} \\ & [z, y, y]^{c_1}[z, y, z]^{c_1'}[y, x, z]^{d_1}[z, x, y]^{d_1}x^{a_2}y^{a_2}z^{a_2}[y, x]^{b_2}[z, x]^{b_2}[z, y]^{b_2} \\ & [y, x, x]^{c_2}[y, x, y]^{c_2'}[z, x, x]^{c_2}[z, x, z]^{c_2'}[z, y, y]^{c_2}[z, y, z]^{c_2'}[y, x, z]^{d_2} \\ & [z, x, y]^{d_2} \\ =\ & x^{a_1+a_2} \ldots [y, x]^{b_1+b_2+a_1a_2} \ldots [y, x, x]^{c_1+c_2+b_1a_2+\frac{1}{2}a_1a_2(a_2-1)} \ldots \\ & \ldots [y, x, y]^{c_1'+c_2'+a_2b_1+a_1a_2^2+\frac{1}{2}a_1a_2(a_1-1)} \ldots [y, x, z]^{d_1+d_2+a_1^2a_2+a_1a_2^2} \\ & [z, x, y]^{d_1+d_2+2b_1a_2+a_1a_2^2}. \end{aligned} \tag{10}$$

Therefore it is enough to show that D_1 and D_2 equal 0 modulo q, where

$$\begin{aligned} D_1 &= [c_1+c_2+a_2b_1+\tfrac{1}{2}a_1a_2(a_2-1)]-[c_1+c_2+a_1b_2+\tfrac{1}{2}a_1a_2(a_1-1)] \\ &= \tfrac{1}{2}(a_2k_1-a_1k_2)n, \\ D_2 &= [c_1'+c_2'+b_1a_2+a_1a_2^2+\tfrac{1}{2}a_1a_2(a_1-1)]-[c_1'+c_2'+a_1b_2+a_1^2a_2+\tfrac{1}{2}a_1a_2(a_2-1)] \\ &= \tfrac{1}{2}(a_2k_1-a_1k_2)n. \end{aligned}$$

If the natural number n is odd, then $D_1 = D_2 \equiv 0$ (q), since the number $\frac{1}{2}(a_2k_1-a_1k_2)$ is an integer and $q|n$.

If n is even, then, in view of Theorem 4, a_1, a_2 are also even. Hence $D_1 = D_2 \equiv 0$ (q).

It should be noticed that the form of the product of two symmetric words of two variables in G is the same as (10), provided the commutators which contain the variable z are dropped. Thus our discussion shows also that $S^{(2)}(G)$ is an Abelian group. The result now follows from the corollary.

Remark. There are non-nilpotent groups for which $S^{(r)}$ is non-Abelian for some r. For example, one can prove (cf. [3]) that for the normal product of Z_p (p-prime) and Z_2 we have $S^{(2)}(Z_pZ_2) = Z_pZ_2 \times Z_p$.

5. Free nilpotent groups. Using the result of the previous section, one can easily get a more accurate description of the group $S^{(r)}(G)$, G being a free nilpotent group.

THEOREM 6. *If G is a free nilpotent group of class 2, then all the groups $S^{(r)}(G)$ are cyclic infinite.*

Proof. From Theorem 2 we deduce that $w \in S^{(2)}(G)$ if and only if w is of the form

$$w = x^{2k}y^{2k}[y, x]^{2k^2},$$

for some integers k. Therefore if we put $w_0 = x^2y^2[y, x]^2$, then $w = w_0^k$. The result now follows from the corollary.

THEOREM 7. *Let G be a free nilpotent group of class 3. Then*

1. *The words $u_1 = x^2y^2[y, x]^2[y, x, x]^2$, $u_2 = [y, x, x][y, x, y]^{-1}$ are the free generators of the group $S^{(2)}(G)$.*

2. *If $r \geqslant 3$, then the words*

$$v_{1r} = \prod_{1\leqslant i\leqslant r} x_i^6 \prod_{1\leqslant j<i\leqslant r} [x_i, x_j]^{18}[x_i, x_j, x_j]^{90} \prod_{\substack{1\leqslant j<i\leqslant r\\ j<k\leqslant r}} [x_i, x_j, x_k]^{72}$$

$$v_{2r} = \prod_{1\leqslant j<i\leqslant r} [x_i, x_j, x_j][x_i, x_j, x_i]^{-1}$$

are the free generators of the group $S^{(r)}(G)$.

Proof. 1. It follows from Theorem 3 that every symmetric word of three variables in G is of the form

$$w = x^{2k}y^{2k}[y, x]^{2k^2}[y, x, x]^c[y, x, y]^{2k^2(2k-1)-c}$$

where k, c are integers. We have

$$u_1^k = x^{2k}y^{2k}[y, x]^{2k}[y, x, x]^c[y, x, y]^{2k^2(2k-1)-p(k)}$$

for a certain integer $p(k)$. Then $w = u_1^k u_2^{c-p(k)}$. Indeed,

$$u_1^k u_2^{c-p(k)} = x^{2k}y^{2k}[y, x]^{2k^2}[y, x, x]^{p(k)}[y, x, y]^{2k^2(2k-1)-p(k)}$$
$$[y, x, x]^{c-p(k)}[y, x, y]^{p(k)-c} = w.$$

Since the exponent k and $c-p(k)$ are uniquely determined by the word w, u_1 and u_2 are free generators of $S^{(2)}(G)$, as required.

2. In view of the Corollary it is sufficient to consider the case $r = 3$. By Theorem 4, $w \in S^{(3)}(G)$ if and only if w is of the form

$$w = x^{6k}y^{6k}z^{6k}[y, x]^{18k^2}[z, x]^{18k^2}[z, y]^{18k^2}[y, x, x]^c[y, x, y]^{18k^2(6k-1)-c}$$
$$[z, x, x]^c[z, x, z]^{18k^2(6k-1)-c}[z, y, y]^c[z, y, z]^{18k^2(6k-1)-c}[y, x, z]^{72k^3}[z, x, y]^{72k^3},$$

where k, c are integers. If $q(k)$ is the exponent at $[y, x, x]$ in the reduced form of v_{13}^k, then, as is easy to verify,

$$w = v_{13}^k v_{23}^{c-q(k)},$$

and the theorem follows.

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