On a class of multi-valued vector fields in Banach spaces

by

J. Bryszewski (Gdańsk)

Abstract. An upper-semicontinuous map \( \Phi : X \to Y \) is called an \( n \)-admissible map, \( n \geq 1 \), provided there exist two maps \( p : Z \to X \) and \( q : Z \to Y \) such that:

1. \( p \) is a Vietoris \( n \)-map or, in particular, a Vietoris map,
2. \( q(x^{-1}(x)) \subseteq \Phi(x) \) for all \( x \in X \).

In this paper we consider \( n \)-admissible compact fields in Banach spaces, i.e., maps of the type \( p = f - \Phi \), where \( \Phi \) is a compact \( n \)-admissible map from a subset of a Banach space \( E \) into \( E \). The topological degree theory for \( n \)-admissible compact fields \( p : A \to E \) is given in the following two cases:

1. \( A \) is a closed ball in \( E \),
2. \( A \) is the boundary of a closed ball in \( E \).

This theory is applied to the fixed point theory of \( n \)-admissible fields and to the proofs of the theorem on antipodes and the theorem on the invariance of domains for some \( n \)-admissible fields.

Note that the class of \( n \)-admissible fields is essentially larger than the class of acyclic fields and, in particular, the class of convex-valued fields.

In this paper we continue the study of \( n \)-admissible maps which was undertaken in [3]. We note that the class of \( n \)-admissible maps contains \( n \)-acyclic maps (see [7], [11]) and hence acyclic maps.

The object of this paper is to extend the classical Brouwer-Lebesgue-Schauder degree theory from \( n \)-admissible maps in finite-dimensional vector spaces to \( n \)-admissible compact vector fields in Banach spaces.

From this theory we obtain some fixed-point theorems and a theorem on antipodes for \( n \)-admissible compact fields. Moreover, we obtain a theorem on antipodes for admissible compact vector fields and a theorem on the invariance of domain for strongly admissible compact vector fields.

We would like to recall that this theory was considered with by:

2. S. A. Williams [15], M. Furi and M. Martelli [6] for acyclic compact vector fields,
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1. Preliminaries. Let \( H \) denote the Šček cohomology functor with integer coefficients \( Z \) from the category of metric spaces and continuous maps to the category of graded abelian groups and homomorphisms of degree zero. Thus, for a space \( X \),

\[
H(X) = [H(X)]_{i \geq 0}
\]

is a graded abelian group, and for a continuous map \( f: X \to Y \), \( H(f) \) is the induced homomorphism

\[
H(f) = f^*: H(Y) \to H(X),
\]

where \( f^*: H^k(Y) \to H^k(X) \).

A non-empty space \( X \) is called 0-acyclic if \( H^0(X) = Z \); let us call \( X \) \( k \)-acyclic, \( k \geq 1 \), provided \( H^k(X) = 0 \); call \( X \) acyclic if \( X \) is \( k \)-acyclic for each \( k \geq 0 \).

Let \( A \) be a subset of a space \( X \). Denote by \( rd_A \) the relative dimension of \( A \) in \( X \). From the definition given in [13] we have:

\[
rd_A = \sup \{ C \in A \}
\]

where \( C \) is a closed subset of \( X \) and by \( \dim C \) we denote the topological dimension of \( C \). We assume that \( rd_A < 0 \) if the set \( A \) is empty and in this case we put \( rd_A = -\infty \). We observe that:

(1.1) Let \( X_0 \) be a closed subset of \( X \). Assume further that \( A \) is a subset of \( X_0 \), \( B \) is a subset of \( X \) and \( A \subset B \). Then we have

\[
rd_A \leq rd_B.
\]

A continuous map \( f: Y \to X \) is proper if for each compact subset \( A \subset X \) the counter-image \( f^{-1}(A) \) of \( A \) under \( f \) is compact; \( f \) is closed provided for each closed subset \( B \subset X \) the image of \( B \) under \( f \) is closed in \( X \).

The following fact is evident:

(1.2) If \( f: Y \to X \) is a proper map, then \( f \) is closed.

The Victorić-Begele Theorem (see [13]) and (1.2) gives

(1.3) Theorem. Let \( f: Y \to X \) be a proper and surjective map, let \( Mf \) be the set of all \( x \in X \) such that \( f^{-1}(x) \) fails to be \( i \)-acyclic. Let \( n = 1 + \max(rd_{Mf[i]} + 1) \). Then for each \( k \geq n \) the induced homomorphism \( f^*: H^k(X) \to H^k(Y) \) is an isomorphism.

Remark. We observe that if \( Mf \) is the empty set for each \( i \geq 0 \), then \( Mf[i] = -\infty \) and hence Theorem (1.3) implies that

\[
f^*: H^k(X) \to H^k(Y) \text{ is an isomorphism.}
\]

(1.4) Definition. A map \( p: Y \to X \) is called a Victorić \( n \)-map provided the following two conditions are satisfied:

(i) \( p \) is a proper and surjective map,

(ii) \( rd_{Mf[i]} \leq n - 1 \) for each \( i \geq 0 \).

Definition (1.4) implies that if \( p \) is a Victorić 1-map, then \( rd_{Mf[i]} \leq 0 \) for each \( i \geq 0 \), and from the above remark we deduce that \( p^*: H^k(X) \to H^k(Y) \) is an isomorphism.

In what follows a Victorić 1-map is called simply a Victorić map.

Finally, we note that if \( p \) is a Victorić \( n \)-map, then we have

\[
1 + \max(rd_{Mf[i]} + 1) \leq 1 + \max((n - 1) + 1) = n - 1
\]

and (1.3) implies that \( p^*: H^k(X) \to H^k(Y) \) is an isomorphism for each \( k \geq n \).

From (1.1) we deduce

(1.5) Let \( p: Y \to X \) be a Victorić \( n \)-map and \( A \subset X \) is closed, then the map \( \beta: p^{-1}(A) \to A \) is a Victorić \( n \)-map, where \( \beta \) is given by \( \beta(y) = p(y) \) for all \( y \in p^{-1}(A) \).

(1.6) Let \( p: Y \to X \) be a Victorić map, then for each \( A \subset X \) the map \( p^{-1}(A) \to A \) is a Victorić map, where \( \beta \) is given by \( \beta(y) = p(y) \) for all \( y \in p^{-1}(A) \).

From (1.6) we deduce

(1.7) If \( p: Y \to X \) and \( p_1: Y \to Z \) are Victorić maps, then the composition \( p_2 \circ p_1: Y \to Z \) of \( p_1 \) and \( p_2 \) is also a Victorić map.

(1.8) Definition. Let \( X \) be a space and let \( A, B \) be two subsets of \( X \). Denote by \( i: (A, A \subset B) \to (X, B) \) and \( j: (B, A \subset B) \to (X, A) \) the inclusion maps. A triple \((X, A, B)\) is called a \( k \)-triad, \( k \geq 0 \), provided:

(i) \( X = A \cup B \)

and \( j^*: H^k(X, A) \to H^k(B, A \subset B) \)

and \( i^*: H^k(A, \subset B) \to H^k(X, \subset B) \)

are isomorphisms for each \( \geq k + 1 \).

If \( (X, A, B) \) is a 0-triad, then \((X, A, B)\) is called simply a triad.

(1.9) Mayer-Vietoris Theorem. Let \((X, A, B)\) be a \( k \)-triad. Then the sequence

\[
\xrightarrow{d_{k+1}} H^k(A \cup B) \xrightarrow{d_k} H^{k+1}(X) \xrightarrow{d_{k+1}} H^{k+1}(A \cup B) \xrightarrow{d_k} H^{k+1}(A \cap B) \xrightarrow{d_{k+1}} ...
\]

in which \( A, \alpha, \beta \) are Mayer-Vietoris homomorphisms, is exact.

The proof of (1.9) is analogous to the proof of the Mayer-Vietoris Theorem for triads (comp. [5], pp. 57-67).

Theorem (1.9) is similar to [5]. If \( f: (X, A, B) \to (Y, C, D) \) is a map of \( k \)-triads, i.e., \( f^*: X \to Y \) is a continuous map and \( f(A) = C, f(B) = D \), then \( f \) induces a homomorphism of the respective Mayer-Vietoris sequences.

Let \( E \) be a Banach space and let \( E^{k+1} \subset E^{k+2} \) be two subspaces of \( E \) with dimensions \( k + 1 \) and \( k + 2 \) respectively. Denote by \( E^{k+2} \) and \( E^{k+2} \) the two closed half-spaces of \( E^{k+2} \) such that \( E^{k+1} = E^{k+2} \cap E^{k+2} \) and \( S^{k+1} = S^{k+1} \cap S \) and \( S^{k+1} = S^{k+1} \cap S \), where \( S \) is the unit sphere in \( E \). Clearly, \( S = S \cap E^{k+1} = S^{k+1} \cap S^{k+1} \).

We note that \((S^{k+1}, S^{k+1}, S^{k+1})\) is a triad and the Mayer-Vietoris homomorphism \( A: H^k(S) \to H^{k+1}(S^{k+1}) \) is an isomorphism.
The following lemma is of importance:

(1.10) Lemma. Let \( p, q : X \to S^{k+1} \) be two continuous maps such that: (i) \( p \) is a Vietoris n-map, \( n < k \), (ii) \( q^{-1}(S^{k+1}) \subseteq S_{k+1} \) and (iii) \( q^{-1}(S^{k+1}) \subseteq S_{k+1} \). Then the following diagram commutes:

\[
\begin{array}{ccc}
H^k(S^1) & \xrightarrow{\delta} & H^{k+1}(S^{k+1}) \\
\phi \cup (\beta \circ q^{-1}) & \circ & \phi \cup (\beta \circ q^{-1}) \\
H^k(S^1) & \xrightarrow{\delta} & H^{k+1}(S^{k+1})
\end{array}
\]

where \( \beta : S^{k+1} \to S^k \) are contractions of \( q \) and \( p \), respectively.

Proof. Let \( Y_+ = p^{-1}(S^{k+1}) \) and \( Y_- = q^{-1}(S^{k+1}) \). Then from (1.5) and (1.3) we deduce that the triple \((Y_+, Y_-, \Delta Y)\) is a k-triad. Observe that \( Y_+ \cap Y_- = p^{-1}(S^k) \). By assumptions (i) and (ii) we infer that \( p, q \) are maps from the k-triad \((Y_+, Y_-, \Delta Y)\) to the k-triad \((S^{k+1}, S^k, S^{k+1})\). Finally, from the naturality of the Mayer–Vietoris Theorem for k-triads we deduce (1.10).

The following fact is well known (comp. [9] pp. 24–25).

(1.11) Approximation Theorem. Let \( E \) be a Banach space and let \( q : Y \to E \) be a compact map. Then for every \( \varepsilon > 0 \) there exists a continuous finite dimensional map \( \varphi : Y \to E^{\varepsilon} \) such that

\[ ||q(y) - \varphi(y)|| < \varepsilon \]

for every \( y \in Y \).

In this case the map \( \varphi \) is called an \( \varepsilon \)-approximation of \( q \).

2. Multi-valued maps. Let \( X \) and \( Y \) be two spaces and assume that for every point \( x \in X \) a non-empty subset \( \varphi(x) \) of \( Y \) is given; in this case, we say that \( \varphi \) is a multi-valued map from \( X \) to \( Y \) and we write \( \varphi : X \to Y \). In what follows the symbols \( \varphi, \varphi, x \) will be reserved for multi-valued maps; the single-valued maps will be denoted by \( f, g, h, p, q \).

Let \( \varphi : X \to Y \) be a multi-valued map. We associate with \( \varphi \) the following diagram of continuous maps:

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi} & Y \\
\Gamma_\varphi = \{(x, y) \in X \times Y; y \in \varphi(x)\} & \xrightarrow{\text{graph}} & \varphi \end{array}
\]

in which \( \Gamma_\varphi \) is the graph of \( \varphi \) and \( \varphi \) is natural projections \( \varphi_1, \varphi_2 \) are given by \( \varphi_1(x, y) = x \) and \( \varphi_2(x, y) = y \).

The image of a subset \( A \subseteq X \) under \( \varphi \) is \( \varphi(A) = \bigcup \varphi(a) \). The counter-image of a subset \( B \subseteq Y \) under \( \varphi \) is \( \varphi^{-1}(B) = \{x \in X; \varphi(x) \subseteq B\} \).

Let \( \varphi : X \to Y \) be a multi-valued map, \( A \) is a subset of \( X \) and \( B \) a subset of \( Y \). If \( \varphi(A) \subseteq B \) then the contraction of \( \varphi \) to the pair \( (A, B) \) a multi-valued map \( \varphi^* : A \to B \) defined by \( \varphi^*(a) = \varphi(a) \) for each \( a \in A \). The contraction of \( \varphi \) to the pair \( (A, Y) \) is simply the restriction \( \varphi|_A \) of \( \varphi \) to \( A \).

Let \( \varphi, \varphi : X \to Y \) be two multi-valued maps such that \( \varphi(x) \subseteq \varphi(x) \) for every \( x \in X \); in this case we say that \( \varphi \) is a selector of \( \varphi \) and indicate this by \( \varphi = \varphi \).

Let \( \varphi : X \to Y \) be a multi-valued map and let \( A \) be a subset of \( X \). A point \( x \in X \) is called a fixed point for \( \varphi \) whenever \( x \in \varphi(x) \); if moreover \( x \in A \), we say that \( \varphi \) has a fixed point in \( A \).

A multi-valued map \( \varphi : X \to Y \) is called upper semi-continuous (u.s.c.) provided:

(i) \( \varphi(x) \) is compact for each \( x \in X \) and (ii) for each open set \( \mathcal{V} \subseteq Y \) the counter-image \( \varphi^{-1}(\mathcal{V}) \) is an open subset of \( X \).

(2.1) Proposition. If \( \varphi : X \to Y \) is a u.s.c. multi-valued map, then the graph \( \Gamma_\varphi \) of \( \varphi \) is a closed subset of \( X \times Y \).

A u.s.c. multi-valued map \( \varphi : X \to Y \) is called compact provided the image \( \varphi(X) \) of \( X \) under \( \varphi \) is contained in a compact subset of \( Y \).

A u.s.c. multi-valued map \( \varphi : X \to Y \) is said to be acyclic provided the set \( \varphi(x) \) is acyclic for every point \( x \in X \).

(2.2) Definition. A map \( \varphi : X \to Z \) is called n-admissible, \( n \geq 1 \), provided there exists a metric space \( Y \) and a pair of single-valued (continuous) maps of the form \( \varphi : X \to Y \subseteq Z \) such that the following two conditions are satisfied:

(i) \( \varphi \) is a Vietoris n-map,
(ii) \( q(p^{-1}(\varphi(x))) = \varphi(x) \) for each \( x \in X \).

In this case the pair of maps \( (p, q) \) is called a selected pair of \( \varphi \), written \( (p, q) \circ \varphi \).

The class of all n-admissible maps from \( X \) to \( Z \) will be denoted by \( C^n(X, Z) \). A map \( \varphi \in C^n(X, Z) \) is called simply admissible.

(2.3) Definition. An admissible map \( \varphi : X \to Z \) is called strongly admissible (s-admissible) provided there exists a selected pair \( (p, q) \circ \varphi \) such that:

\[ q(p^{-1}(\varphi(x))) = \varphi(x) \]

for each \( x \in X \).

If a selected pair \( (p, q) \) satisfies the above condition, then we write \( (p, q) \circ \varphi \). The class of all s-admissible maps from \( X \) to \( Z \) will be denoted by \( C_s(X, Z) \).

Remark 1. We observe that if \( \varphi : X \to Z \) is a compact acyclic map, then, for example, the pair \((p_0, q_0)\) is a selected pair of \( \varphi \) and hence \( (p_0, q_0) \circ \varphi \).

2. Denote by \( C(X, Z) \) the class of all compact acyclic maps from \( X \) to \( Z \); then we have the following inclusions:

\[ C(X, Z) \subseteq C(X, Z) \subseteq C(X, Z) \subseteq C(X, Z) \ldots \]

(2.4) Proposition. If \( \varphi \in C(X_1, X_2) \) and \( \varphi_2 \in C(X_2, X_3) \), then \( \varphi_2 \circ \varphi_1 \in C(X_1, X_3) \) and for every \( (p, q) \circ \varphi \), \( (p_1, q_2) \in \varphi \) there exists a pair \( (p, q) \) such that:

\[ (p')^{-1}q' = (p_1)^{-1}q_1(p)^{-1}q_2. \]

(2.5) Proposition. If \( \varphi \in C(X, Z) \), then for every selected pair \( (p, q) \circ \varphi \) we have \((p')^{-1}q' = (p_1)^{-1}q_1(p)^{-1}q_2\).
For the proof of (2.4) and (2.5) see (3.4) and (3.5) in [3].

It is well known (see example (3.6) in [3]) that \( C(X, Z) \neq C_s(X, Z) \).

(2.6) Definition. Two maps \( \varphi, \psi \in C_s(X, Z) \), \( n \geq 1 \), \( \varphi, \psi \in C(X, Z) \) are called homotopic (\( \varphi \sim \psi \)) if there exists a map

\[
\chi \in C(X \times [0, 1], Z) | \chi(x, 0) = \varphi(x) \text{ and } \chi(x, 1) = \psi(x) \text{ for each } x \in X.
\]

where \([0, 1]\) is the unit interval, such that \( \chi(x, 0) \sim \varphi(x) \) and \( \chi(x, 1) \sim \psi(x) \) for each \( x \in X \).

(2.7) Proposition. If two maps \( \varphi, \psi \in C(X, Z) \), \( \varphi, \psi \in C_s(X, Z) \) are homotopic, then there are selected pairs \((p_0, q_0) \sim \varphi \) and \((p_1, q_1) \sim \psi \) such that for each \( k \geq n \)

\[
\left( p_0^{(k)} \right)^{-1} q_0^k = \left( p_1^{(k)} \right)^{-1} q_1^k
\]

(for each \( k \geq 0 \), \( p_0^{(k)} \sim q_0^k \)).

The proof of (2.7) see [3] (Proposition (3.8)). Consider two continuous maps of the form \( X \to Y \to Z \), where \( p \) is a Victorin \( n \)-map. We define a multi-valued map \( \varphi_{n,p} : X \to Z \) by putting \( \varphi_{n,p}(x) = \varphi(x) \) for each \( x \in X \). Evidently, \( \varphi_{n,p} \) is an \( n \)-admissible and u.s.c. map. Moreover, if \( q \) is a compact map, then \( \varphi_{n,p} \) is also compact.

3. Admissible maps of subsets of finite dimensional vector spaces. Let \( S' \) be the unit sphere in the \((n+1)\)-dimensional vector space \( E^{n+1} \) and \( K^{n+1} \) be the unit closed ball in \( E^{n+1} \); by \( E^{n+1} \) we denote the space \( E^{n+1} \) without the point \( 0 \).

In this section we define the degree of an \( n \)-admissible map \( \varphi : S'_1 \to S'_2 \) where \( S'_i \) \( (i = 1, 2) \) are two spaces which have the cohomology of an \( n \)-sphere \( S^n \). We orient \( S'_i \) by choosing generators \( \beta_i = H(S'_i) \).

Let \( S'_1 \to S'_2 \) be an \( n \)-admissible map and let \((p, q)\) be a pair of the form \( S'_1 \to Y \to S'_2 \) such that \((p, q) \sim \varphi \).

(3.1) Definition. The degree \( \deg(p, q) \) of the pair \((p, q)\) is the unique integer which satisfies

\[
\left( p^{(n)} \right)^{-1} q_0^n \sim \beta_1)
\]

(3.2) Definition. Let \( \varphi : S'_1 \to S'_2 \) be an \( n \)-admissible map. By the degree \( \deg(\varphi) \) of \( \varphi \) we understand the following set of integers:

\[
\deg(\varphi) = \{ \deg(p, q) \mid (p, q) \sim \varphi \}.
\]

(3.3) Let \( \varphi, \psi \in C(S'_1, S'_2) \). Then (see (4.2) in [3]):

(i) \( \varphi \sim \psi \) implies that \( \deg \varphi \cap \deg \psi \neq \emptyset \),

(ii) \( \varphi \sim \psi \) implies that \( \deg \varphi \cap \deg \psi \).

From Remark 1 in Section 2 we obtain:

(3.4) If \( \varphi \) is acyclic, then the set \( \deg \varphi \) is a singleton, in particular, \( \varphi \sim \psi \) implies that \( \deg \varphi = \deg \psi \).

Let \( \varphi : K^{n+1} \to E^{n+1} \) be an \( n \)-admissible map and assume that \( \varphi(S^n) \subset E^{n+1}\setminus \{x_0\} \). By \( \varphi_{n,p} : S^n \to E^{n+1}\setminus \{x_0\} \) we denote the contraction of \( \varphi \) to the pair \((S^n, E^{n+1}\setminus \{x_0\})\). From (1.1) we infer that \( \varphi_{n,p} \in C(S^n, E^{n+1}\setminus \{x_0\}) \). In this case with every selected pair \((p, q) \subset \varphi \) we associate a pair \((p_1, q_1) \subset \varphi_{n,p} \) as follows:

Let \( p : Y \to K^{n+1} \), \( q : Y \to E^{n+1} \) be two maps such that \( p, q \sim \varphi \) and \( p \sim \varphi_{n,p} \).

Then \( p : Y \to E^{n+1} \) is an \( n \)-admissible map such that \( \varphi(S^n) \subset E^{n+1}\setminus \{x_0\} \) for some point \( x_0 \in E^{n+1} \). If \( \deg(\varphi, x_0) \neq 0 \), then there exists a point \( x \in K^{n+1} \) such that \( x_0 \in \varphi(x) \).

(3.7) Theorem (4.5) in [3]. Let \( \varphi : K^{n+1} \to E^{n+1} \) be an \( n \)-admissible map such that \( \varphi(S^n) \subset K^{n+1} \). Then \( \varphi \) has a fixed point.

(3.8) Let \( \varphi : S^n \to S^n \) be an \( n \)-admissible map such that \( \deg \varphi \neq 0 \). Then \( \varphi(S^n) = S^n \).

Proof. The proof of (3.8) is analogous to the proof (6.4) in [3].

(3.9) Theorem (Theorem on antipodes for \( n \)-admissible maps [3], Theorem 5.1). Let \( \varphi : S^n \to S^n \) be an \( n \)-admissible map. If for every \( x \in S^n \) there exists an \( n \)-subspace of \( E^{n+1} \) strictly separating \( \varphi(x) \) and \( \varphi(-x) \), then \( 0 \notin \deg \varphi \).

(3.10) Theorem (Theorem on antipodes for admissible maps). Let \( \varphi : S^n \to S^n \) be an admissible map such that following condition is satisfied: every radius with origin of the zero point of \( E^{n+1} \) has an empty intersection with the set \( \varphi(x) \) or \( \varphi(-x) \) for each \( x \in S^n \). Then \( 0 \notin \deg \varphi \).

Theorem (3.10) clearly follows from Theorem (6.2) in [3].

An \( s \)-admissible map \( \varphi : X \to Z \) is called an \( s \)-map provided the condition \( \varphi(x) \not\subset \varphi(z) \neq \emptyset \) implies \( d(x, x') < \varepsilon \) for each \( x, x' \in X \).

(3.11) Lemma. Let \( \varphi : K^{n+1} \to E^{n+1} \) be an \( s \)-map. Then

(i) \( \varphi(S^n) \subset E^{n+1}\setminus \{x_0\} \) for each \( x_0 \in \varphi(0) \).

(ii) \( 0 \notin \deg(\varphi, x_0) \).

Lemma (3.11) clearly follows from Lemma (7.3) in [3].

(3.12) Theorem (7.5) in [3]. Let \( c > 0 \) be a positive real number. If \( \varphi : E^{n+1} \to E^{n+1} \) is an \( s \)-map, then \( \varphi(E^{n+1}) \) is an open subset of \( E^{n+1} \).

(3.13) Theorem (7.6) in [3]. Let \( U \) be an open subset of \( E^{n+1} \) and \( \varphi : U \to E^{n+1} \) an \( s \)-admissible map. Assume further that for any points \( x_1, x_2 \in U \) the condition \( x_1 \neq x_2 \) implies \( \varphi(x_1) \cap \varphi(x_2) = \emptyset \). Then \( \varphi(U) \) is an open subset of \( E^{n+1} \).

4. Admissible compact vector fields in Banach spaces. Let \( E \) be a Banach space, \( X \) a subset of \( E \) and \( \Phi : X \to E \) a multi-valued map. We define a multi-valued map \( \varphi : X \to E \) by putting \( \varphi = I - \Phi \), where

\[
(I - \Phi)(x) = \{ x - y \mid y \in \Phi(x) \}
\]

for every \( x \in X \).
(4.1) Definition. A map \( \varphi: X \to Y \subset E \) is called an \( n \)-admissible (\( s \)-admissible) compact vector field if and only if there exists an \( n \)-admissible (\( s \)-admissible) compact map \( \Phi: X \to E \) such that \( \varphi = I - \Phi \).

If \( \Phi \) is an admissible compact map, then \( \varphi = I - \Phi \) is called an admissible compact vector field.

A point \( x_0 \in X \) is called a singular point of the vector field \( \varphi: X \to Y \) if the image \( \varphi(x_0) \) contains the origin \( 0 \) of \( E \). If there are no singular points, we say that \( \varphi \) is singularity free (written \( \varphi: X \to F \)).

(4.2) Definition. Two \( n \)-admissible (\( s \)-admissible) compact vector fields \( \varphi_1 = I - \Phi_1 \), \( \varphi_2 = I - \Phi_2 \), provided they exist a map \( \chi: X \times J \to E \), where \( J \) is a unit interval, which can be represented in the form \( \chi(x, t) = x - \chi(x, 0) \), where \( X \times J \to E \) is an \( n \)-admissible (\( s \)-admissible) compact homotopy between \( \varphi_1 \) and \( \varphi_2 \).

The following evident remark is of importance:

(4.3) Let \( A \) be a closed subset of \( E \) and let \( \varphi: A \to E \) be an \( n \)-admissible compact vector field. Then the image \( \varphi(A) \) is a closed subset of \( E \).

(4.4) Consider two maps of the form \( X \to Y \to E \) such that \( X \) is a subset of \( E \) and \( p \) is a Borel borel map of \( X \) and \( Y \). Consider the map \( \varphi: Y \to E \) by putting \( \varphi(y) = p(y) - g(y) \). Then \( I - \varphi = \varphi_\alpha \), (for the definition of \( \varphi_\alpha \) see Section 2) and hence (i) if \( \varphi \) is a compact map, then \( I - \varphi = \varphi_\alpha \) is an \( n \)-admissible compact vector field, (ii) every \( n \)-admissible compact field is an \( n \)-admissible map.

5. Degree of \( n \)-admissible compact vector fields in Banach spaces. Let \( E \) be a Banach space and let \( \varphi: S \to P \) be an \( n \)-admissible compact vector field from the unit sphere \( S \) to \( P = E \setminus \{0\} \).

Consider an arbitrary but fixed selected pair \((p, q) \in \Phi \) of the form \( X \to Y \to E \).

First, for such a pair \((p, q) \in \Phi \) we define an integer \( \deg(p, q) \) which is called the degree of \((p, q) \).

From Lemma (4.3) we obtain a positive number \( \delta \) such that \( d(0, \varphi(S)) = \delta \). We observe that \( d(0, (I - \varphi)p(S)) = \delta \). Let \( \varepsilon \) be a positive number such that \( \varepsilon < \delta \). Since \( \Phi \) is a compact map we infer that \( \varphi \) is also compact. Applying the Approximation Theorem to the map \( \varphi \) and the number \( \varepsilon \), we obtain a map \( \varphi: \varphi = Y \to E^{++} \) such that \( d(0, \varphi(q)) < \varepsilon \) for every \( y \in Y \). We may assume without loss of generality that \( k + 1 \geq 2 \) and \( k + 1 > n \).

Let \( Y_{\varepsilon} = p^{-1}(S^0) \), where \( S^0 = S \subset E^{++} \). Consider the diagram

\[
S^0 \xrightarrow{\varphi} Y_{\varepsilon} \xrightarrow{\pi} E^{++},
\]

in which \( \pi_{\varepsilon} = p^{-1}(S^0) \). We observe that \( \varphi_{\varepsilon} \neq 0 \) for every \( y \in Y_{\varepsilon} \). Indeed, since \( d(0, \varphi_{\varepsilon}) = \delta - \varepsilon > 0 \) for every \( y \in S^0 \), we have

\[
\|\varphi_{\varepsilon}(y)\| = \|p(y) - q_{\varepsilon}(y)\| = \|x - q_{\varepsilon}(y)\| = \|x - q_{\varepsilon} - (q_{\varepsilon} - q_{\varepsilon}(y))\| \\
> \|x - q_{\varepsilon}(y)\| - \|q_{\varepsilon} - q_{\varepsilon}(y)\| \geq \delta - \varepsilon > 0.
\]

We define \( \deg(p, q) \) of the pair \((p, q) \in \Phi \) by putting

\[
\deg(p, q) = \deg(p_{\varepsilon}, q_{\varepsilon}).
\]

where \( \deg(p_{\varepsilon}, q_{\varepsilon}) \) is given in (3.1).

(5.1) Lemma. Let \( \varphi = I - \Phi : S \to P \) be an \( n \)-admissible compact vector field and let \((p, q) \) be a selected pair of \( \Phi \) of the form \( S \to Y \to E \). Assume further that \( q_{\varepsilon} : Y \to E^{++} \) are two \( \varepsilon \)-approximations of \( q \). Then \( \deg(p_{\varepsilon}, q_{\varepsilon}) = \deg(p_{\varepsilon}, q_{\varepsilon}). \)

Proof. Define the map \( h : Y \times [0, 1] \to E^{++} \) by putting

\[
h(y, \varepsilon) = \varphi_{\varepsilon}(y) + (1 - \varepsilon)\varphi_{\varepsilon}(y).
\]

Then \( h \) is a homotopy between \( \varphi_{\varepsilon} \) and \( \varphi_{\varepsilon} \). We prove that \( h(y, \varepsilon) \neq 0 \) for each \( y \in Y_{\varepsilon} \) and \( \varepsilon \in [0, 1] \).

\[
\|\varphi_{\varepsilon}(y) + (1 - \varepsilon)\varphi_{\varepsilon}(y)\| = \|p(y) - q_{\varepsilon}(y) + (1 - \varepsilon)p(y) - q_{\varepsilon}(y))\| \\
= \|p(y) - \varphi_{\varepsilon}(y) + (1 - \varepsilon)\varphi_{\varepsilon}(y)\| \\
= \|p(y) - \varphi_{\varepsilon}(y) - (q_{\varepsilon}(y) - p(y)) + (1 - \varepsilon)(q_{\varepsilon}(y) - p(y))\| \\
> \delta - \varepsilon > 0.
\]

Therefore we have \( \varphi_{\varepsilon} \neq 0 \) and the proof is completed.

(5.2) Lemma. Let \( \varphi = I - \Phi : \Phi \subset S \to P \) be as in (5.1). Assume further that \( E^{++}, E^{++} \) are two subspaces of \( E \) such that \( E^{++} = E^{++} \). If \( q_{\varepsilon} : Y \to E^{++} \) is an \( \varepsilon \)-approximation of \( q \) and \( q_{\varepsilon} : Y \to E^{++} \) is the map given by \( q_{\varepsilon}(y) = q_{\varepsilon}(y) \) for every \( y \in Y \), then \( \deg(p_{\varepsilon}, q_{\varepsilon}) = \deg(p_{\varepsilon}, q_{\varepsilon}). \)

Proof. Define a map \( r : E^{++} \to E^{++} \) by putting \( r(y) = \varepsilon/\delta \). We orient \( S^{++} \) and \( \pi^{++} \) so that \( \deg(\pi_{\varepsilon}, \pi_{\varepsilon}) = \deg(q_{\varepsilon}, r_{\varepsilon}). \) Applying Lemma (1.10) to the pair \((\pi_{\varepsilon}, \pi_{\varepsilon}) \), we obtain (5.2).

Finally, from (5.1) and (5.2) we deduce that \( \deg(p, q) \) of the pair \((p, q) \) is well defined.

Now, we define \( \text{Deg}(I - \Phi) \) of an \( n \)-admissible compact vector field \( \varphi = I - \Phi : S \to P \) by putting

\[
\text{Deg}(I - \Phi)(p, q) = \deg(p, q) \in \Phi.
\]

(5.4) Proposition. Let \( \varphi, \psi : S \to P \) be two \( n \)-admissible compact vector fields. Then

(i) \( \varphi \approx \psi \) implies \( \text{Deg}(\varphi) \cap \text{Deg}(\psi) \neq \emptyset \).

(ii) \( \varphi \subset \psi \) implies \( \text{Deg}(\varphi) \subset \text{Deg}(\psi) \).
Proof. Let $\chi = I - X$ be a homotopy between $\varphi$ and $\psi$ and let $(p, q)$ be a selected pair of $\chi$. The set $\chi(S \times [0, 1])$ is closed and does not contain the origin. Then from the above construction of the degree for the selected pairs (2.7) and (3.3) we obtain (i).

The proof of (ii) is evident.

(5.5) Proposition. Let $\varphi = I - \Phi: S \rightarrow P$ be an admissible compact vector field. If $\Phi$ is an acyclic map, then the set $\text{Deg}(\varphi)$ is a singleton.

Proof. Let $(p, q) \in \Phi$ be a selected pair of $\varphi$ of the form $S \times Y \rightarrow E$. Consider the commutative diagram

\[
\begin{array}{ccc}
S & \xrightarrow{p} & \Gamma_{\varphi} \xrightarrow{q} & E \\
\downarrow f & & \downarrow & \\
Y & \xrightarrow{p} & \Phi & \\
\end{array}
\]

in which $f(y) = (p(y), q(y))$ for each $y \in Y$. Let $(q_0)_p: \Gamma_{\varphi} \rightarrow E^{p+1}$ be an $e$-approximation of $q_0$. For the proof we take an $e$-approximation of $q_0$ such that $q_0(y) = (q_0)_p f(y)$ for each $y \in Y$. Denote by $\Gamma_{\Phi}$ the graph of $\Phi_{|S} (S^k = S \cap E^{p+1})$. Let $(p_0)_q: \Gamma_{\Phi} \rightarrow S^k$ and $(q_0)_q: \Gamma_{\Phi} \rightarrow E^{p+1}$ be contractions of $p_0$ and $(q_0)_q$, respectively. Finally, we obtain (comp. the definition of $\text{deg}(p, q)$ in this section) the commutative diagram

\[
\begin{array}{ccc}
S & \xrightarrow{p} & \Gamma_{\Phi} \xrightarrow{q} & S^k \\
\downarrow f & & \downarrow & \\
S^k & \xrightarrow{p} & p^{k+1} & \\
\end{array}
\]

in which $p$ and $f$ are contractions of $p$ and $\varphi$, respectively, and the map $\Phi_{|S^k}$ is given by $(q_0)_p(x, y) = (p_0)_q(x, y) - (q_0)_q(x, y)$ for each $(x, y) \in \Gamma_{\Phi}$. The map $q_0: S^k \rightarrow p^{k+1}$ is given by $q_0(x) = p_0(x) - q_0(x)$ for each $x \in S^k$. Now, the proof of (5.5) is strictly analogous to the proof of (3.5) in [3].

(5.6) Example. Let $E$ be a Banach space and $\Phi: S \rightarrow E$ be a point such that $\|\Phi\| > 1$. Consider the map $\Phi: S \rightarrow E$ given by $\Phi(x) = (0, \gamma)$ for each $x \in S$. Clearly, $\Phi$ is an admissible and compact map. We have the following selected pairs of $\Phi$:

1) $(d_1, f) = \Phi$, where $f: S \rightarrow E$ is given by $f(x) = 0$ for each $x \in S$, and $d_2, g$, where $g: S \rightarrow E$ is given by $g(x) = g_0$ for each $x$.

Moreover, we infer that $\text{deg}(d_1, f) \neq 0$ and $\text{deg}(d_2, g) = 0$ and hence $\text{Deg}(\Phi)$ is not singleton.

The following theorem is a generalization of the following classical result:

If $f: S^k \rightarrow S^k$ is a continuous single-valued map on the $n$-dimensional unit sphere with a non-vanishing degree, then $f$ must be surjective.

(5.7) Theorem. Let $\varphi: S \rightarrow P$ be an $n$-admissible compact field such that $\text{Deg}(\varphi) \neq 0$. Then for each $x \in S$ there is a positive real number $\lambda$ such that $\lambda \varphi(x) \in \varphi(S)$. 

Proof. Suppose that there exists an $x_0 \in S$ such that

\[
L_{x_0} = \{\lambda x_0: \lambda > 0\} \cap \varphi(S) = \emptyset.
\]

Let $\epsilon = \frac{1}{2} \min \{d(\varphi(S), L_{x_0}), d(0, \varphi(S))\}$. We observe that $d(\varphi(S), L_{x_0}) > 0$, and by assumption we have $\epsilon > 0$.

Let $(p, q) \in \Phi$ be a selected pair of the form $S \times Y \rightarrow E$ such that $\text{deg}(p, q) \neq 0$. We take an $e$-approximation $q_0: Y \rightarrow E^{p+1}$ such that $x_0 \in E^{p+1}$.

Consider the diagram

\[
\begin{array}{ccc}
S^k & \xrightarrow{p} & \Gamma_{\Phi} \xrightarrow{q} & S^k \\
\downarrow f & & \downarrow & \\
S^k & \xrightarrow{p} & p^{k+1} & \\
\end{array}
\]

(comp. the definition of $\text{deg}(p, q)$ in this section). Since $q_0$ is an $e$-approximation of $q$, we obtain

\[
q_0(p_0^{-1}(S^k)) \subseteq O_e(\varphi(S)),
\]

where $O_e(\varphi(S)) = \{y \in E; \text{there is a point } z \in \varphi(S) \text{ such that } \|z - y\| < \epsilon\}$ and hence

\[
L_{x_0} \cap q_0(p_0^{-1}(S^k)) = \emptyset.
\]

Consider the map $\psi: S^k \rightarrow p^{k+1}$ given by $\psi(x) = q_0(p_0^{-1}(x))$ for each $x \in S^k$. Then $(p_0, q_0) \psi$ is a selected pair of $\psi$ and hence $\psi$ is an $n$-admissible map. Moreover, $\text{deg}(p_0, q_0) = \text{deg}(p, q) \neq 0$.

Let $r: p^{k+1} \rightarrow S^k$ be retraction $r(x) = x/\|x\|$. Then $\text{Deg}(\psi) \neq 0$ and from (1) we have $x_0 \notin (r \psi)(S^k)$, but this contradicts (3.8). The proof of (5.7) is completed.

Let $\varphi = I - \Phi: K \rightarrow E$ be an $n$-admissible compact field such that $\varphi(S) \subseteq P$, where $S$ is the boundary of the closed ball $K$. By $\phi_{\Phi}: S \rightarrow P$ we denote the contraction of $\varphi$ to the pair $(S, P)$. From (1.6) we infer that $\varphi(S) \subseteq P$ is an $n$-admissible compact vector field on $S$. In this case with every selected pair $(p, q) \Phi$ we associate a pair $(p_1, q_1) \Phi_{|S}$ as follows: Let $p: S \rightarrow K; q: E \rightarrow E$ be two maps such that $(p, q) \Phi$; then $p: S \rightarrow S^k$ and $q: S^{k+1} \rightarrow E$ are given as contractions of $p$ and $q$, respectively. Evidently $(p_1, q_1) \Phi_{|S}$ is a deformation. We define degree $\text{Deg}(\varphi, 0)$ of $\varphi$ by putting

\[
\text{Deg}(\varphi, 0) = \{\text{deg}(p_1, q_1); (p, q) \Phi_{|S}\}.
\]

Clearly, $\text{Deg}(\varphi, 0) \neq \text{Deg}(\varphi_0)$.

Let $\varphi: K \rightarrow E$ be an $n$-admissible compact field such that $\varphi(S) \subseteq P$. By $(\varphi - z_0): K \rightarrow E$ we denote the $n$-admissible compact field given by

\[
(\varphi - z_0)(x) = \{y - z_0, y \in \varphi(x)\}
\]

for each $x \in K$. Observe that $(\varphi - z_0)(S) \subseteq P$. We define $\text{Deg}(\varphi, z_0)$ by putting

\[
\text{Deg}(\varphi, z_0) = \text{Deg}(\varphi - z_0, 0).
\]

The following lemma is of importance:

(5.10) Lemma. Let $\varphi: K \rightarrow E$ be an $n$-admissible compact vector field such that $\varphi(S) \subseteq P$. If $\text{Deg}(\varphi, 0) \neq 0$, then there exists a point $x_0 \in K$ such that $0 \in \varphi(x_0)$.
Proof. Let $\varphi = I - \Phi$, where $\Phi : K \to E$ is an $n$-admissible compact map. Assume that $0 \notin \varphi(x)$ for all $x \in K$. From Lemma (4.3) we obtain a positive number $\delta$ such that $d(0, \varphi(x))/\delta = 0$. Let $\varepsilon$ be a positive number such that $\varepsilon < \delta$. Let $(p, q)$ be a selected pair of $\varphi$ of the form $K \to Y \to E$. Let $q_1 : Y \to E^{k+1}$ be an $\varepsilon$-approximation of $q$. Then, as in the definition of degree $\text{deg}(p, q)$, we obtain the following diagram:

\[
K^{k+1} \overset{P}{\to} Y \overset{q_1}{\to} E^{k+1}.
\]

Consider the map $\psi : K^{k+1} \to p^{k+1} \overset{p}{\to} P^{k+1}$ given by $\psi(x) = q_1(p^{-1}(x))$ for each $x \in K^{k+1}$. Applying Lemma (3.6) to the map $\psi$ we have $\text{Deg}(\psi, 0) = [0]$. Consequently, $\text{deg}(p_1, q_1) = 0$, where $(p_1, q_1)$ is the pair associated with $(p, q)$. Since $(p, q)$ is an arbitrary selected pair of $\Phi$, we obtain $\text{Deg}(\varphi, 0) = [0]$ and the proof is completed.

The following theorem is an extension to the case of $n$-admissible maps of a well-known theorem of Rothe (cf. [9, 6]).

(5.1) Theorem. If $\Phi : K \to E$ is an $n$-admissible compact map such that $\Phi(S) = K$, then $\Phi$ has a fixed point.

Proof. Let $\varphi : K \to E$ be an $n$-admissible compact field given by $\varphi = I - \Phi$. We may assume without loss of generality that $\varphi(S) \subset F$ and by Lemma (5.10) it suffices to prove that $\text{Deg}(\varphi, 0) \neq [0]$. For this purpose let

\[
\psi(x, t) = x - t\varphi(x) \quad \text{for an arbitrary } x \in S, \quad 0 \leq t \leq 1.
\]

It follows from our assumption that for an arbitrary $z \in \psi(x, t)$ we have

\[
[z] = [x - y] \geq [x - z] \geq [y - t \varphi(x)] > 0 \quad \text{for } 0 \leq t < 1
\]

and thus $\psi : S \times [0, 1] \to F$. It is evident that $\psi(S \times [0, 1])$ is a closed subset of $E$ and $d(0, \psi(S \times [0, 1])) = \delta > 0$.

Let $(p, q) \in \Phi$ be a selected pair of the form $K \overset{\varphi}{\to} Y \overset{\psi}{\to} E$ and let $S^{k+1} \overset{\varphi^{-1}(S)}{\to} K \overset{q_1}{\to} E^{k+1}$ be an $\varepsilon$-approximation of $q_1$, where $0 < \varepsilon < \delta$. We put $S^k = S \cap E^{k+1}$ and $Y^k = \varphi^{-1}(S^k)$. We have the diagram

\[
S^k \overset{p_0}{\to} Y_k \overset{q_1}{\to} E^{k+1},
\]

in which $p_0$, $q_1$ are contractions of $p_1$ and $q_1$, respectively.

Define the map $\tilde{q}_1 : Y_k \to E^{k+1}$ by putting $\tilde{q}_1(y) = q_1(p_0(y)) - q_1(y)$ for each $y \in Y_k$. We claim that $\text{deg}(p_1, q_1) = \text{deg}(p_0, \tilde{q}_1) = 0$. In this order, consider the map $f : Y_k \overset{\varphi}{\to} P^{k+1}$ given by $f(y) = p_0(y)$ and a homotopy $h : Y_k \times [0, 1] \overset{\varphi}{\to} P^{k+1}$ given by $h(y, t) = p_0(y) - t\tilde{q}_1(y)$. Since $\varphi(S) \subset P$ and $q_1$ is an $\varepsilon$-approximation of $q_1$, $0 < \varepsilon < \delta$, we deduce that $h(Y_k \times [0, 1]) \subset P^{k+1}$. Then the maps $f$ and $\tilde{q}_1$ are homotopic and hence $f^* = \tilde{q}_1^*$. Finally, we obtain

\[
\text{deg}(p_1, q_1) = \text{deg}(p_0, \tilde{q}_1) = \text{deg}(p_0, f) \neq 0,
\]

and the proof is completed.

In fact, from the above proof we infer that $\text{Deg}(\varphi, 0)$ is a singleton and $0 \neq \text{Deg}(\varphi, 0)$.

6. Theorems on antipodes. Let $A$ be a subset of Banach space $E$ and let $\eta$ be a positive real number. Define the set $O_\eta(A)$ by putting $O_\eta(A) = \{x \in E; \text{there is an } x \in A \text{ such that } \|x - y\| < \eta\}$.

The following theorem is a certain extension of the theorem on antipodes ([9], [12]) to $n$-admissible fields.

(6.1) Theorem. Let $\varphi = I - \Phi : S \to P$ be an $n$-admissible compact field. Suppose that there exists a real positive number $\eta$ such that for each $x \in S$ there exists a subspace $E_x$ of $E$, of codimension equal to 1, strictly separating $O_\eta(\varphi(x))$ and $O_\eta(\varphi(-x))$. Then $0 \notin \text{Deg}(\varphi)$.

Proof. Consider a selected pair $(p, q) \in \Phi$ of the form $S \overset{\varphi}{\to} Y \overset{\psi}{\to} E$. Let $s_0 = \min(\eta, d(0, \varphi(S)))$. By assumption $s_0 > 0$. We take an $\varepsilon$-approximation $q_1 : Y \overset{\psi}{\to} E^{k+1}$ of $q$, with $0 < \varepsilon < s_0 (k \geq n)$. Consider the diagram

\[
S^k \overset{p^{-1}(S)}{\to} Y \overset{q_1}{\to} E^{k+1},
\]

where $p_0$ is the contraction map of $p$ to the pair $(S^k, p^{-1}(S^k))$ and $q_1(y) = p_0(y) - q_1(y)$ for each $y \in S^k$. Then we have an $n$-admissible map $\psi : S \overset{p^{-1}(S)}{\to} P^{k+1}$ given as the composition $\psi = q_1 p^{-1}$. Let $E_x^k = E_x \cap E^{k+1}$. Observe that $\dim E_x^k = k$. Since $q_1$ is an $\varepsilon$-approximation of $q$, $0 < \varepsilon < s_0$, by assumption we have $\varphi(x) \in O_\eta(\varphi(x))$ for each $x \in S^k$. This implies that $E_x^k$ strictly separates $\varphi(x)$ and $\psi(-x)$ for each $x \in S^k$. Applying Theorem (3.8) to $\psi$, we obtain $0 \notin \text{Deg}(\psi)$ and hence $0 \neq \text{deg}(p, q)$. Since $(p, q)$ is an arbitrary selected pair of $\Phi$, we have $0 \neq \text{Deg}(\varphi)$ and the proof is completed.

From (5.10) and (6.1) we obtain

(6.2) Corollary. Let $\varphi : K \to E$ be an $n$-admissible compact field such that $\varphi|_K$ satisfies all the assumptions of (6.1). Then there is a point $x_0 \in K \cap S$ such that $0 \in \varphi(x_0)$. Now, for admissible compact fields we prove a stronger version of Theorem (6.1).

(6.3) Theorem. Let $\varphi : S \to P$ be an admissible compact field. Suppose that there exists a positive real number $\eta > 0$ such that the following condition is satisfied:

\[
\text{every half-ray } L_x = \{x \in E; \text{there is } y \in E \text{ such that } \|x - y\| < \eta\} \text{ has an empty intersection with the set } O_\eta(\varphi(x)) \text{ or } O_\eta(\varphi(-x)) \text{ for each } x \in S.
\]

Then $0 \notin \text{Deg}(\varphi)$.

Outline of the proof. Consider the admissible map $\psi$ given in the same way as in the proof of (6.1). Applying Theorem (3.9) to the map $\psi$, we deduce (6.3).
From (5.10) and (6.3) we infer

(6.4) Corollary. Let \( \varphi: K \to E \) be an admissible compact field such that \( \varphi |_{S} \) satisfies all the assumptions of (6.3). Then there is a point \( x_0 \in K \) such that \( 0 \in \varphi(x_0) \).

7. Theorem on the invariance of domain. In this section we denote by \( K_t \) a closed ball in a Banach space \( E \) with the centre 0 and radius \( t \), and by \( S_t \) the boundary of \( K_t \) in \( E \). Let \( A \) be a subset of \( E \).

A compact \( t \)-admissible field \( \varphi: A \to E \) is called an \( t \)-field provided the condition

\[
\varphi(x) \cap \varphi(x') \neq \emptyset \implies |x - x'| < \epsilon
\]

is satisfied for any \( x, x' \in A \).

A compact \( t \)-admissible field \( \varphi: A \to E \) is called an \( t \)-field in the narrow sense if for some constant \( \eta > 0 \) the condition

\[
O(\varphi(x)) \cap O(\varphi(x')) \neq \emptyset \implies |x - x'| < \epsilon \eta, \quad x, x' \in A
\]

is satisfied for any \( x, x' \in A \).

The proof of the theorem on the invariance of domain for \( t \)-fields in the narrow sense is based on the following lemmas.

(7.1) Lemma. Let \( \varphi: K \to E \) be an \( t \)-field in the narrow sense. Then:

(i) \( \varphi(K) \subset E \) for every \( t \neq 0 \)

and

(ii) \( 0 \notin \text{Deg}(\varphi, x_0) \).

Proof. For the proof of (i) we observe that \( \varphi(t) \neq \emptyset \neq \emptyset \) for some \( t \neq 0 \), then \( O(\varphi(0)) \cap O(\varphi(t)) \neq \emptyset \) and this implies that \( x \neq 0 \); hence \( \varphi(t) \neq \emptyset \).

Let \( \Phi: K \to E \) be a compact parametrized in \( E \), i.e., \( \Phi = \Phi_{t} \). Consider a selected pair \( (p, q) \in \Phi \) of the form \( \Phi_{t} = \Phi_{t+1} \to E \) and a point \( y_0 \in E \) such that \( y_0 \notin \Phi(0) \).

Let \( \delta = \min \left( \delta(t), \delta(\varphi(x)) \right) \), where \( \delta \) is the \( t \)-field in the narrow sense with the constant \( \eta \). It is evident that \( \delta \) is a positive real number. We take a \( \delta \)-approximation \( q_{x}: Y \to E^{k+1} \) of the compact map \( q \) such that \( y_0 \in E^{k+1} \).

Let \( K^{k+1} = K_{t} \cap E^{k+1} \) and \( Y_{k} = p^{-1}(K^{k+1}) \). We have the diagram

\[
K^{k+1} \xrightarrow{q_{x}} Y_{k} \xrightarrow{p} E^{k+1},
\]

in which \( p_{k} \) is the contraction of \( p \) to the pair \( (Y_{k}, X^{k+1}) \) and \( q_{x} \) is given by \( q_{x}(y) = q_{x}(y) \).

Let \( \varphi: K^{k+1} \to E^{k+1} \) be a multi-valued map given by \( \varphi(x) = \Phi_{t}^{-1}(x) \) for each \( x \in X^{k+1} \). From (4.4) we deduce that \( \varphi \) is an \( t \)-admissible map. We assert that \( \varphi \) is an \( e \)-map. Indeed, because \( 0 < \epsilon \leq \eta \) we have \( \varphi(x) = \Phi_{t}^{-1}(x) \) for each \( x \in X^{k+1} \) and hence the condition \( \varphi(x) \cap \varphi(x') \neq \emptyset \) implies \( \Phi_{t}^{-1}(x) \cap \Phi_{t}^{-1}(x') \neq \emptyset \). Then, by assumption, we obtain \( |x - x'| < \epsilon \) and \( \varphi \) is an \( e \)-map.

Applying Lemma (3.10) to the map \( \varphi \), we obtain \( 0 \notin \text{Deg}(\varphi, y_0) \) and hence \( \text{Deg}(\varphi, y_0) \). Since \( (p, q) \) is an arbitrary selected pair of \( \Phi \), we obtain (ii) and the proof is completed.

(7.2) Lemma. If \( \varphi: K \to E \) is an admissible compact field and \( y_0 \notin \Phi(S_0) \), then for every \( y_1 \in E \) such that \( |y_0 - y_1| < d(y_0, \Phi(S_0)) \) we have \( \text{Deg}(\varphi, y_0) \cap \text{Deg}(\varphi, y_1) \neq \emptyset \).

Proof. Consider the map \( \chi: K \times [0, 1] \to E \) given by \( \chi(x, t) = \chi(x, 0) + (1 - t)y(t) \), where \( \chi(x, t) = \Phi(x, t) + (1 - t)y(t) \) and \( t \in [0, 1] \). It is evident that \( \chi(S_k \times [0, 1]) \subset E \) and \( \chi(S_k \times [0, 1]) \subset E \) is a closed subset of \( E \). Therefore \( \chi \) is a homotopy between \( \varphi \) and \( \varphi - y_1 \), and by similar arguments to those used in the proof of (5.4) we deduce (7.2).

Remark. It is possible to prove that \( \text{Deg}(\varphi, y_0) \) depends on \( y_0 \) as in (7.2) but we only need (7.2).

We now prove the main result of this section.

(7.3) Theorem. If \( \varphi: E \to E \) is an \( e \)-field in the narrow sense, then \( \text{Deg}(\varphi, E) \) is an open subset of \( E \).

Proof. Let \( y_0 \notin \Phi(E) \) be a point in \( \Phi(E) \). Consider the closed ball \( K_t = K(y_0, t) \) and let \( \Phi = \Phi_{y_0} \) be the restriction of \( \Phi \) to \( K_t \). Then \( \Phi: K_t \to E \) is an \( e \)-field in the narrow sense. Applying Lemma (7.1) to \( \Phi \), we obtain \( 0 \notin \text{Deg}(\Phi, y_0) \). Let \( y_1 \in E \) be a point such that \( |y_0 - y_1| < d(y_0, \Phi(S_0)) \). Then from (7.2) we infer \( \text{Deg}(\varphi, y_1) \neq 0 \) and, in view of (5.10), we have \( y_2 \in \Phi(K_t) \). This implies that \( \Phi_{t}^{-1}(y_2) \subset \Phi_{t}^{-1}(E^{k+1}) \) and \( d(y_2, y_1) < d(y_0, \Phi(S_0)) \). The proof of (7.3) is completed.

Because \( E \) is a connected space, from Theorems (7.3) and (4.3) we deduce (4.4)."
Symmetric words in nilpotent groups of class \( \leq 3 \)

by

Ernest Plonka (Wroclaw)

Abstract. Let \( G \) be a group. A word \( w = w(x_1, \ldots, x_n) \) is said to be symmetric if \( w(a_1, \ldots, a_n) = w(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) \) for all \( a_1, \ldots, a_n \in G \) and all permutations \( \sigma \) from the symmetric group \( S_n \) on \( n \) letters. In this note we describe symmetric words in nilpotent groups of class \( \leq 3 \).

1. Introduction and notation. Let \( G \) be a group, and let \( F_\phi(x_1, \ldots, x_r) \) be the group freely generated by \( x_1, \ldots, x_r \) in the smallest variety \( \varphi(G) \) of groups containing \( G \). Let \( A \) be the group of automorphisms of \( F_\phi(x_1, \ldots, x_r) \) induced by the mappings

\[
x_i \mapsto x_{i+1}, \quad 1 \leq i \leq r,
\]

\( \mu \) belonging to the symmetric group \( S_r \) on \( r \) letters. Let \( S^{\leq 3}(G) \) be the set of all fixed points of \( A \), i.e.,

\[
S^{\leq 3}(G) = \{ \xi w = w \text{ for all } \xi \in A \}.
\]

The elements of \( S^{\leq 3}(G) \) are called symmetric words (of \( r \) variables) in \( G \).

Clearly, \( S^{\leq 3}(G) \) is a group. The aim of this note is to describe symmetric words in nilpotent groups of class \( \leq 3 \). We prove that in this case \( S^{\leq 3}(G) \) is Abelian.

2. Symmetric words. In an Abelian group every word \( w \) of \( r \) variables is of the form

\[
w = \prod_{1 \leq i < r} x_i^{\xi_i}.
\]

We thus have

**Theorem 1.** If \( G \) is an Abelian group, then \( w \in S^{\leq 3}(G) \) if and only if

\[
w = \prod_{1 \leq i < r} x_i^{x_i}.
\]

In [3] all elements of \( S^{\leq 3}(G) \) for a nilpotent \( G \) of class 2 are described. Namely

**Theorem 2.** If \( G \) is a nilpotent group of class 2, then \( w \in S^{\leq 3}(G) \) if and only if

\[
w = \prod_{1 \leq i < r} x_i^{x_i} \prod_{1 \leq i < j \leq r} [x_i, x_j]^3,
\]