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The compacta X in S^n for which $\mathrm{Sh}(X) = \mathrm{Sh}(S^k)$ is equivalent to $S^n - X \approx S^n - S^k$

by

T. B. Rushing * (Salt Lake City, Utah)

Abstract. The following is the principle result of the paper: If $X \subset S^n$, $n \geqslant 5$, is a compactum, then for $k \neq 1$, $\mathrm{Sh}(X) = \mathrm{Sh}(S^k)$ is equivalent to $S^n - X \approx S^n - S^k$ if X is globally 1-alg (and if $S^n - X$ has the homotopy type of S^1 when $k = n-2$). Introduction of the notion of shape yields this generalization of the weak flattening theorems for spheres by Duvall–Siebenmann (codimensions greater than 2), Hollingsworth–Rushing (codimension 2) and McMillan (codimension 1), while at the same time yields converses for such weak flattening theorems. The proof is elementary in the sense that the main tools involved are classical algebraic topology, Irwin's embedding theorem, Stallings' Engulfing, and for $k = 2$, $n-3$ the simple surgery technique of trading 2-handles.

1. Definitions and notation. For the definition of the *shape* of a compactum and related topics and notation refer to [1], [2] and [13]. Because of [14], we are justified in using Borsuk shape and Mardešić–Segal shape interchangeably. The notation $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ means that X and Y have the same shape. We use reduced Čech homology and cohomology throughout this paper. A set X in an n-dimensional manifold M is *cellular* if $X = \bigcap_{i=1}^{\infty} D_i$, where each D_i is an n-cell such that $D_{i+1} \subset \mathrm{Int}\, D_i$. A compactum $X \subset S^n$ satisfies the *cellularity criterion* if given a neighborhood U of X, there is a neighborhood $V \subset U$ of X such that every loop in $V - X$ is null-homotopic in $U - X$. A compactum X in S^n is 1-uv (1-UV) if given a neighborhood U of X, there exists a neighborhood $V \subset U$ of X such that each loop in V is null-homologous (null-homotopic) in U. A compactum X in S^n has property UV^{∞} if given a neighborhood U of X, there is a neighborhood $V \subset U$ of X such that V is contractible in U. A compactum X in S^n is *globally* 1-alg (*globally* 1-ss) in S^n if given a neighborhood U of X, there is a neighborhood $V \subset U$ of X such that every loop in $V - X$ which is null-homologous in $V - X$ (null-homologous in $S^n - X$) is null-homotopic in $U - X$. Refer to [16] for other standard definitions used herein.

2. Main results and introduction. Before outlining a history of the general problem in shape theory which this paper concerns, we state our main results.

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THEOREM 1. *Let* $X\subset S^n$, $n\geqslant 5$, *be compact. Then, for* $k\neq 1$ $\mathrm{Sh}(X)=\mathrm{Sh}(S^k)$ *is equivalent to* $S^n-X\approx S^n-S^k$ *if X is globally* 1-alg (*and if* S^n-X *has the homotopy type of* S^1 *when* $k=n-2$).

Theorem 1 is a consequence of the next two theorems. Theorem 2 can be proved in a straightforward manner using the Mardešić–Segal definition [13] of shape. A proof based on the Borsuk definition [1] of shape may be accomplished by using a sort of Černavskiĭ meshing technique. We omit these proofs.

THEOREM 2. *Let* $X\subset S^n$, *n arbitrary, be compact. Then, for* $k\neq 1$, $S^n-X\approx S^n-S^k$ *implies* $\mathrm{Sh}(X)=\mathrm{Sh}(S^k)$.

THEOREM 3. *Let* $X\subset S^n$, $n\geqslant 5$, *be a compactum such that* (1) $\mathrm{Sh}(X)=\mathrm{Sh}(S^k)$, $k\neq 1$, (2) *X is globally* 1-alg *in* S^n *and* (3) *if* $k=n-2$, S^n-X *has the homotopy type of* S^1. *Then,* $S^n-X\approx S^n-S^k$ (cf. [10] for a discussion of the case $X\overset{\text{top}}{\approx}S^k$).

The following proposition is a weak application of Theorems 2 and 3.

PROPOSITION. *A finite-dimensional compact metric space X has the shape of* S^k, $k\neq 1$, *if and only if there is an embedding* $h\colon X\to S^n$, *for some n, such that* $S^n-h(X)\approx S^n-S^k$.

The problem of classifying the shape of subcompacta of some ambient space in terms of their complements has been studied by a number of people. In the fundamental paper of Borsuk [1], it was shown that two continua $X, Y\subset R^2$ have the same shape if they decompose R^2 into the same number of complementary domains. Even before the definition of shape was formulated, McMillan [15] essentially proved that if X is a compactum in S^n which satisfies the cellularity criterion, then $\mathrm{Sh}(X)=\mathrm{Sh}(\text{point})$ if and only if $S^n-X\approx S^n-\text{point}$. This result of McMillan's may be regarded as the special case of our Theorem 1 for $k=0$. Lacher [12] obtained other characterizations of finite-dimensional compacta having the shape of a point.

Chapman [5] obtained the pleasing result that two Z-sets in the Hilbert cube have the same shape if and only if their complements are homeomorphic. Also, Chapman [6] proved some finite dimensional results. In particular, he showed that if two compacta X and Y of dimension $\leqslant k$ have the same shape, then they can be put in R^n, $n\geqslant 2k+2$ so that their complements are homeomorphic. Conversely, he showed that there are copies of X and Y in R^n, $n\geqslant 3k+3$, such that if their complements are homeomorphic, then they have the same shape. Geoghegan and Summerhill [9] improved Chapman's theorem by reducing the condition $n\geqslant 3k+3$ to $n\geqslant 2k+2$ and by making explicit which copies of X and Y are acceptable. For example, they showed that if $X, Y\subset R^n$ have dimensions in the trivial range and both R^n-X and R^n-Y are 1-ULC, then $\mathrm{Sh}(X)=\mathrm{Sh}(Y)$ is equivalent to their complements being homeomorphic.

The following trivial range theorem can be proved by techniques of [7], [9] and work of C. T. C. Wall [19]: If $X, Y\subset R^n$, $n\geqslant 5$, are compacta in the trivial range which are globally 1-alg in R^n and which have the shape of a finite complex, then $\mathrm{Sh}(X)=\mathrm{Sh}(Y)$ is equivalent to $R^n-X\approx R^n-Y$. (Ross Geoghegan pointed out the relevance of [19] to this result.) We feel that the recognition of this theorem increases the interest of [7].

There is no hope for proving such theorems for arbitrary compacta above the trivial range. For example, let X be two linked k-spheres in R^{2k+1} and let Y be two unlinked k-spheres in R^{2k+1}. We now give an example to illustrate why we require $k\neq 1$ in Theorem 2. Let Σ be a locally flat $(n-2)$-sphere in S^n such that $\pi_1(S^n-\Sigma)\not\approx Z$ (eg., see [22]). Let $h\colon S^{n-2}\times R^2\to S^n$ be an embedding such that $h(S^{n-2}\times 0)=\Sigma$, and let $T=h(S^{n-2}\times R^2)$. Finally, let $X=S^n-T$. Then, $S^n-X\approx S^n-S^1$. However, $\mathrm{Sh}(X)\neq\mathrm{Sh}(S^1)$ because the shape groups $\underline{\pi}_1(X)\not\approx\underline{\pi}_1(S^1)$ (see 14.6 of [1]).

Recently, two of the author's students, Vo-Thanh Liem and Gerard Venema, have independently proved results from which our Theorem 3 follows in the omitted case $k=1$.

Let us emphasize here that our proofs are elementary in the sense that the main tools involved are classical algebraic topology, Irwin's embedding theorem, Stalling's engulfing, and for $k=2$, $n-3$ the simple surgery technique of trading discs.

We wish to express appreciation to R. J. Daverman for several discussions concerning this paper.

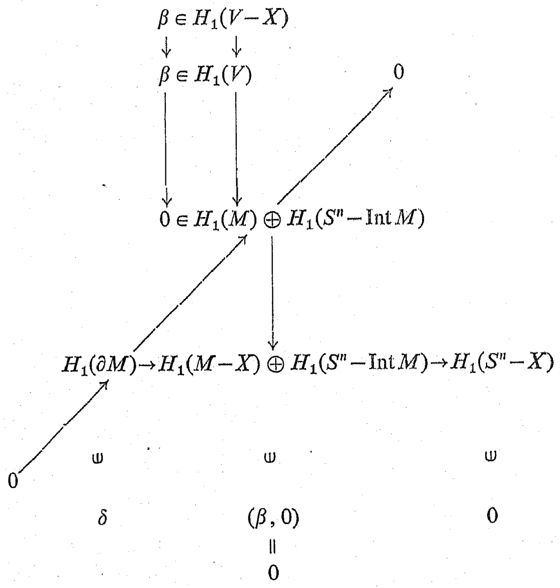
3. Proofs.

LEMMA 1. *Let* $X\subset S^n$ *be a compactum such that* $\mathrm{Sh}(X)=\mathrm{Sh}(S^k)$. *Then, X is* 1-UV *whenever* $k\neq 1$. (*In fact, this proof shows that X is m*-UV *whenever* $\pi_m(S^k)=0$, cf. Theorem 2.1 of [3].)

Proof. Pull a point $\infty\in S^n-X$ out of S^n and consider X to be in R^n. Let $\underline{f}\colon X\to S^k$ and $\underline{g}\colon S^k\to X$ to be fundamental sequences (acting on R^n, [2]) which show $\mathrm{Sh}(X)=\mathrm{Sh}(S^k)$. Since S^k is an ANR, $\underline{f}$ is induced by a map $f\colon X\to S^k$, [1]. Let $\hat{f}\colon R^n\to R^n$ be an extension of f. We may assume that there is a neighborhood W of X in R^n such that $\hat{f}(W)\subset S^k$, since S^k is an ANR. By [1] we may assume that $\underline{f}=\{f_n\}$ where each $f_n=\hat{f}$. Let U be an arbitrary neighborhood of X. Choose $V\subset U\cap W$ and N such that $g_Nf_N|V$ is homotopic to $1|V$ in U. It is easy to see that V satisfies 1-UV w.r.t. U.

LEMMA 2. *Let* $X\subset S^n$ *be a compactum that is* 1-uv. *Then X is globally* 1-ss *in* S^n *if and only if X is globally* 1-alg *in* S^n.

Proof. (This proof is basically the same as Lemma 1 of [10], however we include it here for completeness.) Obviously globally 1-ss always implies globally 1-alg. Let U be an arbitrary neighborhood of X. Choose $M\subset U$ to be a PL-manifold neighborhood of X satisfying global 1-alg w.r.t. U. Choose $V\subset M$ to satisfy 1-uv for X w.r.t. M. Let β be a loop in $V-X$ that is null-homologous in S^n-X. One may conclude that β is null-homologous in $M-X$ by chasing the following diagram:



Hence, by the global 1-alg condition, β is null-homotopic in $U-X$ as desired.

The next lemma is a consequence of Lemmas 1 and 2.

LEMMA 3. *Let* $X\subset S^n$ *be a compactum such that* $\mathrm{Sh}(X) = \mathrm{Sh}(S^k)$, $k \neq 1$. *Then,* X *is globally* 1-alg *in* S^n *if and only if* X *is globally* 1-ss *in* S^n.

LEMMA 4. *Let* $X\subset S^n$ *be a compactum such that* $\mathrm{Sh}(X) = \mathrm{Sh}(S^k)$, $k \neq 1$, $n-2$. *Then,* X *is globally* 1-alg *in* S^n *if and only if* X *satisfies the cellularity criterion.*

Proof. If X is globally 1-alg, then X is globally 1-ss by Lemma 3. But globally 1-ss implies the cellularity criterion since $H_1(S^n-X)\approx H^{n-2}(X)\approx 0$.

Proof of Theorem 3 for $k=0$. We take care of this case first since it is a quick consequence of [15]. Let $\underline{f}\colon X\to S^0$ and $\underline{g}\colon S^0\to X$ be fundamental sequences (acting on $S^n-\infty\approx R^n$, [2]) which show $\mathrm{Sh}(X) = \mathrm{Sh}(S^0)$. Then, we may assume that $\underline{f}$ is induced by a map $f\colon X\to S^0$, [1]. Since $\underline{g}\,\underline{f}\approx \underline{1}$, it follows that each $f^{-1}(-1)$ and $f^{-1}(1)$ are UV^∞. Also, $f^{-1}(-1)$ and $f^{-1}(1)$ satisfy the cellularity criterion by Lemma 4. Thus, $f^{-1}(-1)$ and $f^{-1}(1)$ are cellular [15] and $S^n-X\approx S^n-S^0$ as desired (e.g., Corollary 1.8.2 of [16]).

Proof of Theorem 3 for $2\leqslant k\leqslant n-3$. We begin by proving a proposition. (The last part of the proof of Proposition 1 mimics [17] and [8].)

PROPOSITION 1. *Let* $X\subset S^n$ *be a compactum such that* $\mathrm{Sh}(X) = \mathrm{Sh}(S^k)$, $2\leqslant k \leqslant n-3$, *and such that* X *is globally* 1-alg *in* S^n. *Then, there is a* PL $(n-k-1)$*-sphere* $S\subset S^n$ *such that the inclusion* $S\subset S^n-X$ *is a homotopy equivalence.*

Proof. Lemma 4 says that X satisfies the cellularity criterion. By a standard argument (e.g., [15]), $\pi_1(S^n-X) = 0$, i.e., a PL simple closed curve in S^n-X bounds a PL singular disk in S^n which can be moved off X by taking a fine triangulation and using the cellularity criterion. Alexander Duality and the Hurewicz theorem imply that $\pi_{n-k-1}(S^n-X)\approx\pi_{n-k-1}(S^{n-k-1})\approx Z$. Irwin's embedding theorem [11], yields a PL $(n-k-1)$-sphere $S\subset S^n-X$ which generates $\pi_{n-k-1}(S^n-X)$. From Theorem 3 of [20], we conclude that $S\subset S^n-X$ is a homotopy equivalence. This concludes the proof of Proposition 1.

We may assume that S of Proposition 1 is S^{n-k-1}, [21]. Let T be an ε-neighborhood (ε-small) of S^{n-k-1} which misses X and let T' be a closed ε'-neighborhood ($\varepsilon'<\varepsilon$) of S^{n-k-1}. Then,

$$(T, T', S^{n-k-1})\approx(S^{n-k-1}\times R^{k+1}, S^{n-k-1}\times B_1, S^{n-k-1}).$$

(B_i denotes the closed ball in R^{k+1} of radius i.)

Express S^n-X as a monotone union of compact sets $\emptyset = C_1, C_2, \ldots$ We wish to construct a sequence $h_1, h_2, \ldots$ of homeomorphisms of $S^{n-k-1}\times R^{k+1}$ into S^n-X such that

(1) $$h_1\colon (S^{n-k-1}\times R^{k+1}, S^{n-k-1}\times B_1, S^{n-k-1})\to(T, T', S^{n-k-1}),$$

(2) $$h_1|S^{n-k-1}\times B_{i-1} = h_{i-1}|S^{n-k-1}\times B_{i-1},\ i\geqslant 2,$$

and

(3) $$h_i(S^{n-k-1}\times \mathrm{Int}\,B_i)\supset C_i.$$

Then, $h=\lim h_i\colon S^{n-k-1}\times R^{k+1}\to S^n-X$ will be the required homeomorphism.

Suppose inductively we are given h_i. We can obtain h_{i+1} from an engulfing technique of Stalling's [18] once we establish Propositions 3 and 4 below. (The engulfing technique is applied to a similar situation in the proof of Theorem 4 of [10]; hence we will not repeat it here.)

Denote the pair

$$\big(S^n-(X\cup h_i(S^{n-k-1}\times B_{i+\frac{1}{2}})),\ h_i\big(S^{n-k-1}\times(\mathrm{Int}\,B_{i+1}-B_{i+\frac{1}{2}})\big)\big)\quad\text{by } (M, W).$$

PROPOSITION 2. $\pi_*(M, W) = 0$.

Proof. Since $S^{n-k-1}\subset S^n-X$ is a homotopy equivalence,

$$0\approx H_k(S^n-X, S^{n-k-1})\approx H_k\big(S^n-X, h_i(S^{n-k-1}\times \mathrm{Int}\,B_{i+1})\big).$$

Thus, by excision $H_*(M, W) = 0$ and $H_k(W)\to H_k(M)$ is also an isomorphism. By the proof of Proposition 1 and general position $\pi_1(M)\approx\pi_1\big(S^n-(X\cup S^{n-k-1})\big)$

≈ 0. Also $\pi_1(W) = 0$. Therefore, Theorem 3 of [20] implies that $W \subset M$ is a homotopy equivalence and the conclusion follows.

Let $M' = S^n - (X \cup h_i(S^{n-k-1} \times B_i))$.

PROPOSITION 3. *There are arbitrarily small neighborhoods $V \subset U$ of X in S^n such that if $V' = V - X$ and $U' = U - X$, then $\pi_k(M', V') \approx \pi_k(M', U') \approx 0$ for $k = 0, 1$ and the inclusion induced homomorphism $\pi_2(M', V') \to \pi_2(M', U')$ is trivial.*

Proof. Choose U and V to be connected neighborhoods of X satisfying the cellularity criterion where V is a compact PL n-manifold. It is clear that $\pi_0(M, V') \approx \pi_0(M, U') \approx 0$. Let $f\colon (\Delta^1 \times 0, \partial\Delta^1 \times 0) \to (M', U')$ be a map. In order to extend f to

$$\bar{f}\colon \big(\partial(\Delta^1 \times [0,1]), (\partial\Delta^1 \times [0,1]) \cup (\Delta^1 \times 1)\big) \to (M', U'),$$

let us show that $H_0(U - X) \approx 0$. Since $H_1(U, U - X) \approx H_1(S^n, S^n - X) \approx H^{n-1}(X) \approx 0$, the homology sequence of the pair $(U, U - X)$ implies $H_0(U - X) \approx 0$. $\bar{f}$ is null-homotopic in $S^n - X$ since $\pi_1(S^n - X) \approx 0$ by the proof of Proposition 1. By general positioning with respect to S^{n-k-1} and pushing radially away, we obtain the desired extension

$$\tilde{f}\colon \big(\Delta^1 \times [0,1], (\partial\Delta^1 \times [0,1]) \cup (\Delta^1 \times 1)\big) \to (M', U').$$

Thus, $\pi_1(M', U') \approx 0$. Similarly, $\pi_1(M', V') \approx 0$.

We now give a proof (which requires no surgery) that $\pi_2(M', V') \to \pi_2(M', U')$ is trivial whenever $2 < k < n-3$. Consider the following diagram

$$\begin{array}{ccccc} \pi_2(M') & \to & \pi_2(M', V') & \to & \pi_1(V'), \\ \downarrow & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} \\ \pi_2(M') & \xrightarrow{i} & \pi_2(M', U') & \xrightarrow{\partial} & \pi_1(U'). \end{array}$$

Since j_2 is the zero homomorphism by the cellularity criterion, it will suffice to show ∂ one-to-one. We do this by observing that $\pi_2(M') \approx 0$. By the proof of Proposition 2, $\pi_2(M') \approx \pi_2(W) \approx \pi_2(S^{n-k-1} \times S^k) \approx 0$.

Now let us handle the cases $k = n-3$ and $k = 2$. Since each loop in ∂V is null-homotopic in $U - X$, one can do surgery [4] on ∂V to obtain a simply connected n-manifold neighborhood V_* of X such that $V_* \subset U$ and $\pi_1(V_* - X) \approx 0$. It remains to show that $\pi_2(M', V_* - X) \approx 0$. By the relative Hurewicz theorem and excision, we have the following isomorphisms:

$$\pi_2(M', V_* - X) \approx H_2(M', V_* - X) \approx H_2(S^n - h_i(S^{n-k-1} \times B_i), V_*)$$
$$\approx H_2(S^n - S^{n-k-1}, V_*).$$

Consider the sequence

$$H_2(V_*) \to H_2(S^n - S^{n-k-1}) \to H_2(S^n - S^{n-k-1}, V_*) \to 0.$$

Whenever $k = n-3$ and $k \neq 2$, we have $H_2(S^n - S^{n-k-1}) \approx 0$ and so

$$H_2(S^n - S^{n-k-1}, V_*) \approx 0.$$

Whenever $k = 2$, we have that $H_2(S^n - S^{n-k-1}, V_*) \approx 0$ since $H_2(V_*) \to H_2(S^n - S^{n-k-1})$ is onto. $(H_2(X) \to H_2(S^n - S^{n-k-1})$ is onto by construction, i.e., use properties of linking number of true cycles and the fact that the inclusion $S^{n-k-1} \subset S^n - X$ is a homotopy equivalence.)

Proof of Theorem 3 for $k = n-2$. In view of Lemma 3, this proof can be accomplished by a direct adaptation of the proof of Theorem 4 in [10].

Proof of Theorem 3 for $k = n-1$. This proof follows from the techniques already presented. It is more convenient in the codimension one case to work on one side of X; i.e., show that a specified component of $S^n - X$ is an open n-cell. Notice that the proof suggested here is different from, and somewhat simpler than, the proof given in [15].

Proof of the Corollary. Necessity follows immediately from Theorem 2. Now suppose we have a finite dimensional compact metric space X such that $\mathrm{Sh}(X) = \mathrm{Sh}(S^k)$, $k \neq 1$. Then X can be embedded in S^m for some m. It will suffice by Theorem 3 to show that X satisfies the cellularity criterion in S^{m+3}. Let U be a neighborhood of X in S^{m+3}. By Lemma 1, there is a neighborhood V of X such that each loop in $V - X$ (which we may assume to be a PL simple closed curve) bounds a PL disk in U. By general positioning the interior of the disk with respect to S^m, such loops bound disks in $U - X$ as desired.

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Adjoint groups and the Mal'cev correspondence (a tale of four functors)

by

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Abstract. We make several observations on the connection between the structure of an associative algebra, its Lie algebra, and its adjoint group; with especial reference to the Mal'cev correspondence between Lie algebras and groups. We view in this light the construction, by Levič and Tokarenko, of a locally nilpotent non-Gruenberg Lie algebra.

1. Introduction. The Mal'cev correspondence (Mal'cev [11]) associates to each complete locally nilpotent torsion-free group (in the sense of Kuroš [8] pp. 233, 248) a locally nilpotent Lie algebra over the rational field Q. It defines a pair of mutually inverse *exact* functors

$$\mathscr{L}\colon \mathscr{C}_{\mathscr{G}}\to\mathscr{C}_{\mathscr{L}}, \quad \mathscr{G}\colon \mathscr{C}_{\mathscr{L}}\to\mathscr{C}_{\mathscr{G}}$$

where $\mathscr{C}_{\mathscr{G}}$ and $\mathscr{C}_{\mathscr{L}}$ are the categories of complete locally nilpotent torsion-free groups and of locally nilpotent Lie algebras over Q. A treatment of these results in a manner appropriate to what follows may be found in [15] where the functors are first constructed in the finitely generated ("local") case and then extended to the "global" one.

There is a situation in which standard constructions give rise to groups and Lie algebras of this type. Let R be an associative ring. Under commutation R form a Lie ring $[R]$. Under the operation $\circ$ given by

$$a\circ b = a+b+ab \quad (a, b\in R)$$

R forms a semigroup with 0 as identity. The invertible elements form a group R^0 known as the *adjoint* (or *associated*) group of R. (Compare Kuroš [8] p. 38 where, however, a different definition of $\circ$ is used. This makes no difference since the map $a\to -a$ converts one into the other). If R is a nil ring then every element of R is invertible. Suppose now that $\mathscr{A}$ is the category of locally nilpotent associative algebras over Q, and that $R\in\mathscr{A}$. Then $[R]$ lies in $\mathscr{C}_{\mathscr{L}}$; and it may be shown that R^0 lies in $\mathscr{C}_{\mathscr{G}}$ (cf. Mal'cev [12]). We may therefore form $\mathscr{G}([R])$ and $\mathscr{L}(R^0)$. We shall exhibit isomorphisms

$$R^0\cong \mathscr{G}([R]), \quad [R]\cong\mathscr{L}(R^0).$$