Если рассмотреть структуру выпуклости с леммой 2 в качестве аксиомы, то для нее обобщается лемма 1 и все теоремы от 1 до 10 вместе со следствиями.

2. Для изотонных операций. Пусть X, Y произвольные множества и Φ: $2^X \to 2^Y$ функция, удовлетворяющая условию изотонности

$$A \subset B \Rightarrow \Phi(A) \subset \Phi(B).$$

Для Φ можно рассматривать числа Хелли $h(\Phi)$, Каратеодори $c(\Phi)$ и Радона $r(\Phi)$, а также соответствующие им три вида независимостей точек. Например,

$c(\Phi)$ — это минимальное k, при котором для любого $A \subset X$ и $a \in \Phi(A)$ существуют $x_0, \ldots, x_k \in A$, что $a \in \Phi(\{x_0, .., x_k\})$,

T — h-независимое относительно Φ, если $\bigcap_{a \in T} \Phi(T \setminus \{a\}) = \Phi(\emptyset)$.

Нетрудно заметить, что при этом обобщаются теоремы 7, 8, 9 и следствия 1, 2.

Литература

[1] L. Danzer, B. Grünbaum and V. Klee, *Helly's theorem and its relatives*, Amer. Math. Soc., Procedings of Symposia in Pure Matematics 7 (1963), стр. 101–180 (русский перевод: *Теорема Хелли*, Мир, 1968).

[2] J. Eckhof, *Der Satz von Radon in Konvexen Produktstrukturen*, Monatsh. Math. 72 (1968), стр. 303–314.

[3] Л. Ф. Герман, В. П. Солтан, *О некоторых свойствах, d-выпуклых множеств*, Прикл. Мат. и Программир. 10 (1973), стр. 47–61.

[4] D. Kay and E. Womble, *Axiomatic convexity theory and relationships between the Carathèodory, Helly and Radon Numbers*, Pacific J. Math. 38, (1971), стр. 472–485.

[5] М. Лассак, В. П. Солтан, *Об одной классификации метрических пространств с точки зрения d-выпуклости*, Мат. Исслед. 3 (37) (1975), стр. 90–106.

[6] М. Лассак, *О размерностях Хелли и Каратеодори для конечномерных нормированных пространств*, Мат. Исслед. 3 (37) (1975), стр. 107–114.

[7] F. W. Levi, *On Helly's theorem and the axioms of convexity*, J. Indian Math. Soc. 15 (1951), стр. 65–76.

[8] П. С. Солтан, К. Ф. Присакару, *Задача Штейнера на графах*, ДАН СССР, 205 (3) (1972), стр. 517–519.

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Boolean valued rings

by

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Abstract. We study Kaplansky rings from the viewpoint that it is sometimes more insightful to understand them in terms of truth values taken from their idempotent algebra rather than in terms of ordinary truth values. The connection between these notions of truth is given exactly by Feferman-Vaught. An illustration of these ideas is taken from the arithmetic isolic integers. We show that they form an idempotent valued model of arithmetic and use this to derive the Nerode metatheorems.

0. Introduction. Why is Nerode's decision theorem for the arithmetic isolic integers (cf. [3]) so similar to the Feferman–Vaught decision theorem for reduced direct powers (cf. [1])? This question was asked (to us) by L. Hay in 1972. Our approach to this question was keyed by a fact that we had known for some time that the arithmetic isolic integers could be thought of as a Boolean valued model of classical arithmetic. We were thus led to take a careful look at Boolean valued models, and in particular, of Boolean valued rings.

In Section 1 we examine a model $\mathfrak{A}$ which assumes values in a Boolean algebra A. Let B be a complete subalgebra of A. How can we give a B-valued interpretation to $\mathfrak{A}$? If you like, how can we approximate A-truth by a coarser B-truth? Let $\mathfrak{B}$ be a model with the same universe as $\mathfrak{A}$, but whose truth values on atomic formula is defined by

$$[\![a = b]\!]_{\mathfrak{B}} = \sum^{B}\{x \in B|\ x \leqslant [\![a = b]\!]_{\mathfrak{A}}\}.$$

What could be a more natural approximation? In Section 2 we determine $[\![\cdot]\!]_{\mathfrak{B}}$ in terms of $[\![\cdot]\!]_{\mathfrak{A}}$. The connection is exactly Nerode, is exactly Feferman–Vaught!

Section 3 gathers together some fairly well-known results. We define reduced Boolean power generalizing Mansfield's Boolean ultrapowers (cf. [2]). Finally a connection is made between Boolean valuations and forcing.

In Section 4 we begin our study of rings. A special kind of ring due to Kaplansky is singled out for study and it is shown how they can be considered to be Boolean

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valued where the values are idempotents in the ring itself. Various examples are given and these suggest that it is sometimes more insightful to understand a ring in terms of its idempotent truth values rather than in terms of its 2-values. Note that this claim is made on technical rather than philosophical grounds. The section concludes with an application of our decision theorem to rings.

In Section 5 we apply our ring theory to the arithmetic isolic integers $\Lambda^*(\Lambda)$. The section culminates in showing how to interpret $\Lambda^*(\Lambda)$ as an idempotent valued model of the ordinary ring of integers. This view is exploited in Section 6 where we derive most of the Nerode metatheorems by a uniform Boolean method. It will then be the reader's task to see how well we have answered the original question.

Finally we would like to acknowledge our special debt to L. Hay and R. Larson. Not only were there many joint conversations, but Larson made available to us a set of detailed notes explaining his own sheaf theoretic view of ring theory.

1. Reduced Boolean valued models. If $\mathfrak{A}$ is a mathematical structure let $U_{\mathfrak{A}}$ denote the universe of $\mathfrak{A}$. Let us start with a Boolean algebra $A = (U_A, +, \cdot, -, 0, 1)$. The ingredients of an A-valued structure consist of a non-empty universe $U_{\mathfrak{A}}$; a collection $F_{\mathfrak{A}}$ of functions, each of which maps some finite power of $U_{\mathfrak{A}}$ into $U_{\mathfrak{A}}$; a collection $R_{\mathfrak{A}}$ of A-valued relations, each of which maps some finite power of $U_{\mathfrak{A}}$ into U_A; and a distinguished binary A-valued relation $E_{\mathfrak{A}}$ which serves as the A-valued equality. Distinguished individuals are treated as 0-ary functions. When there is no ambiguity we drop the subscript $\mathfrak{A}$ from these symbols. $\mathfrak{A} = (U, E, F, R, A)$ is an *A-valued structure* if for any $a, b, c \in U$, $f \in F$, and $r \in R$ we have

$$\begin{aligned}
&E(a, a) = 1,\\
&E(a, b) \leqslant E(b, a),\\
(1)\qquad &E(a, b)\cdot E(b, c) \leqslant E(a, c),\\
&E(a, b) \leqslant E[f(a), f(b)],\\
&E(a, b)\cdot r(a) \leqslant r(b).
\end{aligned}$$

The last two conditions of (1) are only given for unary functions and relations. We must also add conditions for the n-ary case. We ask the reader to be prepared for other omissions of this kind.

Let $L_{\mathfrak{A}}$ be a first-order language with equality which is suitable for discussing $\mathfrak{A}$. It contains: Individual variables $v_0, v_1, \ldots$; an equality symbol $=$; a function symbol f for each $f_{\mathfrak{A}} \in F_{\mathfrak{A}}$; and a relation symbol r for each relation $r_{\mathfrak{A}} \in R_{\mathfrak{A}}$. $\boldsymbol{L}_{\mathfrak{A}}$ is obtained from $L_{\mathfrak{A}}$ by adding an individual constant for each $a \in U_{\mathfrak{A}}$. Unless there is ambiguity we shall use the elements of $U_{\mathfrak{A}}$ as their own names. The set of sentences of $L(\boldsymbol{L})$ are denoted by $S(\boldsymbol{S})$ respectively. We evaluate each term τ of $\boldsymbol{S}$ is $\mathfrak{A}$ in the usual way and denote its value by $\tau_{\mathfrak{A}}$. An A-valued structure $\mathfrak{A}$ is an *A-valued model* if there is a function $[\![\cdot]\!]_{\mathfrak{A}}\colon \boldsymbol{S}_{\mathfrak{A}} \to U_{\mathfrak{A}}$ which satisfies

$$\begin{aligned}
&[\![\sigma = \tau]\!] = E(\sigma_{\mathfrak{A}}, \tau_{\mathfrak{A}}),\\
&[\![r(\tau)]\!] = r_{\mathfrak{A}}(\tau_{\mathfrak{A}}),\\
(2)\qquad &[\![\varphi \vee \psi]\!] = [\![\varphi]\!] + [\![\psi]\!],\\
&[\![\sim\varphi]\!] = -[\![\varphi]\!],\\
&[\![(\exists v)\theta]\!] = \sum_{a \in U_{\mathfrak{A}}}^{A} [\![\theta(a)]\!],
\end{aligned}$$

where σ, τ are terms, r is a relation symbol, φ, ψ are sentences, and θ has only v free. $\sum^A$ is the supremum operation of A, and part of our definition is that it exists under the requirements of (2). $[\![\cdot]\!]_{\mathfrak{A}}$ is unique and is called an *A-valuation*. Throughout the rest of this paper $\mathfrak{A}$ will at least be a structure; we shall specify when it is a model.

A model $\mathfrak{A}$ is *rich* if for each formula θ with only v free there is an $a \in U_{\mathfrak{A}}$ such that $[\![(\exists v)\theta]\!] = [\![\varphi(a)]\!]$. Valuations are particularly easy to compute when $\mathfrak{A}$ is rich. We shall therefore give some sufficient conditions for richness. A *partition of A* is a function p mapping some index set I into U_A such that $p_i \cdot p_j = 0$ for $i \neq j$, and $\sum_{i \in I} p_i = 1$. An A-valued structure $\mathfrak{A}$ is *complete* if A is a complete Boolean algebra and for each index set I, each partition $p\colon I \to U_A$, and each I-termed sequence $a\colon I \to U_{\mathfrak{A}}$ there is a $b \in U_{\mathfrak{A}}$ such that $p_i \leqslant E(a_i, b)$ for each $i \in I$.

THEOREM 1. *If $\mathfrak{A}$ is a complete A-valued structure then it is a rich A-valued model.*

Proof. The only bar to being a model is the last clause of (2); the sup might not exist. Thus $\mathfrak{A}$ is a model by the completeness of A. To prove richness we need only find $b \in U_{\mathfrak{A}}$ such that $[\![(\exists v)\theta]\!] \leqslant [\![\theta(b)]\!]$; the converse inequality follows from (2). Let λ be a cardinal number and let $\{a_\xi \mid \xi < \lambda\}$ be a well-ordering of $U_{\mathfrak{A}}$ of type λ. For each $\alpha < \lambda$ let

$$p_\alpha = [\![\theta(a_\alpha)]\!] - \sum_{\xi<\alpha} [\![\theta(a_\xi)]\!]\,.$$

Then $[\![(\exists v)\theta]\!] = \sum_{\xi<\lambda} p_\xi$. By adding an extra element of U_A to p we can convert it to a partition of A. By completeness there is a $b \in U_{\mathfrak{A}}$ such that $p_\xi \leqslant [\![a_\xi = b]\!]$ for each $\xi < \lambda$. Thus $p_\xi \leqslant [\![a_\xi = b]\!]\cdot[\![\theta(a_\xi)]\!]$. By induction on the complexity of θ we can show that

$$(3)\qquad [\![x = y]\!]\cdot[\![\theta(x)]\!] \leqslant [\![\theta(y)]\!] \quad \text{for} \quad x, y \in U_{\mathfrak{A}}\,.$$

Thus $p_\xi \leqslant [\![\theta(b)]\!]$. Summation then gives $\sum_{\xi<\lambda} p_\xi \leqslant [\![\theta(b)]\!]$. ∎

When we come to deal with algebraic operations on structures it is useful to call $\mathfrak{A}$ *discrete* if $E_{\mathfrak{A}}(a, b) = 1$ if and only if $a = b$ for every $a, b \in U_{\mathfrak{A}}$. We give an example of such a structure. Let A be a Boolean algebra and let $a\Delta b$ be the symmetric difference of $a, b \in U_A$. There is a canonical way to interpret A itself as an A-valued structure $\boldsymbol{A}$. Namely let the universe and functions of $\boldsymbol{A}$ be the same as those of A and for $a, b \in U_A$ define

$$(4)\qquad E_A(a, b) = -(a\Delta b)\,.$$

It is trivial to verify the first two clauses of (1) and the third and fourth follow from the fact that $(U_A, \Delta, \cdot, 0, 1)$ is a Boolean ring. Discreteness immediately follows from (4). An example illustrating all of our notions is given in

THEOREM 2. *If A is a complete Boolean algebra then A is a complete discrete rich A-valued model.*

Proof. By Theorem 1 and our preliminary remarks we need only verify that A is complete. Let I be an index set, $p: I \to U_A$ a partition of A, and $a: I \to U_A$ and I-termed sequence of elements from $U_A (= U_A)$. Let $b = \sum_{i \in I}^{A} a_i p_i$. Then

$$p_j \cdot (a_j \Delta b) = a_j p_j \Delta \sum_{i \in I} a_i p_i p_j = a_j p_j \Delta a_j p_j = 0 .$$

Thus $p_j \leqslant -(a_j \Delta b) = E(a_j, b) = [\![a_j = b]\!]$ for $j \in I$ by (4) and (2). ∎

A Boolean algebra B is a *complete subalgebra* of a Boolean algebra A if B is complete, is a subalgebra of A, and $\sum^B$ is the restriction to B of $\sum^A$. Note that A need not be complete in this definition. Now let B be a complete subalgebra of A and let $\mathfrak{A}$ be an A-valued structure. We wish to approximate $\mathfrak{A}$ as closely as possible by a B-valued structure $\mathfrak{B} = \mathfrak{A}(B)$. The universe $U_{\mathfrak{B}}$ and the functions $F_{\mathfrak{B}}$ of $\mathfrak{B}$ are the same as those of $\mathfrak{A}$. For each $r_{\mathfrak{A}} \in R_{\mathfrak{A}}$ and $a, b \in U_{\mathfrak{A}}$ define

$$(5) \qquad \begin{aligned} E_{\mathfrak{B}}(a, b) &= \textstyle\sum^B \{x \in U_B |\ x \leqslant E_{\mathfrak{A}}(a, b)\} , \\ r_{\mathfrak{B}}(a) &= \textstyle\sum^B \{x \in U_B |\ x \leqslant r_{\mathfrak{A}}(a)\} , \end{aligned}$$

and let $R_{\mathfrak{B}}$ be the set of all such $r_{\mathfrak{B}}$. It is easy to show that $\mathfrak{B}$ is a B-valued structure; we must only verify (1). Even better, $\mathfrak{B}$ is a B-valued model since B is complete. Note that if $\mathfrak{A}$ is a complete structure, then so is $\mathfrak{B}$, and that if $\mathfrak{A}$ is discrete, then so is $\mathfrak{B}$. Let $\mathbf{2}$ be the Boolean algebra whose universe is $\{0, 1\}$. $\mathbf{2}$ is a complete subalgebra of every Boolean algebra so that $\mathfrak{A}(\mathbf{2})$ is always defined. $\mathbf{2}$ is written in boldface because we can either think of it as an ordinary Boolean algebra or as the discrete $\mathbf{2}$-valued structure given by (4). The process (5) is called *reduction*. It will be applied to Boolean algebras themselves in the context of $A(B)$. This means that we consider A as an A-valued structure A via (4) and then reduce by (5). The reader should verify that

(6) $\quad A(\mathbf{2})$ is isomorphic to A

in the obvious sense. This will be used later on.

2. Feferman-Vaught like results. Suppose that $\mathfrak{A}$ is an A-valued model and B is a complete subalgebra of A. We wish to compute $[\![\cdot]\!]_{\mathfrak{A}(B)}$. Theorem 3 does this under certain additional assumptions about $\mathfrak{A}$. A complete $\mathfrak{A}$ will satisfy these assumptions, but in general completeness will be much too strong for our applications. An A-valued structure $\mathfrak{A}$ is *pseudo-complete* if for each finite index set I, each partition $p: I \to U_A$, and each I-termed sequence $a: I \to U_{\mathfrak{A}}$ there is a $b \in U_{\mathfrak{A}}$ such that $p_i \leqslant E(a_i, b)$ for each $i \in I$. A model is *perfect* if it is rich and pseudo-complete. Throughout this section $\mathfrak{A}$ will be a perfect A-valued model and B will be a complete subalgebra of A. Note that we have not required A to be a complete Boolean algebra.

Let ω be the non-negative integers. An $\mathfrak{A}$-*assignment* is a function $s: \omega \to U_{\mathfrak{A}}$. If $\varphi \in L_{\mathfrak{A}}$ let $\varphi(s)$ be obtained from φ by replacing each free variable v_n which occurs in φ by s_n. $\varphi \in L_{\mathfrak{A}}$ is *valid* if $[\![\varphi(s)]\!]_{\mathfrak{B}} = 1$ for every model $\mathfrak{B}$ (of the same similarity type as $\mathfrak{A}$) and every $\mathfrak{B}$-assignment s. Lower case letters φ, ψ, θ will range over formulas in $L_{\mathfrak{A}}$ and upper case letters Φ, Ψ will range over formulas of L_A. We use $X_0, X_1, \ldots$ for the variables of L_A. A finite sequence $(\Phi, \theta_0, \ldots, \theta_m)$ is *acceptable* if $\Phi \in L_A$, each $\theta_i \in L_{\mathfrak{A}}$, and the free variables of Φ are among $X_0, \ldots, X_m$. It is a *partitioning* sequence if it is acceptable and $\theta_0 \vee \ldots \vee \theta_m$ and each of the formulas $\sim(\theta_i \wedge \theta_j)$, $i<j$, is valid. We are going to describe an effective method by which we associate with each formula $\varphi \in L_{\mathfrak{A}}$ a *certain* acceptable $\Gamma(\varphi) = (\Phi, \theta_0, \ldots, \theta_m)$ such that a variable is free in φ if and only if it is free in some θ_i. Γ will be the same as an association used by Nerode in [3]. The description of Γ is quite messy, so instead of defining it directly, we let its definition evolve from the proof of Theorem 3, our Boolean formulation of the Feferman–Vaught result (cf. [1]).

THEOREM 3. *Let $\mathfrak{A}$ be a perfect A-valued model and let B be a complete subalgebra of A. If $\varphi \in L_{\mathfrak{A}}$, $\Gamma(\varphi) = (\Phi, \theta_0, \ldots, \theta_m)$, and s is any $\mathfrak{A}$-assignment, then*

$$(7) \qquad [\![\varphi(s)]\!]_{\mathfrak{A}(B)} = [\![\Phi([\![\theta_0(s)]\!]_{\mathfrak{A}}, \ldots, [\![\theta_m(s)]\!]_{\mathfrak{A}})]\!]_{A(B)} .$$

Proof. The proof of (7) and the simultaneous definition of Γ will be by induction on the length of φ. Our cases are only superficially different from those occurring in Feferman–Vaught [1] or Nerode [3].

Case 1. φ is $\sigma = \tau$ where σ and τ are terms of $L_{\mathfrak{A}}$. Let $a, b \in U_{\mathfrak{A}}$ respectively be interpretations of σ, τ for the assignment s. Then

$$(8) \qquad \begin{aligned} [\![\varphi(s)]\!]_{\mathfrak{A}(B)} = E_{\mathfrak{A}(B)}(a, b) &= \textstyle\sum^B \{x \in U_B |\ x \leqslant E_{\mathfrak{A}}(a, b)\} \\ &= \textstyle\sum^B \{x \in U_B |\ x \leqslant [\![E_{\mathfrak{A}}(a, b) = 1]\!]_A\} \\ &= [\![E_{\mathfrak{A}}(a, b) = 1]\!]_{A(B)} = [\![\Phi([\![\varphi(s)]\!]_{\mathfrak{A}})]\!]_{A(B)} \end{aligned}$$

where Φ is $X_0 = 1$. The transition to line (8) is justified by the fact that $[\![x = 1]\!]_A = x$ for any $x \in U_A$. Thus $\Gamma(\varphi) = (X_0 = 1, \varphi)$.

Case 2. φ is $r(\tau)$. Thanks to (5) the same argument as above works with $\Gamma(\varphi) = (X_0 = 1, \varphi)$.

Case 3. φ is $\psi \vee \psi'$. Let $\Gamma(\psi) = (\Phi, \theta_0, \ldots, \theta_m)$ and let $\Gamma(\psi')$ be written in the same form except that everything is primed. For each $i \leqslant m'$ replace the variable X_i in Φ' by X_{m+1+i} and call the resulting formula Φ''. Then it is easy to show (using inductive hypotheses) that (7) holds for $\psi \vee \psi'$ provided we define $\Gamma(\psi \vee \psi') = (\Phi \vee \Phi'', \theta_0, \ldots, \theta_m, \theta'_0, \ldots, \theta'_{m'})$.

Case 4. φ is $\sim\psi$. Just as above we show that (7) holds if

$$\Gamma(\sim\psi) = (\sim\Phi, \theta_0, \ldots, \theta_m) .$$

The most difficult case corresponds to quantifiers and for that we shall need the following:

LEMMA 1. *For each acceptable* $(\Phi, \theta_0, \ldots, \theta_m)$ *we can effectively find a partitioning* $(\Phi', \theta_0', \ldots, \theta_{m'}')$ *such that a variable is free in some* θ_i *if and only if it is free in some* θ_j' *and such that both* $[\![\Phi([\![\theta_0(s)]\!]_{\mathfrak{A}}, \ldots, [\![\theta_m(s)]\!]_{\mathfrak{A}})]\!]_{A(B)}$ *and* $[\![\Phi'([\![\theta_0'(s)]\!]_{\mathfrak{A}}, \ldots, [\![\theta_{m'}'(s)]\!]_{\mathfrak{A}})]\!]_{A(B)}$ *have the same value* (cf. [1]).

Proof. Let $m'+1 = 2^{m+1}$ and let $r_0, \ldots, r_{m'}$ be an enumeration of all subsets of $m+1 = \{0, \ldots, m\}$. Let θ_j' be

$$\bigwedge_{i\in r_j} \theta_i \wedge \bigwedge_{i\in(m+1)-r_j} \sim\theta_i \quad \text{for} \quad j\leqslant m'$$

where $\bigwedge$ stands for repeated conjunction. If $j\leqslant m$ put $q_j = \{k\leqslant m' \mid j\in r_k\}$ and define Φ' to be $\Phi(\sum^*_{i\in q_0} X_i, \ldots, \sum^*_{i\in q_m} X_i)$ where $\sum^*$ denotes a formal summation. We leave it to the reader to verify that $(\Phi', \theta_0', \ldots, \theta_{m'}')$ is a partitioning sequence. Our result now follows because

$$\sum^A_{i\in q_j} [\![\theta_i'(s)]\!]_{\mathfrak{A}} = [\![\theta_j(s)]\!]_{\mathfrak{A}} . \blacksquare$$

We return to the proof of Theorem 3 with

Case 5. φ is $(\exists v_k)\psi$. Let $\Gamma(\psi) = (\Phi, \theta_0, \ldots, \theta_m)$. By Lemma 1 we may assume that $\Gamma(\psi)$ is partitioning. Let θ_i' be $(\exists v_k)\theta_i$ and let Φ' be

(9) $\quad (\exists Y_0, \ldots, Y_m)(\sum^*_{i\leqslant m} Y_i = 1 \wedge \bigwedge_{i<j\leqslant m} Y_i\cdot Y_j = 0 \wedge \bigwedge_{i\leqslant m} Y_i \leqslant X_i \wedge \Phi(Y_0, \ldots, Y_m))$.

Let $a_i = [\![\theta_i(s)]\!]_{\mathfrak{A}}$. Then $\sum^A_{i\leqslant m} a_i = 1$ and $a_i\cdot a_j = 0$ for $i<j\leqslant m$ since $\Gamma(\psi)$ is partitioning. Thus

$$[\![\sum^*_{i\leqslant m} a_i = 1]\!]_{A(B)} = \sum^B\{x\in U_B \mid x\leqslant [\![\sum^*_{i\leqslant m} a_i = 1]\!]_A\}$$
$$= \sum^B\{x\in U_B \mid x\leqslant \sum^A_{i\leqslant m} a_i\} = 1$$

and

$$[\![a_i\cdot a_j = 0]\!]_{A(B)} = \sum^B\{x\in U_B \mid x\leqslant [\![a_i\cdot a_j = 0]\!]_A\}$$
$$= \sum^B\{x\in U_B \mid x\leqslant -(a_i\cdot a_j)\} = 1$$

for $i<j$ since for any $x\in U_A$, $[\![x = 0]\!]_A = -x$. Also

$$a_i = [\![\theta_i(s)]\!]_{\mathfrak{A}} \leqslant [\![(\exists v_k)\theta_i(s)]\!]_{\mathfrak{A}} = [\![\theta_i'(s)]\!]_{\mathfrak{A}} .$$

Now our inductive hypothesis gives $[\![\psi(s)]\!]_{\mathfrak{A}(B)} = [\![\Phi(a_0, \ldots, a_m)]\!]_{A(B)}$. Thus

$$[\![\psi(s)]\!]_{\mathfrak{A}(B)} \leqslant [\![\sum^*_{i\leqslant m} a_i = 1 \wedge \bigwedge_{i<j\leqslant m} a_i\cdot a_j = 0 \wedge \bigwedge_{i\leqslant m} a_i \leqslant [\![\theta_i'(s)]\!]_{\mathfrak{A}} \wedge \Phi(a_0, \ldots, a_m)]\!]_{A(B)}$$
$$\leqslant [\![\Phi'([\![\theta_0'(s)]\!]_{\mathfrak{A}}, \ldots, [\![\theta_m'(s)]\!]_{\mathfrak{A}})]\!]_{A(B)}$$

by (9) since $[\![x\leqslant y]\!]_A = (-x)+y$ for any $x, y\in U_A$. But $[\![\theta_i'(s)]\!]_{\mathfrak{A}}$ is independent of the value s_k since v_k is not free in θ_i'. Letting s_k vary over $U_{\mathfrak{A}}$ and summing gives

$$[\![(\exists v_k)\psi(s)]\!]_{\mathfrak{A}(B)} \leqslant [\![\Phi'([\![\theta_0'(s)]\!]_{\mathfrak{A}}, \ldots, [\![\theta_m'(s)]\!]_{\mathfrak{A}})]\!]_{A(B)} .$$

Up to now we have not directly used the assumption that $\mathfrak{A}$ is perfect. We use it now to prove the converse inequality. For any $a_0, \ldots, a_m\in U_A$ let

(10) $\quad b = [\![\sum^*_{i\leqslant m} a_i = 1 \wedge \bigwedge_{i<j\leqslant m} a_i\cdot a_j = 0 \wedge \bigwedge_{i\leqslant m} a_i \leqslant [\![\theta_i'(s)]\!]_{\mathfrak{A}} \wedge \Phi(a_0, \ldots, a_m)]\!]_{A(B)}$.

Then

(11) $$b\leqslant \sum^A_{i\leqslant m} a_i$$

because

$$b\leqslant [\![\sum^*_{i\leqslant m} a_i = 1]\!]_{A(B)} \leqslant [\![\sum^*_{i\leqslant m} a_i = 1]\!]_A = \sum^A_{i\leqslant m} a_i .$$

Also $b\leqslant [\![a_i\cdot a_j = 0]\!]_{A(B)} \leqslant [\![a_i\cdot a_j = 0]\!]_A = -(a_i\cdot a_j)$ for $i<j\leqslant m$. Thus

(12) $$(a_i\cdot b)\cdot(a_j\cdot b) = 0 \quad \text{for} \quad i<j\leqslant m .$$

$b\leqslant [\![a_i\leqslant [\![\theta_i'(s)]\!]_{\mathfrak{A}}]\!]_{A(B)} \leqslant [\![a_i\leqslant [\![\theta_i'(s)]\!]_{\mathfrak{A}}]\!]_A$ and thus

(13) $$a_i\cdot b\leqslant [\![(\exists v_k)\theta_i(s)]\!]_{\mathfrak{A}} \quad \text{for} \quad i\leqslant m .$$

For any $x\in U_{\mathfrak{A}}$ let s_x be that $\mathfrak{A}$-assignment which agrees with s except at k and assumes the value x there. The richness of $\mathfrak{A}$ gives an $x_i\in U_{\mathfrak{A}}$ such that $[\![(\exists v_k)\theta_i(s)]\!]_{\mathfrak{A}} = [\![\theta_i(s_{x_i})]\!]_{\mathfrak{A}}$. By (12) and pseudo-completeness there is an $x\in U_{\mathfrak{A}}$ such that $a_i\cdot b\leqslant [\![x = x_i]\!]_{\mathfrak{A}}$ for $i\leqslant m$. $a_i\cdot b\leqslant [\![\theta_i(s_{x_i})]\!]_{\mathfrak{A}}$ by (13) and so

(14) $$a_i\cdot b\leqslant [\![\theta_i(s_x)]\!]_{\mathfrak{A}} \quad \text{for} \quad i\leqslant m$$

by (3). If $i, j\leqslant m$ and $i\neq j$ then

$$a_i\cdot b\cdot [\![\theta_j(s_x)]\!]_{\mathfrak{A}} \leqslant [\![\theta_i(s_x)]\!]_{\mathfrak{A}}\cdot [\![\theta_j(s_x)]\!]_{\mathfrak{A}} = 0$$

by (14) and the fact that $\Gamma(\psi)$ is partitioning. Thus

$$b\cdot [\![\theta_i(s_x)]\!]_{\mathfrak{A}} = \sum^A_{j\leqslant m} a_j\cdot b\cdot [\![\theta_i(s_x)]\!]_{\mathfrak{A}} = a_i\cdot b\cdot [\![\theta_i(s_x)]\!]_{\mathfrak{A}} \leqslant a_i\cdot b$$

by (11). (14) gives $a_i\cdot b = [\![\theta_i(s_x)]\!]_{\mathfrak{A}}\cdot b$ and hence $b\leqslant E_A(a_i, [\![\theta_i(s_x)]\!]_{\mathfrak{A}})$. But $b\in U_B$ giving

(15) $$b\leqslant [\![a_i = [\![\theta_i(s_x)]\!]_{\mathfrak{A}}]\!]_{A(B)} .$$

Now $b\leqslant [\![\Phi(a_0, \ldots, a_m)]\!]_{A(B)}$ by (10) and thus

$$b\leqslant [\![\Phi([\![\theta_0(s_x)]\!]_{\mathfrak{A}}, \ldots, [\![\theta_m(s_x)]\!]_{\mathfrak{A}})]\!]_{A(B)} = [\![\psi(s_x)]\!]_{\mathfrak{A}(B)}$$
$$\leqslant [\![(\exists v_k)\psi(s_x)]\!]_{\mathfrak{A}(B)} = [\![(\exists v_k)\psi(s)]\!]_{\mathfrak{A}(B)}$$

by (3), (15), and our inductive hypothesis. It follows from (9), (10) and the fact that $a_0, \ldots, a_m$ were arbitrary elements of U_A that

$$[\![\Phi'([\![\theta_0'(s)]\!]_{\mathfrak{A}}, \ldots, [\![\theta_m'(s)]\!]_{\mathfrak{A}})]\!]_{A(B)} \leqslant [\![(\exists v_k)\psi(s)]\!]_{\mathfrak{A}(B)},$$

completing our induction.

It is now clear from Cases 1 thru 5 how to define Γ. Perhaps the most uniform way is to use Lemma 1 at each stage so that $\Gamma(\varphi)$ is always a partitioning sequence. ∎

3. Algebraic operations on structures. There are several elementary operations on Boolean valued structures which are needed for applications. Suppose that A, B are Boolean algebras, $\mathfrak{A}$ is an A-valued structure, and h is a homomorphism of A onto B. With $\mathfrak{A}$ we associate a B-valued "homomorphic image" $\mathfrak{B} = h(\mathfrak{A})$. The universe $U_{\mathfrak{B}}$ and the functions $F_{\mathfrak{B}}$ of $\mathfrak{B}$ are the same as those of $\mathfrak{A}$. For each $r_{\mathfrak{A}} \in R_{\mathfrak{A}}$ and $a, b \in U_{\mathfrak{A}}$ define

$$E_{\mathfrak{B}}(a, b) = hE_{\mathfrak{A}}(a, b), \quad r_{\mathfrak{B}}(y) = hr_{\mathfrak{A}}(a), \tag{16}$$

and let $R_{\mathfrak{B}}$ be the set of all such $r_{\mathfrak{B}}$. It is easy to see that $\mathfrak{B}$ is a B-valued structure since the inequalities of (1) are preserved under homomorphism. However, in general, discreteness is not preserved under this operation.

THEOREM 4. *If $\mathfrak{A}$ is a perfect A-valued model and h is a homomorphism of A onto B then $\mathfrak{B} = h(\mathfrak{A})$ is a perfect B-valued model. Moreover, $[\![\varphi]\!]_{\mathfrak{B}} = h[\![\varphi]\!]_{\mathfrak{A}}$ for each $\varphi \in S_{\mathfrak{A}}$.*

Proof. We start by showing that $\mathfrak{B}$ is pseudo-complete. Suppose that I is a finite index set, $p: I \to U_B$ is a partition of B, and $a: I \to U_{\mathfrak{B}}$ is an I-termed sequence. For each $i \in I$ we choose $q_i \in U_A$ such that $h(q_i) = p_i$. Define $q_i' = q_i - \sum_{j \neq i}^{A} q_j$. Then

$$h(q_i') = h(q_i) - \sum_{j\neq i}^{B} h(q_i \cdot q_j) = h(q_i).$$

Since $q_i' \cdot q_j' = 0$ for $i \neq j$, we can extend q': $I \to U_A$ to a finite partition of A. But $\mathfrak{A}$ is pseudo-complete so there is a $b \in U_{\mathfrak{A}} = U_{\mathfrak{B}}$ such that $q_i' \leqslant E_{\mathfrak{A}}(a_i, b)$ for $i \in I$. Then $p_i = h(q_i') \leqslant hE_{\mathfrak{A}}(a_i, b) = E_{\mathfrak{B}}(a_i, b)$ and we are done.

Next we show that $h[\![\cdot]\!]_{\mathfrak{A}}$: $S_{\mathfrak{B}} \to B$ is a function satisfying (2). This is done by direct case by case computation. The atomic case is taken care of by (16) and the propositional cases are trivial. Suppose v is the only free variable of θ. Since $\mathfrak{A}$ is rich, there is an $a \in U_{\mathfrak{A}}$ such that $[\![(\exists v)\theta]\!] = [\![\theta(a)]\!]$. Then

$$h[\![(\exists v)\theta]\!] = h[\![\theta(a)]\!] \leqslant \textstyle\sum^{B}\{h[\![\theta(x)]\!] \mid x \in U_{\mathfrak{B}}\} \leqslant h[\![(\exists v)\theta]\!], \tag{17}$$

which takes care of the quantifier case. Define $[\![\cdot]\!]_{\mathfrak{B}} = h[\![\cdot]\!]_{\mathfrak{A}}$. We have shown that $[\![\cdot]\!]_{\mathfrak{B}}$ is a B-valuation. $\mathfrak{B}$ is rich by (17). ∎

Having lost discreteness in the course of a homomorphism, we regain it by the following general method. Suppose that $\mathfrak{A}$ is not discrete. If $a, b \in U_{\mathfrak{A}}$ define $a \sim b$ if $E_{\mathfrak{A}}(a, b) = 1$ and then verify that $\sim$ is an equivalence relation. Let $\bar{x}$ be the equivalence class containing x. We define an A-valued structure $\mathfrak{B} = \overline{\mathfrak{A}}$ as follows. The universe $U_{\mathfrak{B}} = \{\bar{x} \mid x \in U_{\mathfrak{A}}\}$. Show that $a \sim b$ implies $f_{\mathfrak{A}}(a) \sim f_{\mathfrak{A}}(b)$ and $r_{\mathfrak{A}}(a) \leqslant r_{\mathfrak{A}}(b)$ for each $f_{\mathfrak{A}} \in F_{\mathfrak{A}}$, $r_{\mathfrak{A}} \in R_{\mathfrak{A}}$ and then define

$$E_{\mathfrak{B}}(\bar{a}, \bar{b}) = E_{\mathfrak{A}}(a, b), \quad f_{\mathfrak{B}}(\bar{a}) = \overline{f_{\mathfrak{A}}(a)}, \quad r_{\mathfrak{B}}(\bar{a}) = r_{\mathfrak{A}}(a), \tag{18}$$

and let $F_{\mathfrak{B}}(R_{\mathfrak{B}})$ be the set of all such $f_{\mathfrak{B}}(r_{\mathfrak{B}})$ respectively. The following result is obvious.

THEOREM 5. *If $\mathfrak{A}$ is a perfect A-valued model, then $\mathfrak{B} = \overline{\mathfrak{A}}$ is a discrete perfect A-valued model. Moreover, $[\![\varphi(\bar{a})]\!]_{\mathfrak{B}} = [\![\varphi(a)]\!]_{\mathfrak{A}}$ for each $\varphi \in L_{\mathfrak{A}}$ with only one free variable.*

We have now described all of the algebraic operations on structures that are needed for our applications. They are i) reduction (5), ii) homomorphism (16), and iii) discretization (18). Together they will be used in later sections to study the theories of certain rings. We use them now to define the reduced Boolean power. This notion will not be used in our paper, but it seems quite natural and is included here for the sake of completeness. Let $\mathfrak{A}$ be a discrete $\mathbf{2}$-valued structure and let A be a complete Boolean algebra. In [2] Mansfield defines the A-power $\mathfrak{B} = \mathfrak{A}^{(A)}$ as follows. The universe of $\mathfrak{B}$ is the set of all functions g: $U_{\mathfrak{A}} \to U_A$ such that $\sum^A\{g(x) \mid x \in U_{\mathfrak{A}}\} = 1$ and $g(x) \cdot g(y) = 0$ for distinct $x, y \in U_{\mathfrak{A}}$. For each $f_{\mathfrak{A}} \in F_{\mathfrak{A}}$ and $r_{\mathfrak{A}} \in R_{\mathfrak{A}}$, both say unary, $a \in U_{\mathfrak{A}}$ and $g \in U_{\mathfrak{B}}$ define

$$(f_{\mathfrak{B}}(g))(a) = \textstyle\sum^A\{g(x) \mid f_{\mathfrak{A}}(x) = a\}, \quad r_{\mathfrak{B}}(g) = \textstyle\sum^A\{g(x) \mid r_{\mathfrak{A}}(x) = 1\}. \tag{19}$$

Equality is treated as just another relation. Let $F_{\mathfrak{B}}(R_{\mathfrak{B}})$ be the set of all such $f_{\mathfrak{B}}(r_{\mathfrak{B}})$ respectively. It is then shown that (19) can be extended to arbitrary formulas, i.e.,

$$[\![\varphi(g)]\!]_{\mathfrak{B}} = \textstyle\sum^A\{g(x) \mid [\![\varphi(x)]\!]_{\mathfrak{A}} = 1\}. \tag{20}$$

If for each $a \in U_{\mathfrak{A}}$ we define $a^* \in U_{\mathfrak{B}}$ by $a^*(a) = 1$ and $a^*(x) = 0$ for $x \neq a$ then (20)) implies that * is an elementary embedding, i.e., $[\![\varphi(a)]\!]_{\mathfrak{A}} = [\![\varphi(a^*)]\!]_{\mathfrak{B}}$. Further, $\mathfrak{B}$ is, complete.

Our candidate for a *reduced Boolean power* of a discrete $\mathbf{2}$-valued structure $\mathfrak{A}$ is obtained by successively taking a Boolean power of $\mathfrak{A}$ (as in (19)), then a homomorphic image (as in (16)), then discretization (as in (18)), and finally reduction (as in (5)) with B taken to be $\mathbf{2}$. By (20) and Theorems 3, 4 and 5 there is a method for keeping track of the first order theories of the various intermediate structures. Recently Urquhart has generalized the notion of a Boolean power of a discrete

2-valued structure to that of a discrete A-valued structure $\mathfrak{A}$ (cf. [6]). Let B be a complete Boolean algebra extending A such that $\sum^A$ is the restriction to A of $\sum^B$. Our universe $U_{\mathfrak{B}}$ is the set of all functions $g\colon U_{\mathfrak{A}}\to U_{\mathfrak{B}}$ such that $\sum^B\{g(x)|\ x\in U_{\mathfrak{A}}\}=1$, $g(a)\cdot E_{\mathfrak{A}}(a,b)\leqslant g(b)$, and $g(a)\cdot g(b)\leqslant E_{\mathfrak{A}}(a,b)$ for $a,b\in U_{\mathfrak{A}}$. The second of these conditions says that g is a function (in a Boolean sense). Functions and relations are then defined by (19) with A replaced by B, and we can also show that (20) holds with A replaced by B. This new Boolean power can then be combined with (16), (18), and (5) to obtain a more general notion of reduced Boolean power.

An alternative way to handle Boolean valued structures is via "forcing". Let $\mathfrak{A}$ be an A-valued structure. Our forcing conditions is the set $P = U_A-\{0\}$ partially ordered by the Boolean $\leqslant$. Let $\varphi\in S_{\mathfrak{A}}$ and $b\in P$. If φ is atomic we say that b *forces* φ (in symbols $b\Vdash\varphi$) if $b\leqslant[\![\varphi]\!]$ (recall that $[\![\varphi]\!]$ is defined for atomic φ). We complete our definition with the standard clauses

$$b\Vdash\sim\varphi \quad\text{if}\quad (\forall b'\leqslant b)b' \text{ not } \Vdash\varphi,$$
$$b\Vdash\varphi\vee\psi \quad\text{if}\quad (\forall b'\leqslant b)(\exists b''\leqslant b')b''\Vdash\varphi \text{ or } b''\Vdash\psi,$$
$$b\Vdash(\exists v)\theta \quad\text{if}\quad (\forall b'\leqslant b)(\exists b''\leqslant b)(\exists a\in U_{\mathfrak{A}})b''\Vdash\theta(a).$$

Notice that forcing is defined for any structure. Its connection with $[\![\cdot]\!]$ is given by

THEOREM 6. *If $\mathfrak{A}$ is a model, then $b\Vdash\varphi$ if and only if $b\leqslant[\![\varphi]\!]$.*

Proof. By induction on the complexity of φ. For atomic φ the result is true by definition.

Case 2. $b\Vdash\sim\varphi$ iff $(\forall b'\leqslant b)(b'$ not $\Vdash\varphi$ iff $(\forall b'\leqslant b)b'\nleqslant[\![\varphi]\!]$ iff $b\leqslant-[\![\varphi]\!] = [\![\sim\varphi]\!]$.

Case 3. $b\Vdash\varphi\vee\psi$ iff $(\forall b'\leqslant b)(\exists b''\leqslant b')b''\Vdash\varphi$ or $b''\Vdash\psi$ iff $(\forall b'\leqslant b)(\exists b''\leqslant b')b''\leqslant[\![\varphi]\!]$ or $b''\leqslant[\![\psi]\!]$ iff $b\leqslant[\![\varphi]\!]+[\![\psi]\!] = [\![\varphi\vee\psi]\!]$.

Case 4. $b\Vdash(\exists v)\theta$ iff $(\forall b'\leqslant b)(\exists b''\leqslant b')(\exists a\in U_{\mathfrak{A}})b''\leqslant[\![\theta(a)]\!]$ iff $b\leqslant \sum^A\{[\![\theta(a)]\!]|\ a\in U_{\mathfrak{A}}\} = [\![(\exists v)\theta]\!]$. ■

When $\mathfrak{A}$ is a structure but not a model it is possible to assign the value $\{b|\ b\Vdash\varphi\}$ to φ. Indeed, this might be taken as our definition of a Boolean valuation: the only trouble is that we have assigned a regular open subset of P to φ and not an element of P. This regular open set does belong to a Boolean algebra, namely, the regular open completion $\bar{A}$ of A. In the next section we see that one objection to doing this is that the nature of our theory requires A to be a definable object in $\mathfrak{A}$.

4. Rings with enough idempotents. Let $\mathfrak{A}$ be a commutative ring with unit, i.e., $\mathfrak{A}$ is a discrete 2-valued structure (model) which satisfies the axioms of a commutative ring with unit. An idempotent is an element $a\in U_{\mathfrak{A}}$ such that $a^2 = a$. Let I be the set of all idempotents of $\mathfrak{A}$. On I we define operation $\wedge$, $\vee$, and $\neg$ by

$$a\wedge b = ab\,,$$
$$a\vee b = a+b-ab\,,$$
$$\neg a = 1-a\,.$$

By computation we can show that I is closed and is a Boolean algebra under these operations. Call this algebra A. It is the *idempotent algebra* of $\mathfrak{A}$. We wish to interpret $\mathfrak{A}$ as an A-valued structure.

We say that $\mathfrak{A}$ has *enough idempotents* if there is a function $e\colon U_{\mathfrak{A}}\to U_{\mathfrak{A}}$ such that

(21) (i) $e(0) = 0$,
(ii) $e(xy) = e(x)e(y)$,
(iii) $e(e(x)) = e(x)$,
(iv) $xe(x) = x$.

LEMMA 2. *Let e defined on $U_{\mathfrak{A}}$ satisfy* (21). *Then*

(22) (i) $e(x)^2 = e(x)$,
(ii) *if* $x^2 = x$ *then* $e(x) = x$,
(iii) $xy = 0$ *iff* $xe(y) = 0$,
(iv) $e(-x) = e(x)$,
(v) $e(x+y)\leqslant e(x)\vee e(y)$

where $\leqslant$ is the canonical inequality of A.

Proof. (i) $e(x) = e(x)e(e(x)) = e(x)e(x)$ by (21) (iv), (iii).

(ii) If $x^2 = x$ then $e(x)e(\neg x) = e(x)e(1-x) = e(x-x^2) = e(0) = 0$ (21) (ii), (i) so that $e(x)\leqslant\neg e(\neg x)$. $x\leqslant e(x)$, $\neg x\leqslant e(\neg x)$ by (21) (iv) and thus $x\leqslant e(x)\leqslant\neg e(\neg x)\leqslant\neg\neg x = x$.

(iii) If $xy = 0$ then $e(x)e(y) = e(xy) = e(0) = 0$ by (21) (ii), (i) so that $xe(y) = xe(x)e(y) = 0$ by (21) (iv). Conversely if $xe(y) = 0$ then $xy = xe(y)y = 0$ by (21) (iv).

(iv) $e(x)(\neg e(x)) = 0$ so $x(\neg e(x)) = 0$ by (22) (iii). Thus $(-x)(\neg e(x)) = 0$ and $e(-x)(\neg e(x)) = 0$ by another use of (22) (iii) so that $e(-x)\leqslant\neg\neg e(x) = e(x)$. Replacing x by $-x$ gives the converse inequality.

(v) As in (22) (iv) we have $x(\neg e(x)) = 0$, $y(\neg e(y)) = 0$. Thus $(x+y)(\neg e(x))(\neg e(y)) = 0$ so that $e(x+y)(\neg e(x))(\neg e(x)) = 0$ by (22)(iii). Taking complements we obtain $e(x+y)\leqslant e(x)\vee e(y)$. ■

COROLLARY 1. *If $\mathfrak{A}$ has enough idempotents, then the function e of* (16) *is unique and called $e_{\mathfrak{A}}$.*

Proof. $x(\neg e(x)) = 0$ by (22)(iii) since $e(x)(\neg e(x)) = 0$. If $y\in U_A$ and $xy = 0$ then $e(x)y = 0$ by (22)(iii) so that $y\leqslant\neg e(x)$. Thus $e(x)$ is the complement in A of a maximal idempotent which kills x. ■

If $\mathfrak{A}$ has enough idempotents, we define $\mathfrak{B} = \mathfrak{A}^*$ as follows. The universe $U_{\mathfrak{B}}$ and the functions $F_{\mathfrak{B}}$ of $\mathfrak{B}$ are the same as those of $\mathfrak{A}$. The valuation algebra A of $\mathfrak{B}$ is the idempotent algebra of $\mathfrak{A}$. For $a,b\in U_{\mathfrak{B}}$ let

(23) $$E_{\mathfrak{B}}(a,b) = \neg e(a-b)\,.$$

In the following theorem we evaluate $[\![\varphi]\!]_{\mathfrak{B}}$ for certain sentences φ even though $\mathfrak{B}$ is not a model. This can be done because the φ in question have no quantifiers.

THEOREM 7. *If $\mathfrak{A}$ has enough idempotents, then $\mathfrak{B} = \mathfrak{A}^{\#}$ is an A-valued structure in which each axiom of an integral domain receives a value of* 1.

Proof. The verification that $\mathfrak{B}$ is an A-valued structure is trivial. We remark that (22)(iv) is used to show $E(a, b) \leqslant E(b, a)$ and (22)(v) is used to show $E(a, b) \wedge E(b, c) \leqslant E(a, c)$. Each of the commutative ring with unit axioms is an identity and hence receives a value of 1. Finally we have the interesting fact that $\mathfrak{A}$ becomes an integral domain in the A-sense because

$$[\![ab = 0]\!]_{\mathfrak{B}} = \neg e(ab) = \neg(e(a)e(b)) = (\neg e(a)) \vee (\neg e(b)) = [\![a = 0 \vee b = 0]\!]_{\mathfrak{B}}. \blacksquare$$

If $\mathfrak{A}$ has enough idempotents, then $e_{\mathfrak{A}}$ is *discrete* in the sence that $e_{\mathfrak{A}}(x) = 0$ implies $x = 0$. We note that $e_{\mathfrak{A}}$ is discrete if and only if $\mathfrak{A}^{\#}$ is discrete. A simple example can be constructed from a Boolean algebra A (with operation $\vee, \wedge, \neg$). Convert A into a Boolean ring A' by defining $x+y = x\Delta y$ and $x \cdot y = x \wedge y$. The identity function is witness to A' having enough idempotents by (22) (ii) and of course is discrete. We can then form $A'^{\#}$ by (23), and then form an A-valued structure $A^{\#}$ by using the original operations of A with the equality of $A'^{\#}$. What we get is precisely A of Theorem 2 because

$$E_{A^{\#}}(a, b) = \neg e(a-b) = \neg(a\Delta b) = E_A(a, b).$$

If $\mathfrak{A}$ has enough idempotents then $\mathfrak{B} = \mathfrak{A}^{\#}(2)$ is a discrete 2-valued structure and

(24) $\mathfrak{A}^{\#}(2)$ is isomorphic to $\mathfrak{A}$

because $E_{\mathfrak{B}}(a, b) = 1$ iff $E_{\mathfrak{A}^{\#}}(a, b) = 1$ iff $e_{\mathfrak{A}}(a-b) = 0$ iff $a = b$. Thus $\mathfrak{A}$ and $\mathfrak{B}$ have the same equality. Incidentally, this proves (6) since we have identified A with $A^{\#}$.

Any sentence in $S_{\mathfrak{A}}$ is called *parameter free*. $\mathfrak{A}$ is *almost 2-valued* if it is a model such that $[\![\varphi]\!]_{\mathfrak{A}} \in 2$ for every parameter free φ. The *parameter free theory of* $\mathfrak{A}$ is the set of all parameter free φ such that $[\![\varphi]\!]_{\mathfrak{A}} = 1$. Almost 2-valued models are not as rare as one might think. They most naturally arise when $\mathfrak{A}$ can be represented as a direct product.

THEOREM 8. *For $i<2$ assume*

i) *$\mathfrak{A}_i$ is a commutative ring with unit,*
ii) *$\mathfrak{A}_i$ has enough idempotents,*
iii) *$\mathfrak{A}_i^{\#}$ is an almost 2-valued perfect model,*
iv) *A_i the idempotent algebra of $\mathfrak{A}_i$ is atomless.*

Conclusion. *If $\mathfrak{A}_0^{\#}$ and $\mathfrak{A}_1^{\#}$ have the same parameter free theory, then so do $\mathfrak{A}_0$ and $\mathfrak{A}_1$, i.e., $\mathfrak{A}_0$ and $\mathfrak{A}_1$ are elementarily equivalent.*

Proof. It is well known that any two atomless Boolean algebras have the same (2-valued) parameter free theory. By (6) and (24) we have $A_i(2) \cong A_i$ and

$\mathfrak{A}_i^{\#}(2) \cong \mathfrak{A}_i$. Let φ be any parameter free sentence with $\Gamma(\varphi) = (\Phi, \theta_0, \ldots, \theta_m)$. It follows from Theorem 3 that

$$\begin{aligned}[\![\varphi]\!]_{\mathfrak{A}_i} &= [\![\varphi]\!]_{\mathfrak{A}_i^{\#}(2)} \\ &= [\![\Phi([\![\theta_0]\!]_{\mathfrak{A}_i^{\#}}, \ldots, [\![\theta_m]\!]_{\mathfrak{A}_i^{\#}})]\!]_{A_i(2)} \\ &= [\![\Phi([\![\theta_0]\!]_{\mathfrak{A}_i^{\#}}, \ldots, [\![\theta_m]\!]_{\mathfrak{A}_i^{\#}})]\!]_{A_i}.\end{aligned}$$

By our hypotheses, the last term in this chain is independent of i which proves our theorem. $\blacksquare$

Theorem 8 is used in the following way. For many rings which occur in "nature", i.e., function rings, the hypotheses of Theorem 8 are met and the parameter free theory of $\mathfrak{A}^{\#}$ is much easier to understand than that of $\mathfrak{A}$. Even when the idempotent algebra of $\mathfrak{A}$ is not atomless, Theorem 3 reduces questions about $\mathfrak{A}$ to ones about $\mathfrak{A}^{\#}$ and A. Since the theory of any one Boolean algebra is decidable (cf. [5]), and since we have assumed a fair amount of knowledge about $\mathfrak{A}^{\#}$, a considerable simplification has been achieved.

The function e was introduced by Kaplansky in his study of minimal prime ideals. Let $\mathfrak{A}$ be a commutative ring with unit, with idempotent algebra A, with enough idempotents witnessed by e, and let $P \subseteq U_{\mathfrak{A}}$ be a minimal prime ideal. The following lemma gives details to a sketch in [4].

LEMMA 3 (Scott [4]). (i) $e(P) \subseteq P$,
(ii) *$e(P)$ is a prime ideal in A,*
(iii) *$e(P)$ uniquely determines P,*
(iv) *$e(P)$ is proper if and only if P is proper.*

Proof. Let $Q = \{x \in U_{\mathfrak{A}} |\ e(x) \in P\}$. Our plan is to show that Q is a prime ideal contained in P. Let $x \in Q$. Then $x = xe(x) \in P$ by (21)(iv) since $e(x) \in P$. Thus $Q \subseteq P$. If $xy \in Q$ then $e(x)e(y) = e(xy) \in P$ by (21)(ii) and thus one of $e(x), e(y)$ is in P. This implies that either x or y is in Q. If $x \in Q$ and $y \in U_{\mathfrak{A}}$ then $e(xy) = e(x)e(y) \in P$ and thus $xy \in Q$. If $x, y \in Q$ then $e(x-y) = e(x-y)[e(x) \vee e(y)]$ by (22) (iv), (v) and $e(x) \vee e(y) = e(x)+e(y)-e(x)e(y) \in P$ giving $x-y \in Q$. Our plan complete, $P = Q$ by the minimality of P. This gives (i).

We start (ii) with the claim that $e(P) = P \cap U_A$. By (i) $e(P) \subseteq P$ and by (22) (i), $e(P) \subseteq U_A$. Conversely if $x \in P \cap U_A$, then $x = e(x) \in e(P)$ by (22) (ii) and (i). Our claim proven, it is easy to verify (ii). We claim that $P = U_{\mathfrak{A}} \cdot e(P)$. If $x \in P$ then $x = xe(x) \in U_{\mathfrak{A}} \cdot e(P)$ by (21) (iv). Thus $P \subseteq U_{\mathfrak{A}} \cdot e(P)$ and the converse inclusion follows from (i) and the fact that P is an ideal.

Our claim proven, (iii) is immediate.

Finally, for (iv) notice that P is proper iff $1 \notin P$ iff $1 \notin e(P)$ iff $e(P)$ is proper because $P \cap U_A = e(P)$. $\blacksquare$

Let $\mathfrak{A}/P$ be the usual quotient of a ring by an ideal and if $x \in U_{\mathfrak{A}}$ let x/P be the element determined by x in $\mathfrak{A}/P$. Now let us assume that $\mathfrak{A}$ has enough idempotents with witness e and that $P \subseteq U_{\mathfrak{A}}$ is a minimal prime ideal. If $\mathfrak{A}$ has idempotent algebra A then by the proof of Lemma 3(ii) we know that $e(P) = P \cap U_A$ is a prime ideal in A. Let $h: U_A \to 2$ be a homomorphism whose kernel is $e(P)$, i.e., these are the elements that h takes into 0. $\mathfrak{A}/P$ and $h(\mathfrak{A}^{\#})$ are both discrete 2-valued structures. We claim that

(25) $\mathfrak{A}/P$ is isomorphic to $\overline{h(\mathfrak{A}^{\#})}$.

For if $\mathfrak{B} = h(\mathfrak{A}^{\#})$ it suffices to show that $\overline{\mathfrak{B}}$ and $\mathfrak{A}/P$ have the same equality. If $a, b \in U_{\mathfrak{A}}$ then $a/P = b/P$ iff $a-b \in P$ iff $e(a-b) \in P$ iff $\neg E_{\mathfrak{A}^{\#}}(a, b) \in e(P)$ iff $hE_{\mathfrak{A}^{\#}}(a, b) = 1$ iff $E_{\mathfrak{B}}(a, b) = 1$ iff $E_{\overline{\mathfrak{B}}}(\bar{a}, \bar{b}) = 1$ iff $\bar{a} = \bar{b}$ by (i) of Lemma 3, (23), (16), and (18). Our claim is proven.

Theorem 9. *Assume that*

(i) $\mathfrak{A}$ *is a commutative ring with unit and with enough idempotents,*

(ii) $\mathfrak{A}^{\#}$ *is an almost 2-valued perfect model,*

(iii) P *is a proper minimal prime ideal in* $\mathfrak{A}$.

Conclusion. $\mathfrak{A}^{\#}$ *and* $\mathfrak{A}/P$ *have the same parameter free theory.*

Proof. Let A be the idempotent algebra of $\mathfrak{A}$. By Lemma 3 we know that $P \cap U_A$ is a proper prime ideal of A. Let $h: U_A \to 2$ be a homomorphism whose kernel is $P \cap U_A$. If $\mathfrak{B} = h(\mathfrak{A}^{\#})$ then $\mathfrak{A}/P \cong \overline{\mathfrak{B}}$ by (25). Then for any parameter free sentence φ we have $[\![\varphi]\!]_{\mathfrak{A}^{\#}} = 1$ iff $h[\![\varphi]\!]_{\mathfrak{A}^{\#}} = 1$ iff $[\![\varphi]\!]_{\mathfrak{B}} = 1$ iff $[\![\varphi]\!]_{\overline{\mathfrak{B}}} = 1$ by Theorems 4 and 5. The properness of P was used to justify the first "iff" in the preceding chain. ■

5. The arithmetic isolic integers. The elaborate Boolean machinery developed in the preceding sections would be quite useless were it not for some good examples. An excellent one is the arithmetic isolic integers. For the reader who is unfamiliar or uninterested in this example, we rest our case. However, the reader who pursues this section will see that the example is a good one because (i) all of our methods work, and (ii) little is known about the structure of these integers. We begin with a brief review.

Let ω^* (sometimes written as Z) be the rational integers conceived of as a discrete 2-valued structure whose functions are all the arithmetic functions and whose relations are all the arithmetic relations. For this section Λ^* (extending ω^*) is the arithmetic isolic integers conceived of as a discrete 2-valued structure. Each $f \in F_Z$ and $r \in R_Z$, both say unary, are *canonically* extended to $f_{\Lambda^*}: \Lambda^* \to \Lambda^*$ and $r_{\Lambda^*} \subseteq \Lambda^*$. These are the functions and relations of Λ^*. We follow the usual custom of writing $\omega^*(\Lambda^*)$ to mean either the structure or its universe respectively. Let $L = L_Z$ and $S = S_Z$. We call these the *parameter free formulas* and *parameter free sentences* of Λ^* (note that both may contain parameters from ω^*). We interpret $\varphi \in S$ in ω^* in the obvious way and φ in Λ^* by replacing functions and relations occurring in φ by their canonical extensions. When dealing with discrete 2-valued $\mathfrak{A}$ we sometimes write $\mathfrak{A} \models \varphi$ instead of $[\![\varphi]\!]_{\mathfrak{A}} = 1$. An important technical feature of Λ^* is (cf. [3]).

(26) if $\varphi \in S$ is an arbitrarily quantified Horn sentence and $\omega^* \models \varphi$ then $\Lambda^* \models \varphi$.

The plus and times operations of ω^* are arithmetical functions. Use the same symbols $+$, $\cdot$ to represent them as well as their canonical extensions to Λ^*. Now each of the commutative ring with unit axioms can be expressed as a Horn sentence (in fact as an identity) and hence by (26) Λ^* is a commutative ring with unit. Another technical feature of Λ^* is [cf. [3])

(27) the idempotent algebra A of Λ^* is atomless.

Let $e: \omega^* \to \omega^*$ be defined by $e(0) = 0$ and $e(x) = 1$ for $x \neq 0$. e satisfies (21) in ω^*, and since each of the conditions in (21) is an identity, e_{Λ^*} satisfies (21) in Λ^*. Thus Λ^* has enough idempotents. Now by Theorem 7 we may regard Λ^* as an A-valued structure $\mathfrak{A}$. The universe of $\mathfrak{A}$ is just Λ^*, the functions of $\mathfrak{A}$ are $+$ and $\cdot$, and its equality is given by (23), that is, $\mathfrak{A}$ is at least Λ^{**}. We are still not done because we want to interpret Λ^* as an A-valued structure (in fact a model) with respect to all of its functions and relations. Let $f \in F_Z$ and $r \in R_Z$, both say unary. With f we associate $f_{\mathfrak{A}}$ which by definition is f_{Λ^*}. Let c^r be the characteristic function of r, i.e., c^r assumes the value 1 on r and the value 0 on its complement. Then for $a \in \Lambda^*$ define

(28) $$r_{\mathfrak{A}}(a) = c^r_{\Lambda^*}(a).$$

An application of (26) shows that $r_{\mathfrak{A}}(a) \in U_A$ as it should be. Let $F_{\mathfrak{A}}(R_{\mathfrak{A}})$ be the set of all such $f_{\mathfrak{A}}(r_{\mathfrak{A}})$ respectively. In order to justify $\mathfrak{A}$ as an A-valued structure we must verify (1). We defer this for a more general result. But first several observations are in order. For the moment let r be a unary relation of ω^*, $=$ is the equality of ω^* considered as a binary relation, $E_{\mathfrak{A}}$ is the ring theoretic equality of $\mathfrak{A}$ given by (23), and $a, b \in U_{\mathfrak{A}} = \Lambda^*$. In general we write $r_{\Lambda^*}(a)$ for $a \in r_{\Lambda^*}$. Then by several applications of (26) we obtain

(29) $$\begin{gathered} r_{\mathfrak{A}}(a) = 1 \text{ iff } r_{\Lambda^*}(a), \\ E_{\mathfrak{A}}(a, b) \text{ is } =_{\mathfrak{A}}(a, b), \\ =_{\Lambda^*}(a, b) \text{ iff } a = b. \end{gathered}$$

Combining these results implies that the equality of $\mathfrak{A}$ can be handled in exactly the same way as any of its relations.

We are going to show that $\mathfrak{A}$ admits an A-valuation. Let $\varphi \in L$ and $s: \omega \to U_{\mathfrak{A}}$. Define $\varphi(s) \in L_{\mathfrak{A}}$ by replacing each function f (relation r) of φ by $f_{\mathfrak{A}}(r_{\mathfrak{A}})$ respectively and replacing each free v_i of φ by s_i. Let n be any number such that the free variables of φ are in $\{v_0, \ldots, v_{n-1}\}$ and let $r_\varphi = r(\varphi)$ be the n-ary arithmetic relation such that

(30) $$\varphi \leftrightarrow r_\varphi(v_0, \ldots, v_{n-1})$$

is valid in ω^*. Finally, as a candidate for our valuation let

$$[\![\varphi(s)]\!]_{\mathfrak{A}} = r(\varphi)_{\mathfrak{A}}(s_0, \ldots, s_{n-1}).$$

One of the first things we must check is that $[\![\varphi(s)]\!]_{\mathfrak{A}}$ is independent of n (provided the free variables of φ are in $\{v_0, \ldots, v_{n-1}\}$). This is an easy consequence of (26).

LEMMA 4. $[\![\cdot]\!]_{\mathfrak{A}}$ *satisfies* (2).

Proof. This is done by direct case by case computation.

Case 1. φ is $p(\tau)$ where p is a unary relation symbol and τ is a term whose only free variable is v_0. Let f be an arithmetic function such that $\tau = f(v_0)$ is valid in ω^*. Now by (30) we have $p(f(v_0)) \leftrightarrow r_\varphi(v_0)$ valid in ω^* and hence the composition $c^p \circ f = c^{r(\varphi)}$. In [3] it is shown that

(31) the composition of arithmetic functions commutes with their extension to Λ^*.

If $\tau_{\mathfrak{A}}(s)$ is the denotation of τ in $\mathfrak{A}$ when s_0 is assigned to v_0, then (31) gives $\tau_{\mathfrak{A}}(s) = f_{\mathfrak{A}}(s_0)$ and

$$[\![\varphi(s)]\!]_{\mathfrak{A}} = r(\varphi)_{\mathfrak{A}}(s_0) = c_{\Lambda^*}^{r(\varphi)}(s_0) = c_{\Lambda^*}^{p}(f_{\Lambda^*}(s_0))$$
$$= p_{\mathfrak{A}}(f_{\mathfrak{A}}(s_0)) = p_{\mathfrak{A}}(\tau_{\mathfrak{A}}(s)).$$

For the rest of this proof we use $\wedge$, $\vee$, $\neg$, $\leqslant$ for the Boolean operations in the idempotent algebra of either ω^* or Λ^*. Whenever there is any danger of confusing these symbols with those denoting logical operations we shall use boldface for the Boolean ones.

Case 2. Suppose that the free variables of $\varphi \vee \psi$ are in $\{v_0, \ldots, v_{n-1}\}$. Let $r(\varphi)$, $r(\psi)$ and $r(\varphi \vee \psi)$ all be n-ary. Then $r(\varphi) \cup r(\psi) = r(\varphi \vee \psi)$ which implies that

$$c^{r(\varphi)}(v_0, \ldots, v_{n-1}) \vee c^{r(\psi)}(v_0, \ldots, v_{n-1}) = c^{r(\varphi \vee \psi)}(v_0, \ldots, v_{n-1})$$

is valid in ω^* and hence by (26) in Λ^*.

$$[\![\varphi(s)]\!]_{\mathfrak{A}} \vee [\![\psi(s)]\!]_{\mathfrak{A}} = [\![\varphi \vee \psi(s)]\!]_{\mathfrak{A}}$$

then follows from the definition of $[\![\cdot]\!]_{\mathfrak{A}}$.

Case 3. $\sim\psi$ is handled as in the preceding case.

Case 4. φ is $(\exists v_k)\psi$. For ease in writing we suppose that the only free variable in φ is v_0 and that $k = 1$. Take r_φ unary and r_ψ binary. Then

(32) $$r_\psi(v_0, v_1) \rightarrow r_\varphi(v_0),$$

(33) $$r_\varphi(v_0) \rightarrow (\exists v_1) r_\psi(v_0, v_1)$$

are both valid in ω^* and hence by (26) are also valid in Λ^*. (32) gives $c^{r(\psi)}(v_0, v_1) \leqslant c^{r(\varphi)}(v_0)$ valid in Λ^*. If $x \in U_{\mathfrak{A}}$ let s_x be that assignment which agrees with s except at 1 and assumes the value x there. Then

$$[\![\psi(s_x)]\!]_{\mathfrak{A}} \leqslant [\![\varphi(s_x)]\!]_{\mathfrak{A}} = [\![\varphi(s)]\!]_{\mathfrak{A}}$$

since v_1 is not free in φ. For the converse (33) gives

$$(\exists v_1) c^{r(\varphi)}(v_0) \leqslant c^{r(\psi)}(v_0, v_1)$$

valid in Λ^* and hence there is an $a \in U_{\mathfrak{A}}$ such that $[\![\varphi(s)]\!]_{\mathfrak{A}} \leqslant [\![\psi(s_a)]\!]_{\mathfrak{A}}$. Then

$$[\![\varphi(s)]\!]_{\mathfrak{A}} \leqslant [\![\psi(s_a)]\!]_{\mathfrak{A}} \leqslant \textstyle\sum^{\Lambda}\{[\![\psi(s_x)]\!]_{\mathfrak{A}} \mid x \in U_{\mathfrak{A}}\} \leqslant [\![\varphi(s)]\!]_{\mathfrak{A}}$$

and we are done. ∎

THEOREM 10. $\mathfrak{A}$ *is an almost* 2*-valued, discrete, perfect* Λ*-valued model. Moreover* ω^* *and* $\mathfrak{A}$ *have the same parameter free theory.*

Proof. By Lemma 4 $\mathfrak{A}$ is a model, *a fortiori* $\mathfrak{A}$ is a structure. If $\varphi \in S$ let $r(\varphi)$ be unary. If $\omega^* \models \varphi$ then $c^{r(\varphi)}$ is identically 1 so $[\![\varphi]\!]_{\mathfrak{A}} = 1$ and if ω^* not $\models \varphi$ then $c^{r(\varphi)}$ is identically 0 so $[\![\varphi]\!]_{\mathfrak{A}} = 0$, i.e., $\mathfrak{A}$ is almost 2-valued and has the same parameter free theory as ω^* (allowing parameters from ω^*). We have already seen that $\mathfrak{A}$ is discrete. It is rich by the last chain in the proof of Lemma 4. Finally, for pseudo-completeness (in the case that I has 2 elements) note that

$$u_0^2 = u_0 \wedge u_1^2 = u_1 \wedge (u_0 \vee u_1 = 1) \wedge (u_0 \wedge u_1 = 0) \rightarrow (\exists w) u_0 \leqslant \neg e(v_0 - w) \wedge$$
$$\wedge u_1 \leqslant \neg e(v_1 - w)$$

is valid in ω^* and hence in Λ^*. ∎

6. Romping through Λ^*. We obtain two important metatheorems of Nerode (cf. [3]) as consequences of our Boolean valued ring theory. Let C be the filter of cofinite subsets of ω and let $\mathfrak{B}$ be the reduced product (cf. [1]) $(Z)^{\omega}/C$. $\mathfrak{A}$ is as in the last section.

COROLLARY 2. (i) Λ^* *and* $\mathfrak{B}$ *have the same parameter free ring theory. Even better*

(ii) Λ^* and $\mathfrak{B}$, *conceived of as models whose functions and relations are canonical extensions of arithmetic functions and relations, have the same parameter free theory.*

Proof sketch. (i) follows from Theorems 8 and 10. (ii) follows in just about the same way using an easy generalization of Theorem 8 readily provided by Theorem 3. We discuss (i). By Theorem 10, Λ^* satisfies the conditions for one of the $\mathfrak{A}_i$ of Theorem 8. We must show that $\mathfrak{B}$ also satisfies these conditions. This is fairly easy; e.g., to show that $\mathfrak{B}^{\#}$ is an idempotent valued model of Z we simply duplicate the Łoś theorem for ultraproducts. Moreover, the idempotent algebra of $\mathfrak{B}$ is atomless since it is isomorphic to $2^{\omega}/C$. ∎

COROLLARY 3. *If P is a proper minimal prime ideal in Λ^* then*

(i) Λ^*/P *and* Z *(both discrete* 2*-valued models) have the same parameter free ring theory. Even better*

(ii) Λ^*/P *and* Z*, conceived of as models whose functions and relations are canonical extensions of arithmetic functions and relations, have the same parameter free theory.*

Proof. (i) is immediate from Theorem 9 and (ii) is an easy consequence of Theorems 4 and 10. ∎

In actual applications to Λ^*, it is almost impossible to directly use Theorem 3. Corollary 2 is better because we have some intuitive idea of what $\mathfrak{B}$ is like. This association would be perfect if we knew that $\mathfrak{B}$ was isomorphic to Λ^*. Despite some attempts we do not know if this is true. So instead of using $\mathfrak{B}$ as a "thought guide" to Λ^*, we show by example the relevance of $\mathfrak{A}$.

Let $\varphi \in S$ be a sentence written in prenex conjunctive form. By a *metatheorem* (in the style of Nerode) we mean an assertion of the form "φ holds in Λ^* if and only if some Horn reduct of φ holds in ω^*" (cf. [3] for the definition of Horn reduct). Thus when we say that Λ^* has a *universal metatheorem*, what we mean is that a metatheorem of the above sort holds for all universal sentences in S.

First consider the case where φ is a universal Horn sentence. As an illustration let us suppose that it has the form

$$(\forall v)\big(r(v) \wedge s(v) \rightarrow t(v)\big)$$

where v is a variable and r, s, t are relations. If $\omega^* \models \varphi$ then by Theorem 10 we have $[\![\varphi]\!] = 1$. Consequently $[\![r(a)]\!] \wedge [\![s(a)]\!] \leqslant [\![t(a)]\!]$ for any $a \in \Lambda^*$. In particular if $a \in r_{\Lambda^*}$ and $a \in s_{\Lambda^*}$ then by (29), $r_{\mathfrak{A}}(a) = 1$ and $s_{\mathfrak{A}}(a) = 1$ making $t_{\mathfrak{A}}(a) = 1$, i.e., $a \in t_{\Lambda^*}$. Thus $\Lambda^* \models \varphi$ and we have shown that if a universal Horn sentence is true in ω^* then it is true in Λ^*. This has the consequence that if φ is any universal sentence having a Horn reduct true in ω^* then φ is true in Λ^*. We thus have proved one-half of the universal metatheorem.

For the converse suppose that φ is no longer Horn, but still universal. By the preceding paragraph we lose no generality in assuming that φ has the form

$$(\forall v)\big(r(v) \rightarrow \big(s(v) \vee t(v)\big)\big).$$

Suppose that both Horn reducts of φ fail in ω^*. Then $\omega^* \models (\exists v)\big(r(v) \wedge \sim s(v)\big)$, $\omega^* \models (\exists v)\big(r(v) \wedge \sim t(v)\big)$. Theorem 10 then gives $[\![(\exists v)\big(r(v) \wedge \sim s(v)\big)]\!] = 1$ as well as $[\![(\exists v)\big(r(v) \wedge \sim t(v)\big)]\!] = 1$. Richness gives us elements $a_0, a_1 \in \Lambda^*$ such that $[\![r(a_0) \wedge \sim s(a_0)]\!] = 1$ and $[\![r(a_1) \wedge \sim s(a_1)]\!] = 1$. Now let b_0, b_1 be a partition of A, the idempotent algebra of Λ^*, where neither b_0 nor b_1 is the zero of A (cf. (27)). Then by pseudo-completeness there is an $x \in \Lambda^*$ such that $b_i \leqslant [\![a_i = x]\!]$. Now $[\![r(a_i)]\!] = 1$ and hence $b_i \leqslant [\![r(a_i)]\!] \wedge [\![a_i = x]\!] \leqslant [\![r(x)]\!]$ for $i<2$ by (3). Summing on i gives $1 \leqslant [\![r(x)]\!]$ so $x \in r_{\Lambda^*}$ by (29). Also $b_0 \leqslant [\![\sim s(a_0)]\!] \wedge [\![a_0 = x]\!] \leqslant [\![\sim s(x)]\!]$ and so $[\![s(x)]\!] \leqslant -b_0$, $[\![s(x)]\!] \neq 1$, $x \notin s_{\Lambda^*}$. Similarly $x \notin t_{\Lambda^*}$ giving us a counterexample to φ in Λ^*. We have thus shown that Λ^* has a universal metatheorem for sentences having a single conjunct. The full result follows as soon as we recall that universal quantifications distribute over conjunctions.

Say that φ is *disjunctive* if it is arbitrarily quantified and its matrix consists of of a single conjunct. If in addition φ is a Horn sentence, we may suppose it is written in the form

$$(\mathrm{Q}v)\big(r(v) \rightarrow s(v)\big)$$

where $(\mathrm{Q}v)$ is a string of quantified variables. If $\omega^* \models \varphi$ then we can find a sequence of arithmetic Skolem functions for φ. Let ψ be the result of replacing existentially quantified variables in φ by terms denoting these Skolem functions. Then $\omega^* \models \psi$ so $\Lambda^* \models \psi$ by the universal metatheorem. Replacing terms by variables immediately gives $\Lambda^* \models \varphi$. Thus we have shown that if a disjunctive Horn sentence is true in ω^* then it is true in Λ^*. This has the consequence that if φ is any disjunctive sentence having a Horn reduct true in ω^* then φ is true in Λ^*, i.e., one-half of the disjunctive metatheorem.

For the converse suppose that φ is no longer Horn, but still disjunctive. As an illustration let us suppose that φ has the form

$$(\forall v_0)(\exists v_1)(\mathrm{Q}u)(r \rightarrow s \vee t)$$

where $(\mathrm{Q}u)$ is a string of quantified variables. Further, let us suppose that both Horn reducts of φ fail in ω^*. Let b_0, b_1 be a partition of A such that neither b_0 nor b_1 is the zero of A. Then by Theorem 10 we obtain in a trivial sense, that

$$\text{(34)} \qquad \begin{aligned} b_0 &\leqslant [\![(\exists v_0)(\forall v_1)\sim(\mathrm{Q}u)(r \rightarrow s)]\!], \\ b_1 &\leqslant [\![(\exists v_0)(\forall v_1)\sim(\mathrm{Q}u)(r \rightarrow t)]\!]. \end{aligned}$$

Richness then gives $a_0, a_1 \in \Lambda^*$ such that

$$\begin{aligned} b_0 &\leqslant [\![(\forall v_1)\sim(\mathrm{Q}u)\big(r(a_0) \rightarrow s(a_0)\big)]\!], \\ b_1 &\leqslant [\![(\forall v_1)\sim(\mathrm{Q}u)\big(r(a_1) \rightarrow t(a_1)\big)]\!]. \end{aligned}$$

By pseudo-completeness there is an $x \in \Lambda^*$ such that $b_i \leqslant [\![a_i = x]\!]$ for $i<2$. Then proceed as in the universal case to show that

$$\begin{aligned} b_0 &\leqslant [\![(\forall v_1)\sim(\mathrm{Q}u)\big(r(x) \rightarrow s(x)\big)]\!], \\ b_1 &\leqslant [\![(\forall v_1)\sim(\mathrm{Q}u)\big(r(x) \rightarrow t(x)\big)]\!]. \end{aligned}$$

Thus we have shown that there is an $x \in \Lambda^*$ such that for all $y \in \Lambda^*$

$$\text{(35)} \qquad \begin{aligned} b_0 &\leqslant [\![\sim(\mathrm{Q}u)\big(r(x, y) \rightarrow s(x, y)\big)]\!], \\ b_1 &\leqslant [\![\sim(\mathrm{Q}u)\big(r(x, y) \rightarrow t(x, y)\big)]\!]. \end{aligned}$$

Continuing this process which led from (34) to (35) we show that

$$(\exists x \in \Lambda^*)(\forall y \in \Lambda^*)(\exists x_0 \in \Lambda^*) \ldots$$

$$\begin{aligned} b_0 &\leqslant [\![r(x, y, x_0, \ldots) \wedge \sim s(x, y, x_0, \ldots)]\!], \\ b_1 &\leqslant [\![r(x, y, x_0, \ldots) \wedge \sim t(x, y, x_0, \ldots)]\!]. \end{aligned}$$

Then just as in the universal case these inequalities imply that

$$(\exists x \in \Lambda^*)(\forall y \in \Lambda^*)(\exists x_0 \in \Lambda^*) \ldots (x, y, x_0, \ldots) \in r_{\Lambda^*}, \ \notin s_{\Lambda^*}, \text{ and } \notin t_{\Lambda^*}$$

showing that φ fails in Λ^*. Thus we have shown that Λ^* has a disjunctive metatheorem.

Another class of sentences for which we obtain a metatheorem is the positive sentences. By noting that conjunctions such as $r(v) \wedge s(v)$ can be replaced by some $t(v)$, and using propositional distributive laws, we can rewrite the matrix of a positive sentence in disjunctive form. Thus the positive case is subsumed under the disjunctive one. We have obtained metatheorems for the universal, the positive, and the disjunctive sentences. It is known from experience with $(Z)^{\omega}/C$ that this is about as far as we can go. Note that in our proofs we used partitions of A into two pieces. This was because our sentences had two positive disjuncts. Had there been more then we would have had to partition A into more pieces. For the metatheorems to work we need arbitrarily large finite partitions. However, this is incomparably weaker than the fact that A is atomless.

This completes our Boolean valued theory of A^*. We remark that an equivalent treatment of A^* could have been obtained by forcing together with Theorem 6.

References

[1] S. Feferman and R. L. Vaught, *The first order properties of products of algebraic systems*, Fund. Math. 47 (1959), pp. 57–103.
[2] R. Mansfield, *The theory of Boolean ultrapowers*, Ann. Math. Logic 2 (1971), pp. 297–323.
[3] A. Nerode, *Arithmetically isolated sets and nonstandard models*, Recursive Function Theory, Proc. Symp. Pure Math. 5 (1962), pp. 105–116.
[4] D. Scott, *On constructing models for arithmetic*, Infinitistic Methods, Proc. Symp. Found. Math. Warsaw 1959, pp. 235–255.
[5] A. Tarski, *Arithmetical classes and types of Boolean algebras*, Preliminary report, Bull. Amer. Math. Soc. 55 (1949), pp. 64, 1192.
[6] A. Urquhart, *Boolean model theory I and II* (preprint, Univ. of Toronto).

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ERRATA

Page, ligne	Au lieu de	Lire
54^5	[5]	[5], [9]
$66_{2,3}$	[8] П. С. Солтан, К. Ф. Присакару, *Задача...*	[8] J. de Groot, *Some special metrics in general topology*, Coll. Math. 6 (1958), pp. 283–286. [9] W. Nitka, *Remarks on sets convex in the sense of J. de Groot*, Indag. Math. 21 (1959), pp. 36–38.
74_3	$\leqslant \Sigma^b, \{h$	$\leqslant \Sigma^b\{h$
82_8	$r_{\varphi}(v_0, v_1)$	$r_{\psi}(v_0, v_1)$
84_{18}	$s(a_1)$	$t(a_1)$