Theory of equidistance and betweenness relations
in regular metric spaces *

by

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Abstract. The paper contains some partial results concerning theory of metric equidistance
and betweenness relations.

The purpose of this note is, roughly speaking, to describe metric spaces in terms
of two relations: equidistance relation $D$ and betweenness relation $B$. From this
point of view, the class of all the metric spaces over ordered groups seems to be too
large on the one hand, and too small on the other; too large — as regards geometric
properties, and too small — as regards algebraic ones. For this reason, the objects
under consideration are some special metric spaces over rather general algebraic
structures.

To the author’s best knowledge, this is the first paper on this subject. Since many
questions remain open, the author believes this is not the last one.

To avoid any confusion, we give definitions of all the algebraic notions used
in the paper, even those which can be found in the literature.

Algebraic preliminaries. Let us consider a system $\mathcal{G} = (G, G_0, 0, +)$ with
$G_0 \subseteq G - \{0\}$. We shall use $a, b, r, \ldots$ to denote elements of $G_0$ and $x, \lambda, \mu, \ldots$ to
denote arbitrary elements of $G$.

The system $\mathcal{G}$ is said to be a commutative semi-group generated by $G_0$ whenever
$0$ is a neutral element with respect to $+$,

$+$ is associative and commutative,

and

$\bigwedge_{x \neq 0} \bigvee_{a_1, \ldots, a_n} \lambda = a_1 + \ldots + a_n.$

$\mathcal{G}$ is a commutative semi-group with cancellation if additionally

$\lambda + \mu = \lambda' + \mu \Rightarrow \lambda = \lambda'$ for every $\lambda, \lambda', \mu$.

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*** Fundamenta Mathematicae XCVI
Let \( N \) be the set of natural numbers and let \( \bar{N} = N \cup \{0\} \). Put
\[
\begin{align*}
0 \cdot \lambda &= 0 \\
\lambda \cdot n &= (n-1) \cdot \lambda + \lambda & \text{for} & n \in N, \lambda \in G.
\end{align*}
\]
\( \mathcal{G} \) is freely generated by \( G_0 \) whenever
\[
\bigwedge_{(m_1, \ldots, m_k) \in \bar{N}^k} \bigwedge_{n \in \bar{N}} (m_1 \cdot n_1 + \cdots + m_k \cdot n_k = n_1 \cdot x_1 + \cdots + n_k \cdot x_k \Rightarrow m_i = n_i, i = 1, \ldots, k).
\]

Example 1. The semi-group \( \mathcal{N} = (N, 1, 0, +) \) is freely generated by \( \{1\} \).

Example 2. Given an arbitrary set \( X_0 \), let us consider the following set of functions:
\[
C = \{ \lambda \in N^{X_0} : \lambda(a) = 0 \text{ for almost all } a \in X_0 \}.
\]

Let
\[
(\lambda + \mu)(a) = \lambda(a) + \mu(a) \quad \text{for every} \quad a \in X_0;
\]
of course, setting
\[
\lambda = 0 \Leftrightarrow \lambda(a) = 0 \text{ for every } a \in X_0
\]
we get a neutral element of \( + \). Take the subset \( C_0 \) of \( C \) consisting of the non-zero characteristic functions for all the elements of \( X_0 \). The system
\[
\mathcal{G}(X_0) = (C, C_0, 0, +)
\]
is a commutative semi-group freely generated by \( C_0 \). This semi-group will be referred to as the semi-group of chains in \( X_0 \) (over the semi-group \( \mathcal{N} \)).

According to the traditional notation we shall not distinguish between an element \( \alpha \) of \( X_0 \) and its characteristic function, and thus we shall write
\[
\lambda = \sum_{i=1}^{k} m_i \cdot x_i \quad \text{whenever} \quad \lambda(x_i) = m_i \quad \text{for some} \quad i = 1, \ldots, k \quad \text{and} \quad \lambda(x) = 0 \quad \text{otherwise}.
\]

A commutative semi-group is said to be free whenever it is freely generated by some \( G_0 \); it is said to be acyclic whenever
\[
n \cdot \lambda = 0 \Rightarrow n = 0 \Leftrightarrow \lambda = 0.
\]

Let us notice that
\[
\text{Proposition 1. Every free commutative semi-group is acyclic.}
\]

A system \( (G, \geq) \) is a partially pseudo-ordered set provided \( \geq \) is reflexive and weakly antisymmetric, i.e.
\[
\lambda \geq \lambda \quad \text{and} \quad \lambda \geq \lambda' \land \lambda' \geq \lambda \Leftrightarrow \lambda = \lambda' \quad \text{for every} \quad \lambda, \lambda'.
\]

A system \( \mathcal{G}^p = (\mathcal{G}, \geq) \) is a partially pseudo-ordered semi-group whenever
I. \( \mathcal{G} \) is a commutative semi-group,
II. \( (G, \geq) \) is a partially pseudo-ordered set,
III. \( \lambda \geq 0 \) for every \( \lambda \in G_0,
2. \lambda \geq \lambda' \Leftrightarrow \lambda + \lambda' \geq \lambda' + \lambda, \text{ i.e. } + \text{ is monotone with respect to } \geq,
\]

By III.1, 2 together with the weak antisymmetry of \( \geq \), it follows that
\[
\lambda + \mu = 0 \Rightarrow \lambda = 0 \land \mu = 0 \quad \text{for every} \quad \lambda, \mu;
\]
thus
\[
\text{Proposition 2. Every partially pseudo-ordered semi-group is acyclic.}
\]

Equidistance and betweenness relations in arbitrary metric spaces. Let us consider the class
\[
\mathcal{G}_0 = (\mathcal{G}^p = (\mathcal{G}, \geq), \mathcal{G}^p \text{ is a partially pseudo-ordered semi-group with cancellation})
\]
and a system
\[
\mathcal{X} = (X, \mathcal{G}^p, \xi)
\]
consisting of a set \( X \), a semi-group \( \mathcal{G}^p \in \mathcal{G}_0 \) and a function \( \xi : X \times X \rightarrow G \) satisfying the well known metric axioms (*):

M.1. \( \xi(ab) = 0 \Rightarrow a = b, \)
M.2. \( \xi(ab) = \xi(ba), \)
M.3. \( \xi(ab) + \xi(bc) \geq \xi(ac). \)

The system \( \mathcal{X} \) will be referred to as a metric space over \( \mathcal{G}^p \).

Let us define the following two relations in \( \mathcal{X} \):
\[
D_0(abc) \Leftrightarrow \xi(ab) = \xi(cd),
R_0(abc) \Leftrightarrow \xi(ab) + \xi(bc) = \xi(ac).
\]

These two relations will be referred to as metric equidistance and metric betweenness relation.

Let \( \mathcal{G} \subseteq \mathcal{G}_0 \). Given an arbitrary class \( \mathcal{M} \) of metric spaces over \( \mathcal{G} \) (i.e. over semi-groups from \( \mathcal{G}_0 \)), one can consider the following three classes of relational structures:
\[
\mathcal{D}_\mathcal{M} = \{(X, D_\mathcal{M}) : \mathcal{X} \in \mathcal{M}\},
\mathcal{R}_\mathcal{M} = \{(X, R_\mathcal{M}) : \mathcal{X} \in \mathcal{M}\},
\mathcal{D}_\mathcal{M} = \{(X, D_\mathcal{M}) : \mathcal{X} \in \mathcal{M}\}.
\]

The problem arises, for which \( \mathcal{M} \) the classes \( \mathcal{D}_\mathcal{M}, \mathcal{B}_\mathcal{M} \) and \( \mathcal{D}_\mathcal{B}_\mathcal{M} \) are elementary classes. We give a solution for \( \mathcal{D}_\mathcal{B}_\mathcal{M} \) and \( \mathcal{D}_\mathcal{M} \), the classes \( \mathcal{B}_\mathcal{M} \) and \( \mathcal{M}_j \) being defined as follows.

(*). Throughout the paper we omit the universal quantifiers which should be placed in front of a formula to bound all the free variables occurring in it.

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\( \lambda \) is a natural modification of the well known notion of a group of chains over the group of integers.
Consider two additional metric axioms $M_{A_4}$ ($n \in N$) and $M.5$ (the first one is an elementary schema (7)):

$$M_{A_4}: \bigwedge_{n \in N} \bigvee_{k \in N} \bigvee_{l \in N} \big( q(q_0 q_{l-1}) + q(q_{k-1} q_l) = q(q_0 q_l) \big) \land q(q_{1-1} q_1) = q(q_{00} b_{00} b_{10}).$$

$$M.5: q(ab) + q(bc) = q(ac) \land q(ac) = q(ad') \land q(ab) = q(ab') \land q(bc) = q(bc').$$

(The first sentence describes a kind of rectifiability, the second one --- a kind of homogeneity.)

$\mathcal{F} \in M_2$, whenever $\gamma \in G_0$, and all the axioms $M.1$-$M.5$ are satisfied. Such a space $\mathcal{F}$ will be referred to as a regular metric space.

Let $G_{i+1} = \langle \gamma \in G_i; \gamma \cdot \gamma \rangle$ be a partially ordered free commutative semi-group. $\gamma \in M_1$, whenever $\gamma \in G_0$, and the axioms $M.1$-$M.3$ are satisfied.

**Regular DB-structures.** A DB-structure is understood as a model $(\mathcal{X}, D, B)$ of the following axiom system (A.1-$A.14$) (7):

A.1. $D(abab),$
A.2. $D(abcd) \land D(ad'c'd) \Rightarrow D(aba'b'),$
A.3. $D(aabb),$
A.4. $D(abc) \Rightarrow a = b,$
A.5. $D(abba),$
A.6. $B(aab),$
A.7. $B(abc) \Rightarrow B(cha),$
A.8. $B(abc) \land B(acb) \Rightarrow b = c,$
A.9. $B(abd) \land B(bad) \Rightarrow B(abc) \land B(acd),$
A.10. $B(abd) \land B(bad) \land D(abc) \Rightarrow a = b = c = d,$
A.11. $D(ab'b') \land D(bac'c') \land D(ac'a'c') \land B(abc) \Rightarrow D(b'b'c'),$
A.12. $D(ab'b') \land D(ac'a'c') \land B(abc) \land B(b'c') \Rightarrow D(bac'c'),$
A.13. $\bigwedge_{i \in I} [B(p_0 p_{i-1} p_i) \land B(q_0 q_{i-1} q_i)] \land \bigvee_{i \in I} D(p_{i-1} p_i q_{i-1} q_i) = D(p_0 p_i q_0 q_i).$ (7)

A DB-structure is said to be regular provided it satisfies additionally the following two axioms $A.15,$ and $A.16$:

A.15. $\bigvee_{i \in I} [B(q_0 q_{i-1} q_i) \land D(\alpha q_{i-1} q_i b_{00} b_{10})],$
A.16. $B(abc) \land D(ac'a'c') \Rightarrow \bigvee_{b} [D(ab'b') \land D(bcb'c')].$

As direct consequences of the above axioms one obtains

0.1. $B(abab) \Rightarrow a = b$ (by A.6, A.8),
0.2. $B(abp_{i+1}) \land \bigvee_{i \in I} B(p_i p_{i+1}) \Rightarrow \bigvee_{i \in I} B(\alpha p_{i+1}),$ (by A.7, A.9),
0.3. $\bigvee_{i \in I} B(p_0 p_{i-1} p_i) \Rightarrow \bigvee_{i \in I} B(p_0 p_{i+1}),$ (by A.7, A.9),
0.4. $\bigvee_{i \in I} B(p_0 p_{i-1} p_i) \land D(p_0 p_i q_0 q_i) \Rightarrow \bigvee_{i \in I} [B(q_0 q_{i-1} q_i) \land D(p_0 p_{i-1} q_0 q_i) \land D(p_{i-1} p_i q_0 q_i)].$

(by A.11, A.16).

**Theory of equidistance and betweenness relations in regular metric spaces.** We are going to prove the following

**Representation Theorem I.** The class $DB_{\mathcal{H}}$, coincides with the class of models of (A.1-$A.16$), i.e.

$$DB_{\mathcal{H}} = \exists \eb (A.1-$A.16$).$$

This theorem is a corollary of Theorems I.1 and I.2 below.

**Theorem I.1. Every structure $(\mathcal{X}, D_{\mathcal{X}}, B_{\mathcal{X}})$, with $\mathcal{X} \in M_2$, is a model of (A.1-$A.16$).

**Proof.** Let $\mathcal{X} = (\mathcal{X}, \gamma \cdot \gamma) \in M_2$; then, by Proposition 2, $\gamma \cdot \gamma$ is acyclic. The structure $(\mathcal{X}, D_{\mathcal{X}}, B_{\mathcal{X}})$ satisfies A.1-$A.16$; indeed, A.1, A.2, A.11 and A.13, follow by the definitions of $D_{\mathcal{X}}$ and $B_{\mathcal{X}}$; A.3-$A.5$ by M.1 and M.2; A.6 --- by $0 + 0 = 1$; M.2 and commutativity of $+$ imply A.7; the condition M.2 together with acyclicity and cancellation imply A.8; in turn A.9 follows by M.3 together with cancellation, monotony of $+$ and weak antisymmetry of $\geq$; A.10 --- by M.1, acyclicity and cancellation; A.12 --- by cancellation; A.14 --- by M.3, commutativity of $+$ and weak antisymmetry of $\geq$; A.15, --- by M.4, and finally A.16 --- by M.5.}
Theorem 1.1 enables us to define the following function
\[ \Phi_0: \mathcal{M}_0 \to \mathcal{R}(A.1-A.16), \]
\[ \Phi_0(\mathcal{X}) = (\mathcal{X}, D_\mathcal{X}, B_\mathcal{X}) \quad \text{for every } \mathcal{X} \in \mathcal{M}_0. \]

**Theorem 1.2.** For every regular DB-structure \((\mathcal{X}, D, B)\) there is a regular metric space \(\mathcal{X}\) such that \(D_\mathcal{X} = D\) and \(B_\mathcal{X} = B\).

In other words, there is a function
\[ \Psi: \mathcal{R}(A.1-A.16) \to \mathcal{M}_0 \]
such that \(\Phi_0 \circ \Psi(X, D, B) = (\mathcal{X}, D, B)\) for every \((\mathcal{X}, D, B) \in \mathcal{R}(A.1-A.16).\)

The construction of the function \(\Psi\) will be referred to as a metrization of regular DB-structures.

**Metrization of regular DB-structures.** Let us consider an arbitrary regular DB-structure, i.e. any model \((\mathcal{X}, D, B)\) of the axiom system \((A.1-A.16).\) We are going to define a partially pseudo-ordered semi-group with cancellation, \(g_{(x, y, a)} = (g_{(x, y, a)}', ')\), and a function \(\mathcal{V}: \mathcal{X} \times \mathcal{X} \to G_{(x, y, a)}\) such that the system \(\mathcal{X} = (X, g_{(x, y, a)}', ')\) is a regular metric space, and the relations \(D_\mathcal{X}\) and \(B_\mathcal{X}\) coincide with \(D\) and \(B\).

1. The quaternary relation \(D\) in \(\mathcal{X}\) induces the following binary relation \(\approx\) in \(\mathcal{X}^2;\)

\[ ab = cd \Leftrightarrow D(abcd). \]

By A.1-A.5 we get
1. \(\approx\) is an equivalence relation.
2. \(aa = bb,\)
3. \(ab = cc \Rightarrow a = b,\)
4. \(ab = ba.\)

Let \(X_0 = \mathcal{X}^2.\) Consider the semi-group of chains \(C(X_0)\) (see Example 2) and let \(\Lambda\) be the smallest congruence in \(C(X_0)\) which contains \(([\text{iso}, 0]).\) Let
\[ g_{(x, y, a)} = C(X_0)_{\approx}. \]

Notice that
1.5. \(g_{(x, y, a)}\) is a commutative semi-group freely generated by \(X_0.\)

We are going to define (by means of \(B\)) a binary relation \(\approx\) in \(G_{(x, y, a)},\)

satisfying the following five conditions:

(i) \(\approx\) is a congruence with respect to \(+,\)

(ii) \(\lambda \sim 0 \Rightarrow \lambda = 0,\)

(iii) \(\alpha \sim \beta \Rightarrow \alpha = \beta,\)

(iv) \([ab] + [bc] = [ac] \Rightarrow B(abcd),\)

(v) \(\lambda + \beta = \alpha + \beta \Rightarrow \lambda \sim \beta.\)

Let us consider the following function \(S: G_{(x, y, a)} \to Z_{(x, y, a)}.\)

\[ S(\alpha) = \{ [p_0 p_1] + \ldots + [p_{n-1} p_n] : \alpha = [p_0 p_n] \wedge B(p_0 p_{n-1} p_n), \ i = 1, \ldots, n, p_i \in N\}. \]

2.0. \(S \left( \sum_{i=1}^{N} m_i a_i \right) = \left( \sum_{i=1}^{N} m_i a_i \right) \quad \text{for } \lambda = \left( \sum_{i=1}^{N} m_i a_i \right), \quad \lambda = \left( \sum_{i=1}^{N} m_i a_i \right).\)

The relation \(\approx\) is defined by the formula
\[ \lambda \approx \beta \Leftrightarrow \bigvee \lambda, \beta \in S(\alpha). \]

In the sequel, we write simply \(\approx\) whenever there is no danger of a confusion.

The proof of (iii)-(v) is based on the following statements concerning \(S.\)

2.1. \(S(0) = \{0\},\)
2.2. \(0 \in S(\lambda) \Rightarrow \lambda = 0,\) (by 1.5),
2.3. \(\lambda \in S(\lambda),\) (by A.6),
2.4. \(\lambda \in S(\lambda) \Leftrightarrow \bigvee \sum_{i=1}^{N} m_i a_i.\)

2.5. \(x \in S(\lambda + \lambda) \Leftrightarrow x_{\lambda_1, \lambda_2} \Leftrightarrow S(\lambda),\)
2.6. \(S(0) \cap S(\beta) = \emptyset \Rightarrow \beta \neq \beta,\) (by 2.4, 1.5 and A.13),
2.7. \(S(\lambda) = \lambda \Rightarrow \lambda \wedge \beta \) (by 2.2, 2.3, 2.5, 2.6),
2.8. \(\lambda \in S(\lambda) \wedge \lambda \in S(\mu) \Rightarrow \lambda \in S(\mu)\) (by 2.4, 2.5, 1.5, 0.4a and 0.2a).

**Proof.** Let \(\lambda = \sum_{i=1}^{N} a_i,\) and \(\mu = \sum_{j=1}^{N} \beta_j.\) Induction on \(m.\)

1. Let \(m = 1,\) then
2.5. \(x \in S(\lambda) \Leftrightarrow \bigvee_{x_{\lambda_1, \lambda_2}} x_{\lambda_1, \lambda_2} \in S(\lambda).\)

By 2.4,
2.4. \(\bigvee_{x_{\lambda_1, \lambda_2}} x_{\lambda_1, \lambda_2} \in S(\lambda) \Leftrightarrow \bigvee_{x_{\lambda_1, \lambda_2}} x_{\lambda_1, \lambda_2} \in S(\lambda).\)

By 2.4,
2.4. \(\bigvee_{x_{\lambda_1, \lambda_2}} x_{\lambda_1, \lambda_2} \in S(\lambda) \Leftrightarrow \bigvee_{x_{\lambda_1, \lambda_2}} x_{\lambda_1, \lambda_2} \in S(\lambda).\)
and
\[ (2) \quad \bigwedge_{i=1}^{l} \left( \sum_{n_{i} \in \mathcal{E}} q_{i}^{n_{i}} + \left[ q^{l-i} \right] \wedge \beta_{i} = [q^{l-i} \wedge B(q^{l-i} \wedge q^{i})] \right). \]

Thus, by 1.5, one gets
\[ (3) \quad l = n \text{ and there is a bijection } f : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\} \text{ such that} \]
\[ [p_{0}^{l-1} q_{0}] = [q^{l-1} q_{0}] \text{ for } i = 1, \ldots, n. \]

By (1), (3) and \( Q_{m_{k}} (k = 1, \ldots, n) \), there exist \( q_{0}, \ldots, q_{m_{n}} \), such that
\[ (4) \quad B(q^{i} f_{i-1} q^{i}) \quad \text{for } i = 1, \ldots, m_{k} = m_{n}, \quad k = f(i), \quad \text{and} \]
\[ q_{0} = q^{m_{n}}, \quad q_{m_{n}} = q^{i} \text{ for } k = 1, \ldots, n. \]

By (2)-(4), applying 0.2, one gets
\[ (5) \quad B(q^{j} f_{j-1} q^{j}) \quad \text{for } j = 1, \ldots, m_{n} = m_{n}, \quad k = 1, \ldots, n. \]

By (1)-(5), \( x \in S(\beta_{i}) \), i.e. \( x \in S(\mu) \) for \( m = 1. \)

2\( \alpha \quad \text{Assume that the assertion holds for } m \leq m_{n} - 1 \text{ and let } m = m_{n} \text{ i.e. } \mu = \sum_{i=1}^{m_{n}} \beta_{i}. \)

Let \( \mu' = \sum_{i=1}^{m_{n}} \beta_{i} \); then \( \mu = \mu' + \beta_{m_{n}}. \) Since \( \lambda \in S(\mu) \), by 2.5 it follows that
\[ (6) \quad \bigvee_{x, x'} \lambda = \lambda' + \lambda'' \wedge \lambda' \in S(\mu') \wedge \lambda'' \in S(\beta_{m_{n}}) \]

in turn, since \( x \in S(\lambda', \lambda'') \), hence
\[ (7) \quad \bigvee_{x, x'} x = \lambda' + \lambda'' \wedge x' \in S(\lambda') \wedge x'' \in S(\lambda''). \]

By the inductive assumption, (6) and (7) imply \( x' \in S(\mu') \) and \( x'' \in S(\beta_{m_{n}}) \), thus \( x \in S(\mu). \)

As a direct consequence of A.15, one obtains
\[ (8) \quad \bigwedge_{x, x'} x \in S(\mu). \]

The statements 2.1-2.9 imply the following assertions concerning the relation \( \sim. \)

2.10. \( \lambda \sim \lambda' \Leftrightarrow \bigvee_{x, x'} x \in S(\lambda') \) (by 2.8, 2.9).

2.11. \( \lambda \sim \beta \Rightarrow \lambda \in S(\beta) \) (by 2.9, 2.3 and 2.6).

By 2.10 and 1.5 together with A.12, it follows easily that
\[ (2.12) \quad \lambda' + \beta' \sim \beta + \lambda' \Rightarrow \lambda' = \lambda. \]

Now, the condition (i) follows by 2.0, 2.3, 2.5, 2.10; condition (ii) — by 2.1, 2.2; the condition (iii) — by 2.7.

Proof of (iv): (by 2.11, 2.4, A.5-A.7, A.11, 2.0, 2.3). Let \( [a b] + [b c] \sim [a c] \).

If either \( a = b \) or \( b = c \), then by A.6 and A.7 one gets \( B(abc) \). If \( a = c \), then, by (ii), \( a = b \) and thus \( B(abc) \). If \( a \neq b \), then, by 2.11, \( [a b] + [b c] \in S(ac) \). By 2.4, there exist \( a \neq b \) such that \( [a b] + [b c] = [a b'] + [b c'] \), \( [a c] = [a c'] \) and \( B(a b' c) \).

There are two possibilities: either \( a b = a b' \) and \( b c = b' c' \) or \( a b = b' c' \) and \( b c = a b' \).

In the first case A.11 implies \( B(abc) \), in the second one, A.5 and A.11 imply \( B(abc) \).

The converse implication \( B(abc) = [a b] + [b c] \sim [a c] \) follows directly by 2.0 and 2.3.

Proof of (v): (by 2.7, 2.10, 2.3, 2.6, and 2.12). Let \( \lambda + \beta \sim \alpha + \beta \). By 2.10, there is a \( \gamma \) such that \( \lambda + \beta \in S(y) \) and \( \alpha + \beta \in S(y) \). Thus, there are \( p_{0}, \ldots, p_{n} \) such that
\[ B(p_{0} p_{1} \ldots p_{n}) \text{ for } i = 1, \ldots, n. \]

Then, by 1.5, there is a \( f \in \{1, \ldots, n\} \) such that \( \beta = [p_{0} p_{1} \ldots p_{n}] \).

Obviously
\[ (1) \quad \lambda \in S(\lambda). \]

By 2.5, there exist \( a' \) and \( \gamma' \) such that
\[ (2) \quad x \in S(\lambda) \]
and
\[ (3) \quad \alpha' + \beta \in S(\gamma'). \]

By (1), (2) and 2.3, one gets
\[ (4) \quad \lambda + \beta \in S(\alpha' + \beta) \quad \text{and} \quad \alpha + \beta \in S(\alpha' + \beta). \]

By 2.8, the condition (4) implies
\[ (5) \quad \lambda + \beta \in S(\alpha' + \beta), \]
and (5) together with (3) imply \( \lambda + \beta \in S(\gamma') \). By 2.6, \( \lambda + \beta \in S(\gamma') \cap S(\gamma') \) implies \( \gamma' = \gamma \). Thus, by (3), \( \alpha' + \beta \in S(\gamma) \) and \( \alpha + \beta \in S(\gamma) \), i.e. \( \alpha' + \beta \sim \alpha + \beta \). Then by 2.12, \( \alpha' = \alpha \) and thus, by (1), (2) and 2.8, \( \lambda \in S(\alpha) \). Therefore \( \lambda \sim \alpha \).}

3. We are going to define a binary relation \( \geq_{B} \) in \( G_{(x, B)} \), satisfying the following six conditions:

(i) \( \lambda \geq_{B} 0 \) for every \( \lambda \).

(ii) \( \geq_{B} \) is reflexive.

(iii) \( \geq_{B} \) is a congruence with respect to \( \geq_{B} \).

(iv) \( \geq_{B} \) is monotone with respect to \( \geq_{B} \).

(v) \( \lambda \geq_{B} \lambda' \Leftrightarrow \lambda' = \lambda \).

(vi) \( [a b] + [b c] \geq_{B} [a c] \).
Let us consider the following function $T: G_{(x,d)} \to 2^{G_{(x,d)}}$

$$T^+(a) = \{ T^+(p_1) + \ldots + T^+(p_n) : p_1 + \ldots + p_n = a, \ p_i \in N \}$$

$$T^-(\sum_{i=1}^k m_i d_i) = \sum_{i=1}^k m_i T^-(d_i), \quad \text{for } m_i \in N, \quad i = 1, \ldots, k.$$

The relation $\geq$ is defined by means of the auxiliary relation $\geq$:

$$\lambda \geq x \iff \exists \gamma \in T(\lambda) \land x \in S(\gamma),$$

$$\lambda \geq x \iff \exists \gamma \in T(\lambda) \land x \in S(\gamma).$$

Obviously

3.1. $S(\lambda) \subseteq T(\lambda),$

3.2. $\lambda \in T(\lambda),$

and

3.3. $x \in T(\lambda) \land \lambda \in T(\mu) \Rightarrow x \in T(\mu).$

As a consequence of A.14 and 0.4, one gets

3.4. $S(\lambda) \cap T(\mu) \neq \emptyset \neq T(\lambda) \cap S(\beta) \Rightarrow \lambda = \beta.$

By 2.9, 2.8, 3.1 and 3.3 it follows that

3.5. $\lambda \geq x \iff \exists \gamma \in T(\lambda) \land x \in S(\gamma).$

The condition (i) follows by 3.0 and 2.1 together with the reflexivity of $\geq$; the condition (ii) — by 3.2 and 2.3; the condition (iii) — by 2.10, 3.5 and 2.6; the additivity of $S$ and $T$ implies (iv); the statements 2.6, 2.10 and 3.4 imply (v); the condition (vi) follows immediately by 2.9.

4. The condition (i) enables us to define the semi-group $\mathcal{G}_{(x,d)}$ as the quotient algebra:

$$\mathcal{G}_{(x,d)} = \mathcal{G}(x,d)/\sim.$$

By (ii), the class [0] is the neutral element of $\mathcal{G}_{(x,d)}$. The condition (v) together with 2.9 imply the cancellation law. Thus

4.1. $\mathcal{G}_{(x,d)}$ is a commutative semi-group with cancellation.

The condition (iii) enables us to define the relation $\geq$ in $\mathcal{G}_{(x,d)}$ by the formula

$$[x] \geq [y] \iff \lambda \geq \lambda x.$$
Theory of equidistance and betweenness relations. Consider now the class $M_1$ of all the metric spaces over partially ordered free semi-groups. Applying the previous results, we are going to prove

**Representation Theorem II.** The class $D_{M_1}$ coincides with the class of models of (A.1-A.5), i.e.

$$D_{M_1} = \mathcal{R}(A.1-A.5).$$

Since every partially ordered semi-group with cancellation (thus moreover every free semi-group) can be extended to an ordered group, this theorem can be formulated as follows:

The theory of equidistance relation in metric spaces over ordered groups coincides with the theory based on (A.1-A.5).

We have first the following obvious

**Theorem II.1.** Every structure $(X, D, \rho)$, with $X \in M_1$, is a model of (A.1-A.5) (*)

This theorem enables us to define the following function

$$\phi_1: M_1 \to \mathcal{R}(A.1-A.5),$$

$$\phi_1(\mathcal{A}) = (X, D, \rho) \quad \text{for every } \mathcal{A} \in M_1.$$

Now, it remains to prove

**Theorem II.2.** For every model $(X, D)$ of (A.1-A.5) there is a metric space $\mathcal{A} \in M_1$ such that $D_{\mathcal{A}} = D$.

In other words, there is a function $\phi_1: \mathcal{R}(A.1-A.5) \to M_1$ such that

$$\phi_1(\phi_1(\mathcal{A})) = (X, D) \quad \text{for every } (X, D) \in \mathcal{R}(A.1-A.5).$$

**Proof.** Let $(X, D)$ be a model of (A.1-A.5). We are going to define the required metric space $\mathcal{A} = (X, \rho^*, \rho)$, where $\rho^*$ and $\rho$ are as constructed in §1. By 1.5, $\rho^*$ is a free semi-group. Let us extend the structure $(X, D)$ to a $DB$-structure $(X, D, B_0)$ setting

$$B_0(abc) \Leftrightarrow a = b \lor b = c.$$  

Evidently, $(X, D, B_0)$ satisfies A.1-A.6 (moreover, it satisfies all the axioms except A.15).

Look at the relations $\sim$ and $\geq$ in $G_{(X, D)}$. First of them is the identity, thus it satisfies the conditions (i)-(v) (§2) and (ii) (§3). The second one is of the form

$$\lambda \geq \kappa \Leftrightarrow \lambda \in T(\kappa),$$

(*) Here $M_1$ can be replaced by any class of metric spaces over arbitrary semi-groups, even without any ordering relation.