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MADURA COLLEGE
MADURAI UNIVERSITY
Madurai (India)

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The equivalence of absolute almost continuous retracts and ε-absolute retracts

by

Kenneth R. Kellum * (Birmingham, Ala.)

Abstract. In this paper we are concerned with types of generalized retracts and ε-retracts in which the retraction function may or may not be continuous. We first consider a generalized type of retract in which the retraction function belongs to an arbitrary class of functions which is assumed to be closed under composition with continuous functions. Theorems are proved which are generalizations of well-known theorems about AR's and εAR's. These theorems hold if the class in question is the class of the continuous functions, the class of the almost continuous functions, or a new class of functions which we call weakly continuous. These results, together with the proof of the proposition which is our title, lead to a number of other equivalences.

1. Introduction. In [11] I reported that an almost continuous retract of an n-cube need not be compact. These spaces are of interest because they must posses the fixed point property. The present paper is the result of studying the special case of those almost continuous retracts which do happen to be compact. The main result implies that a compact subset Y of an n-cube X is an almost continuous retract of X if and only if Y is an ε-retract of X.

Suppose $Y\subset X$. That Y is a *retract* of X means that there exists a continuous function (called a *retraction*) $r\colon X\to Y$ such that $x = r(x)$ for each $x\in Y$. This important concept is due to Borsuk and has been studied extensively (see [1] and [9]). Recently the notion of a retract has been generalized in two seemingly different ways. First, Noguchi [16] and later Gmurczyk [4], [5] and Granas [6] studied ε- (or approximative) retracts in which the requirement that $x = r(x)$ is weakened. Second, motivated by question 10 of Stallings [17], several authors have studied connectivity and almost continuous retracts in which the requirement that the retraction function be continuous is weakened (see [2], [3], [7], [10], [11] and [12]). Here we show that these two lines of research are in fact closely related.

We adopt the following conventions. All spaces, except the function spaces considered below, are assumed to be separable metric. If x and y are points of a space X, $d(x, y)$ denotes the distance from x to y. If $x\in X$, then

$$N(x, \varepsilon) = \{y\in X\colon\ d(x, y)<\varepsilon\}.$$

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We regard a function as being identical with its graph. By a *map* we mean a continuous function. The Hilbert cube is denoted by the letter Q.

2. Definitions. Suppose $f\colon X\to Y$. That f is *almost continuous* means that if $f\subset D$, where D is an open subset of $X\times Y$, then there exists a map $g\colon X\to Y$ such that $g\subset D$. Now, denote by F the set of all functions from X into Y and let T be a topology for F. We say that $f\in F$ is *almost continuous relative to* T if whenever $f\in U\in T$, then U contains a continuous function in F. In this terminology, an almost continuous function is one which is almost continuous relative to the graph topology introduced by Naimpally [14]. For convenience, we call a function *weakly continuous* if it is almost continuous relative to the compact-open topology. In case X is compact, weak continuity is the same as Γ_1-almost continuity as defined by Naimpally and Pareek [15]. Note that an almost continuous function is weakly continuous.

Since we are interested in types of retracts in which the retraction function may belong to several different classes of functions, in Section 3 we generalize some standard results of retract theory to retracts involving an unspecified class of functions. This motivates the following definitions. Suppose P is a class of functions. We say that f is a *P function* if f belongs to P. That a subset Y of a space X is a *P retract* of X means that there exists a P function $r\colon X\to Y$ such that $x=r(x)$ for each $x\in Y$. That Y is an *ε-P* retract of X means that for each $\varepsilon>0$ there is a P function $r\colon X\to Y$ such that $d(x, r(x))<\varepsilon$ for each $x\in Y$. That Y is an *absolute P retract* (resp. *absolute ε-P retract*) means that Y is a compactum (compact metric space) and that whenever Y is homeomorphic to a closed subset Y' of a space X, then Y' is a P retract (resp. an ε-P retract) of X.

3. *P* retracts. Suppose P is a class of functions. We say that P is O. K. if $fgh\colon X_1\to X_4$ is a P function whenever $f\colon X_1\to X_2$ and $h\colon X_3\to X_4$ are continuous and $g\colon X_2\to X_3$ is a P function, where X_2, X_3, and X_4 are compact.

For the remainder of this section, we use the abbreviations APR and εAPR for the terms absolute P retract and ε-absolute P retract, respectively.

THEOREM 3.1. *Let P be an* O. K. *class of functions. That Y be an* εAPR (*resp.* APR) *it is necessary and sufficient that Y be homeomorphic to a closed ε-P retract* (*resp. P retract of Q*).

Proof. We give only the proof for the ε-P retract case.

Necessity. This follows easily from the fact that Y can be imbedded in Q ([11], p. 241).

Sufficiency. Suppose $h\colon Y\to Y'$ is a homeomorphism, where Y' is a closed ε-P retract of Q. Let $k\colon Y\to Y''$ be a homeomorphism, where Y'' is a closed subset of the space X. Let $\varepsilon>0$. By the generalized Tietze's extension theorem ([11], p. 251), the map $hk^{-1}\colon Y''\to Y'$ has a continuous extension f over X relative to Q. By the uniform continuity of $kh^{-1}\colon Y'\to Y''$, there exist $\delta>0$ such that if $d(y_1, y_2)<\delta$, then

$$d\big(kh^{-1}(y_1), kh^{-1}(y_2)\big)<\varepsilon .$$

Let $r\colon Q\to Y'$ be a P function such that $d(y, r(y))<\delta$ for each $y\in Y$. Then $kh^{-1}rf\colon X\to Y''$ is a P function. Finally, if $y\in Y''$, $f(y)=hk^{-1}(y)$, so

$$d\big(hk^{-1}(y)=f(y), rf(y)\big)<\delta$$

and

$$d\big(kh^{-1}hk^{-1}(y)=y, hk^{-1}rf(y)\big)<\varepsilon .$$

Thus, Y'' is an ε-P retract of X and the proof is completed.

THEOREM 3.2. *Let P be an* O. K. *class of functions. A necessary and sufficient condition that a compactum Y be an* APR (*resp. an* εAPR) *is that if $f'\colon X'\to Y$ is continuous, where X' is a closed subset of a space X, then there exists a P function $f\colon X\to Y$ such that $f(x)=f'(x)$ for each $x\in X'$* (*resp. for each $\varepsilon>0$, there exists a P function $f\colon X\to Y$ such that $d(f(x), f'(x))<\varepsilon$ for each $x\in X$*).

Proof. Again, we give only the proof for the ε-P retract case.

Necessity. This follows easily from Theorem 3.1.

Sufficiency. Let $h\colon Y\to Y'\subset Q$ be a homeomorphism. Let $\varepsilon>0$. By the uniform continuity of h there exists $\delta>0$ such that if $d(x_1, x_2)<\delta$, then $d(h(x_1), h(x_2))<\varepsilon$. By hypothesis, there exists a P function $f\colon Q\to Y$ such that $d(f(x), h^{-1}(x))<\delta$ for each $x\in Y'$. Then $hf\colon Q\to Y'$ is a P function and the proof is completed.

COROLLARY 3.3. *Let P be an* O. K. *class of functions. A closed subset Y of an* APR (*resp.* εAPR) *X is an* APR (*resp.* εAPR) *if and only if Y is a retract* (*resp. ε-retract*) *of X.*

COROLLARY 3.4. *Let P be an O. K. class of functions. A closed subset Y of an* AR (*resp.* εAR) *X is an* APR (*resp.* εAPR) *if and only if Y is a P retract* (*resp. ε-P retract*) *of X.*

4. The weakly continuous functions are O. K. Two examples of O. K. classes of functions are the continuous functions and the Darboux functions. That the almost continuous functions are O. K. follows from Propositions 1 and 4 of [17]. This brief section is devoted to showing that the weakly continuous functions also form an O. K. class. The assumption that all spaces are separable metric is not needed in the next two theorems.

THEOREM 4.1. *Let $f\colon X\to Y$ be weakly continuous and $g\colon Y\to Z$ continuous. Then $gf\colon X\to Z$ is weakly continuous.*

Proof. For $1\leqslant i\leqslant n$, let U_i be a compact subset of X and V_i an open subset of Z such that $gf(U_i)\subset V_i$. Then

$$D=\{h\colon X\to Z\colon h(U_i)\subset V_i;\ 1\leqslant i\leqslant n\}$$

is a basic open neighborhood of gf in the compact-open topology. Now, since

$$D'=\{h\colon X\to Y\colon h(U_i)\subset g^{-1}(V_i);\ 1\leqslant i\leqslant n\}$$

is a neighborhood of f, D' contains a continuous function, $h\colon X\to Y$. Clearly, $gh\in D$, and the proof is completed.

We omit the proof of the following theorem because it is similar to that of Theorem 4.1.

THEOREM 4.2. *Let f: $X\to Y$ be continuous and g: $Y\to Z$ weakly continuous, where X is a compact Hausdorff space and Y is Hausdorff. Then gf: $X\to Z$ is weakly continuous.*

5. The main results.

THEOREM 5.1. *Suppose Y is a compact ε-weakly continuous retract of a space X. Then Y is an ε-retract of X.*

Proof. Let $\varepsilon>0$. Since Y is compact there exist points $x_1, x_2, \ldots, x_n$ of Y such that $\{N(x_i, \frac{1}{3}\varepsilon)\colon 1\leqslant i\leqslant n\}$ covers Y. Let D be the set of all functions h: $X\to Y$ such that

$$h\big(\mathrm{cl}(N(x_i, \tfrac{1}{3}\varepsilon))\cap Y\big)\subset Y-N(x_i, \tfrac{2}{3}\varepsilon)$$

for each $1\leqslant i\leqslant n$. There exists a weakly continuous function r: $X\to Y$ such that $d(x, r(x))<\frac{1}{3}\varepsilon$ for each $x\in Y$. Since, as is easily verified, $r\in D$, and D is open in the compact-open topology, D contains a continuous function g. But then $d(x, g(x))<\varepsilon$ for each $x\in Y$, as required.

Suppose that f: $X\to Y$ is not almost continuous. We say that K is a *minimal blocking set* (MBS) of f if K is a closed subset of $X\times Y$, $f\cap K=\emptyset$, $g\cap K\neq\emptyset$ if g: $X\to Y$ is a map, and no proper subset of K has the preceeding properties.

THEOREM 5.2. *Suppose f: $X\to Y$ is not almost continuous, where X is compact and Y is an εAR. There exists a MBS K of f and $p(K)$ is a non-degenerate continuum, where p: $X\times Y\to X$ is the natural projection.*

Proof. That K exists follows from Theorem 2 of [11]. Also, $p(K)$ is non-degenerate because K meets every constant map from X into Y. Assume $p(K)$ is not a continuum. Then $p(K)=A\cup B$, where A and B are closed, non-empty and disjoint. Now,

$$K_1=K-\big(K\cap p^{-1}(B)\big) \quad\text{and}\quad K_2=K-\big(K\cap p^{-1}(A)\big)$$

are closed, proper subsets of K, so, by the minimality of K, there exist maps g_1: $X\to Y$ and g_2: $X\to Y$ such that $g_1\cap K_1=\emptyset$ and $g_2\cap K_2=\emptyset$. Thus, $p(g_1\cap K)\subset A$ and $p(g_2\cap K)\subset B$. Let $g=g_1|B\cup g_2|A$. Then g: $p(K)\to Y$ is continuous, and g and K are closed, disjoint subsets of the compact space $X\times Y$. Let $\varepsilon=d(g, K)$. By Theorem 4.2, there exists a map g': $X\to Y$ such that $d(g(x), g'(x))<\varepsilon$ for each $x\in p(K)$. But then $g'\cap K=\emptyset$, a contradiction. This completes the proof.

We now extend the custom of abbreviating the terms absolute retract and ε-absolute retract by AR and εAR in a natural way. The terms absolute almost continuous retract, ε-absolute almost continuous retract, absolute weakly continuous retract and ε-absolute weakly continuous retract are abbreviated by AACR, εAACR, AWCR, and εAWCR, respectively.

THEOREM 5.3. *Let Y be a compactum. The following statements are equivalent to each other.*

(1) *Y is an* AACR.

(2) *Y is an* AWCR.

(3) *Y is an* εAWCR.

(4) *Y is an* εAACR.

(5) *Y is an* εAR.

Proof. The implications (1)→(2), (2)→(3), (5)→(4), and (4)→(3) are all immediate. That (3)→(5) follows from Theorem 5.1. We will establish the equivalence of the five statements by proving that (5)→(1).

Let Y be an εAR. By Theorem 4.1, we may assume that $Y\subset Q$. Denote by Θ the set of all closed subsets S of $Q\times Y$ such that $p(S)$ contains c-many points not in Y. Using transfinite induction we may define a function r: $Q\to Y$ such that if $x\in Y$, then $x=r(x)$ and if $S\in\Theta$, then $r\cap S\neq\emptyset$. (For more detail on the construction of r see the proof of Theorem 1 of [10].) We complete the proof by showing that r is almost continuous. Assume that it is not. Then, by Theorem 5.2, there exists a MBS K of r and $p(K)$ is a non-degenerate continuum. By the construction of r, we must have that $p(K)\subset Y$. Let $\varepsilon=d(K, r|Y=Y^2)$. There exists a map f: $Q\to Y$ such that $d(x, f(x))<\varepsilon$ for each $x\in Y$. But then $f\cap K=\emptyset$. This contradiction completes the proof.

The following two corollaries are consequences of Theorems 3.1, 3.2 and 5.3.

COROLLARY 5.4. *Let Y be a compactum. The following statements are equivalent to each other.*

(1) *Y is an* AACR.

(2) *If f': $X'\to Y$ is continuous where X' is a closed subset of a space X, there exists an almost continuous function f: $X\to Y$ such that $f(x)=f'(x)$ for each $x\in X'$.*

(3) *If f': $X'\to Y$ is continuous where X' is a closed subset of a space X, for each $\varepsilon>0$ there exists an almost continuous function f: $X\to Y$ such that $d(f(x), f'(x))<\varepsilon$ for each $x\in X'$.*

(4) *Y is homeomorphic to a closed almost continuous retract of Q.*

(5) *Y is homeomorphic to a closed ε-almost continuous retract of Q.*

COROLLARY 5.5 *Corollary 5.4 holds if "almost continuous" is replaced by "weakly continuous" throughout.*

It is natural to ask if the requirement that the function f' in parts (2) and (3) of Corollary 5.4 be continuous can be replaced by the requirement that f' be almost continuous. We now show that this is the case, obtaining a cute analogy to a standard theorem of retract theory.

THEOREM 5.6. *Let Y be a compactum. The following statements are equivalent to each other.*

(1) *Y is an* AACR.

(2) *If f': $X'\to Y$ is almost continuous, where X' is a closed subset of a space X,*

there exists an almost continuous function $f\colon X\to Y$ such that $f(x) = f'(x)$ for each $x\in X'$

(3) *If $f'\colon X'\to Y$ is almost continuous where X' is a closed subset of a space X, for each $\varepsilon>0$ there exists an almost continuous function $f\colon X\to Y$ such that $d(f(x), f'(x))<\varepsilon$ for each $x\in X'$.*

Proof. Clearly (2)→(3) and (3) implies statement (3) of Corollary 5.4, so we need only prove that (1)→(2).

Let $h\colon X\to Z\subset Q$ be a homeomorphism and let $Z' = h(X')$. We may define a function $g\colon Q\to Y$ such that if K is a closed subset of $Q\times Y$ such that $p(K)$ has c-many points not in Z', then $g\cap K\neq\emptyset$ and such that $g(x) = f'h^{-1}(x)$ for each $x\in Z'$. Now we show that g is almost continuous. Assume that it is not and let K be a MBS of g. By Theorem 5.2 and the definition of g, we have that $p(K)\subset Z'$. Now, $f'h^{-1}|Z'\colon Z'\to Y$ is almost continuous by Propositions 4 and 2 of [15]. Since $K\cap f'h^{-1}|Z' = \emptyset$, there exists a map $j'\colon Z'\to Y$ such that $j'\cap K = \emptyset$. Since Y is an AACR, there exists an almost continuous function $j\colon Q\to Y$ such that $j(x) = j'(x)$ for each $x\in Z'$. But then $j\cap K = \emptyset$, leading to a contradiction. Thus g is almost continuous. Let $f = gh\colon X\to Y$, and the proof is completed.

6. Questions and remarks.

1. Since an n-cube is an AR, by Corollary 3.4 and Theorem 5.3, the compact almost continuous retracts of an n-cube are precisely its compact ε-retracts. The problem of characterizing the non-compact almost continuous retracts of an n-cube remains open. Since an open interval is an ε-retract of its closure, it is clear that the methods of this paper will not work in the non-compact case.

2. I conjecture that the analogue for Theorem 5.3 holds for neighborhood retracts. The implications

$$(1)\to(2)\to(3)\leftrightarrow(4)\leftrightarrow(5)$$

will all hold in much the same way, but I have not found a proof for (5)→(1). By further restricting the class P, results similar to those of Section 3 can be obtained for neighborhood retracts.

3. Under what circumstances is the composition of two weakly continuous functions weakly continuous?

4. A function $f\colon X\to Y$ is said to be a *connectivity* function if $f|C$ is a connected subset of $X\times Y$ wherever C is a connected subset of X. We use the abbreviation "ACR" for "absolute connectivity retract". In light of Theorem 3 of [2] and the results of this paper, I conjecture that an ACR is an AR and that an εACR is an εAR. The connectivity functions are not an O. K. class of functions. I do not know to what extent the theorems of Section 4 will hold for connectivity retracts.

5. One consequence of Theorem 6.3 is that a compact weakly continuous retract of an n-cell has the fixed point property. In [9] an example is given of a weakly continuous function on the unit interval which fails to have a fixed point. I do not know whether a non-compact weakly continuous retract of an n-cell must have the fixed point property.

6. In [10] I asked if an almost continuous retract of an almost continuous retract of an n-cell X is an almost continuous retract of X. By Theorem 5.3, the answer to this question is "yes", if the retracts are assumed to be compact. The question is still open for the non-compact case.

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DEPARTMENT OF MATHEMATICS
MILES COLLEGE
Birmingham, Alabama

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