Scattered compactification for the Arens' space S_2

by

M. Jayachandran and **M. Rajagopalan** * (Memphis, Tenn.)

Abstract. We prove, using (CH), that the scattered Hausdorff space S_2, introduced by R. Arens n 1950, admits a scattered Hausdorff compactification. This gives an affirmative answer to a problem aised by Z. Semadeni in 1959.

Introduction. The class of scattered spaces has been recently studied by several mathematicians like V. Kannan [2], S. Mrówka, M. Rajagopalan, T. Soundararajan ([3], [5]), C. Ryll-Nardzewski, R. Telgarsky [6], Z. Semadeni [7] and W. Sierpiński [8]. It has been recently proved by P. J. Nyikos [4] and R. C. Solomon [9] that there exists a completely regular scattered space which does not admit a scattered compactification. However, the problem of deciding whether or not a given completely regular scattered space admits a scattered Hausdorff compactification is a nontrivial one. In this connection, S. P. Franklin has raised the following question: "Does the Aren's space S_2 (definition follows) admit a Hausdorff scattered compactification?" Z. Semadeni [7] has raised the question whether a particular subspace of S_2 has a scattered compactification. The aim of this paper is to give an affirmative answer to these questions, by constructing a suitable quotient space X of βN, the Stone–Čech compactification of the set N of natural numbers, such that X is scattered and X contains a homeomorph of S_2 as a dense subspace.

NOTATION 1. N denotes the discrete space of natural numbers and βN, its Stone–Čech compactification. ω denotes the least infinite ordinal and Ω denotes the first uncountable ordinal. For any subset $A \subseteq N$, A^* denotes the set

$$(\mathrm{cl}_{\beta N} A) \cap (\beta N - N) = (\mathrm{cl}_{\beta N} A) - A .$$

DEFINITION 2. We denote by $S_1 = N \cup \{\infty\}$, the one-point compactification of N. S_1 is also called a convergent sequence and the only non-isolated point ∞ of S_1 is called its *suspension point*.

DEFINITION 3. For each $n \in N$, let X_n denote a homeomorphic copy of S_1. Let $X_n \cap X_m = \emptyset$ for all $n, m \in N$ such that $n \neq m$. Also, let $x_n \in X_n$ be the suspen-

* The second author acknowledges gratefully a grant from Memphis State University during the preparation of this paper.

sion point of X_n for each $n \in N$, Let $C = \{c_1, c_2, \ldots, c_n, \ldots\} \cup \{c_\infty\}$ be a convergent sequence such that c_∞ is the suspension point of C and $C \cap X_n = \emptyset$ for all $n \in N$. Let B denote the free union $(\bigcup_{n=1}^{\infty} X_n) \cup C$ of the spaces X_n with C. Define a partition π in B by declaring $\{c_n, x_n\}$ as a member of π for each $n \in N$ and $\{x\}$ as a member of π for all other $x \in B$. The quotient space B/π is called the *Arens' space* S_2 (see [1]).

Construction 4 (CH). Take a sequence $A_1, A_2, \ldots, A_n, \ldots$ of mutually disjoint countably infinite subsets of N whose union is N. Then each A_n^* is clopen in $\beta N - N$ and $A_n^* \cap A_m^* = \emptyset$ for $n \neq m$, $n, m = 1, 2, \ldots$ Let $M = \mathrm{cl}_{\beta N}(\bigcup_{n=1}^{\infty} A_n^*)$ and $K = M - \bigcup_{n=1}^{\infty} A_n^*$. Then K is a non-empty closed set in $\beta N - N$ and it is not open in $\beta N - N$. Let $\mathscr{F} = \{S \subseteq N / S$ is infinite and $S \cap A_n$ is finite $\forall n = 1, 2, \ldots\}$. Then, $\forall S \in \mathscr{F}$, S^* is clopen in $\beta N - N$ and $S^* \cap A_n^* = \emptyset$ for all $n = 1, 2, \ldots$ Now, well order $\mathscr{F}$. Let us assume continuum hypothesis. Then $\mathscr{F}$ can be written as $\mathscr{F} = \{S_\alpha |\ \alpha \in [1, \Omega)\}$. Put $T_\alpha = S_\alpha^*$ for all $\alpha \in [1, \Omega)$. For $\alpha \in [1, \Omega)$, define the sets $F_\alpha \subset \beta N - N$ as follows: Put $F_1 = T_1$. Having chosen F_i for each i, $1 \leqslant i < \gamma$, $\gamma \in [1, \Omega)$, choose F_γ such that F_γ is clopen in $\beta N - N$, $F_\gamma \cap M = \emptyset$, and $F_\gamma \supseteq \bigcup_{1 \leqslant i < \gamma} F_i \cup T_\gamma$. This is possible, since the Boolean algebra of clopen sets in $\beta N - N$ is Dubois–Reymond separable [10]. Our construction ends here.

Theorem 5. *The space S_2 admits a scattered Hausdorff compactification.*

Proof. Let the sets $S_\alpha, T_\alpha, F_\alpha$ ($\alpha \in [1, \Omega)$), M, K and A_n be as constructed above. Then,

Claim 1. $\bigcup_{\alpha \in [1, \Omega)} F_\alpha = \beta N - N - M$. For, it is clear that $\bigcup_{\alpha \in [1, \Omega)} F_\alpha \subset \beta N - N - M$. To get the other inclusion, let $x_0 \in \beta N - N - M$. Now, $\beta N - N$ is zero dimensional, M is closed in $\beta N - N$ and $x_0 \notin M$. Therefore, there exists a clopen set V in $\beta N - N$ such that $x_0 \in V$ and $V \cap M = \emptyset$. Also, $V = A^*$ for some infinite subset A of N and $V \cap A_n^* = \emptyset$ for all $n = 1, 2, \ldots$ Hence, $A \cap A_n$ is finite for all $n = 1, 2, \ldots$ So, $A = S_\alpha$ for some $\alpha \in [1, \Omega)$ and hence, $V = T_\alpha \subseteq F_\alpha$. This implies $x_0 \in F_\alpha$ and our claim is justified.

Claim 2. There exists a compact, Hausdorff space X and a map $q\colon \beta N \to X$ such that

(i) q is a quotient map and

(ii) $q(\beta N - N)$ is homeomorphic to the quotient space obtained by taking the free union of $[1, \omega]$ and $[1, \Omega]$ with their usual order topologies and identifying ω and Ω.

To justify this claim, we can assume, without loss of generality, that the sets F_α of our construction are all distinct. Now, put $H_1 = F_1$ and $H_\alpha = F_\alpha - \bigcup_{1 \leqslant i < \alpha} F_i$ for all α such that $2 \leqslant \alpha < \Omega$. Then $\bigcup_{\alpha \in [1, \Omega)} F_\alpha = \bigcup_{\alpha \in [1, \Omega)} H_\alpha$. Further, the disjoint collection of sets $\{H_\alpha\}_{\alpha \in [1, \Omega)}$, $\{A_n^*\}_{n \geqslant 1}$, $\{K\}$ and $\{n\}_{n \in N}$, $n \in N$ gives a partition of βN by closed sets in βN. Let the quotient space induced by this partition be denoted by X. Let $q\colon \beta N \to X$

be the corresponding quotient map. Then, it is clear that X is compact and Hausdorff. Let $q(A_n^*) = \{l_n\}$ for all $n \in N$; $q(K) = \{l_\infty\}$; $q(H_\alpha) = \{l_\alpha\}$ for all $\alpha \in [1, \Omega)$. Then, we have $q(\beta N - N) = \{l_1, l_2, \ldots, l_n, \ldots\} \cup \{l_\infty\} \cup \{l_\alpha\}_{1 \leqslant \alpha < \Omega}$ and it can be verified that q satisfies the conditions (i) and (ii) of Claim 2.

Claim 3. If $q(M \cup N) = Y$, then Y is homeomorphic to S_2. To justify this claim, let us take $A_k = \{a_{k_1}, a_{k_2}, \ldots, a_{k_n}, \ldots\}$ for all $k = 1, 2, \ldots$; $q(A_n) = B_n$ for all $n \in N$ and $q(a_{k_n}) = l_{k_n}$ for all $k, n \in N$. Then, each $\{l_{m_n}\}$, $m, n \in N$, is clopen in Y. Also, we note that $l_n \to l_\infty$ as $n \to \infty$. Further, the compactness of $q(A_n \cup A_n^*)$ implies that $l_n = \lim_{k \to \infty} l_{n_k}$ for all $n = 1, 2, \ldots$

We will now show that an open set in Y containing l_∞ is an open set containing l_∞ in the topology of S_2 and *vice versa*. Let 0 be an open set in Y such that $l_\infty \in 0$. Since, $l_n \to l_\infty$ as $n \to \infty$, there exist $n_0 \in N$, such that $l_n \in 0$ for all $n \geqslant n_0$. Since $l_{n_k} \to l_n$ as $k \to \infty$, for all $n \in N$, it follows that $(Y - 0) \cap B_n$ is finite for all $n \geqslant n_0$. Conversely, let $S \subseteq q(N)$ be such that $S \cap B_n$ is finite for all $n \in N$. Then $q^{-1}(S) \cap A_n$ is finite for all $n \in N$. Therefore, $q^{-1}(S) = S_\alpha$ for some $\alpha \in [1, \Omega)$. Also $S_\alpha^* = T_\alpha \subseteq F_\alpha$ and so

$$\mathrm{cl}_{\beta N}(S_\alpha \cup F_\alpha) = \mathrm{cl}_{\beta N}(S_\alpha) \cup F_\alpha = S_\alpha \cup S_\alpha^* \cup F_\alpha = S_\alpha \cup F_\alpha.$$

Therefore, $S_\alpha \cup F_\alpha$ is closed in βN and hence $q(S_\alpha \cup F_\alpha)$ is closed in X. Therefore, $(S \cup q(F_\alpha)) \cap Y$ is closed relative to Y. But $F_\alpha \cap M = \emptyset$ for all $\alpha \in [1, \Omega)$ implies that $q(F_\alpha) \cap Y = \emptyset$. Therefore, $S \cap Y = S$ is closed relative to Y. Hence, $Y - S$ is open in Y and $l_\infty \in Y - S$. Therefore, the relative topology of Y from X is homeomorphic to S_2.

Since N is dense in βN, we also have $q(M \cup N) = Y$ is dense in X. Since, the relati ology of $q(N)$ from X is discrete and that of $q(\beta N - N)$ is homeomorphic to th : obtained by taking the free union of $[1, \omega]$ and $[1, \Omega]$ and identifying ω and ιim 2), it follows that X is scattered. This completes the proof of the

References

.. Arens, *Note on convergence in topology*, Math. Mag. 23 (1950), pp. 229–234.

V. Kannan and M. Rajagopalan, *On scattered spaces*, Proc. Amer. Math. Soc. 43 (1974), pp. 402–408.

S. Mrówka, M. Rajagopalan and T. Soundararajan, *A characterization of compact scattered spaces through chain limits*, Topology 72, Springer-Verlag, Lecture Notes in Mathematics, Vol. 378 (1974), pp. 288–298.

·] P. J. Nyikos, *Not every scattered space has a scattered compactification*, Notices of Amer. Math. Soc. 21 (1974), p. 570.

[5] M. Rajagopalan, *Sequential order and spaces S_n*, Proc. Amer. Math. Soc. 54 (1976), pp. 433–438.

[6] C. Ryll-Nardzewski and R. Telgarsky, *On scattered compactification*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 18 (1970), pp. 233–234.

[7] Z. Semadeni, *Sur les ensembles clairsemés*, Dissertationes Math. 19 (1959), p. 39.
[8] W. Sierpiński and S. Mazurkiewicz, *Contribution à la topologie des ensembles dénombrables*, Fund. Math. 1 (1920), pp. 11–16.
[9] R. C. Solomon, *Scattered spaces and their compactifications*, to appear.
[10] R. C. Walker, *The Stone-Čech compactification*, (Thesis) Department of Mathematics, Carnegie-Mellon University, (1972).

MADURA COLLEGE
MADURAI UNIVERSITY
Madurai (India)

Accepté par la Rédaction le 23. 7. 1975

The equivalence of absolute almost continuous retracts and ε-absolute retracts

by

Kenneth R. Kellum * (Birmingham, Ala.)

Abstract. In this paper we are concerned with types of generalized retracts and ε-retracts in which the retraction function may or may not be continuous. We first consider a generalized type of retract in which the retraction function belongs to an arbitrary class of functions which is assumed to be closed under composition with continuous functions. Theorems are proved which are generalizations of well-known theorems about AR's and εAR's. These theorems hold if the class in question is the class of the continuous functions, the class of the almost continuous functions, or a new class of functions which we call weakly continuous. These results, together with the proof of the proposition which is our title, lead to a number of other equivalences.

1. Introduction. In [11] I reported that an almost continuous retract of an n-cube need not be compact. These spaces are of interest because they must posses the fixed point property. The present paper is the result of studying the special case of those almost continuous retracts which do happen to be compact. The main result implies that a compact subset Y of an n-cube X is an almost continuous retract of X if and only if Y is an ε-retract of X.

Suppose $Y\subset X$. That Y is a *retract* of X means that there exists a continuous function (called a *retraction*) $r\colon X\to Y$ such that $x = r(x)$ for each $x\in Y$. This important concept is due to Borsuk and has been studied extensively (see [1] and [9]). Recently the notion of a retract has been generalized in two seemingly different ways. First, Noguchi [16] and later Gmurczyk [4], [5] and Granas [6] studied ε- (or approximative) retracts in which the requirement that $x = r(x)$ is weakened. Second, motivated by question 10 of Stallings [17], several authors have studied connectivity and almost continuous retracts in which the requirement that the retraction function be continuous is weakened (see [2], [3], [7], [10], [11] and [12]). Here we show that these two lines of research are in fact closely related.

We adopt the following conventions. All spaces, except the function spaces considered below, are assumed to be separable metric. If x and y are points of a space X, $d(x, y)$ denotes the distance from x to y. If $x\in X$, then

$$N(x, \varepsilon) = \{y\in X\colon\ d(x, y)<\varepsilon\}.$$

* Author supported in part by NSF Grant GY-10729.