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# Boolean matrices, subalgebras and automorphisms of complete Boolean algebras

by

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**Abstract.** We shall prove several theorems on complete Boolean algebras motivated by the theory of Boolean valued models. Section 1 deals with Boolean matrices that correspond to collapsing mappings in Boolean extensions of the universe. An analogue of Cantor–Bernstein theorem for Boolean matrices is proved. The notion of maximal subalgebras is introduced in Section 2. It is shown that a complete Boolean algebra is rigid iff it does not contain any proper maximal subalgebra. The last Section is devoted to the problem of existence of rigid (non-complete) Boolean algebras of power $\aleph_1$. It is shown that such algebras can exist independently on the Continuum hypothesis (CH). Namely, the statement "$\neg$CH+there is a rigid Boolean algebra of power $\aleph_1$ the completion of which is rigid as well" is consistent relatively to ZFC. Only the proof of Lemma 3 Section 3 makes use of Boolean valued models explicitly, the other proofs are algebraical.

§ 0. **Preliminaries.** Standard set theoretical notation and terminology is used through the paper. Ordinal numbers are denoted $\alpha, \beta, \gamma, \ldots$ and an ordinal coincides with the set of all smaller ordinals. Infinite cardinals are denoted by $\varkappa, \lambda, \ldots$ and are identified with initial ordinals. The cardinality of a set $x$ is denoted by $|x|$. A Boolean algebra $\boldsymbol{b}$ is the structure $\langle b, \vee, \wedge, -, 0, 1\rangle$ satisfying the usual axioms. We use bold face letters to distinguish Boolean algebras from their universes. Every Boolean algebra is partially ordered by $\leqslant$ and 1 is the greatest and 0 the least element. It should be noted that the operations are definable in terms of $\leqslant$ and vice versa. We say that $\boldsymbol{b}$ is a complete Boolean algebra if the operations $\vee$ and $\wedge$ corresponding to supremum and infimum with respect to $\leqslant$ can be extended to any subset of $b$. As customary, these infinite operations are then denoted by $\bigvee, \bigwedge$. For any Boolean algebra $\boldsymbol{b}$, let $\mathrm{Sp}\,\boldsymbol{b}$ denote the *set of all subsets $a$* of $b$ such that $\sup a$ exists. Thus $\boldsymbol{b}$ is a *complete Boolean algebra* iff $\mathrm{Sp}\,\boldsymbol{b} = P(b)$ (the power set of $b$).

Let $\boldsymbol{b}$ be a complete Boolean algebra. We say that $\boldsymbol{b}_1$ is a (*complete*) *subalgebra* of $\boldsymbol{b}$ if $\boldsymbol{b}_1$ is closed under infinite operations and under —. For any $u \in b$ define $\pi_{b_1}(u)$ as follows

$$\pi_{b_1}(u) = \bigwedge \{v \in b_1;\ v \geqslant u\}\,.$$

For any $u \in b$, let $\boldsymbol{b}|u$ denote the partial algebra with the universe $b|u = \{v \in b;\ v \leqslant u\}$ and operations $-_u v = u - v$, $\bigvee_u a = u \wedge \bigvee a$, $\bigwedge_u a = \bigwedge a$ for any $v \in b|u$ and $a \subseteq b|u$. Clearly, $\boldsymbol{b}|u$ is a complete Boolean algebra.

We say that a subset $a$ of $b$ is *dense* if for every non-zero $u \in b$, there is a $v \in a$, $v \neq 0$ such that $v \leqslant u$. Two elements $u, v \in b$ are *disjoint* if $u \wedge v = 0$. A set $a \subseteq b$ is a *partition* of $b$ if the elements of $a$ are parwise disjoint and if $\bigvee a = 1$. We say that $a$ is a *partition* of $u \in b$ if instead $\bigvee a = u$. A partition $a_2$ is *finer* than a partition $a_1$ if every element of $a_1$ is partitioned by a subset of $a_2$. We say that $a_2$ is *strictly finer* than $a_1$ if any element of $a_1$ is partitioned by $a_2$ at least to two non-zero elements.

The celularity $\mu(b)$ of the algebra $b$ is the least cardinal greater than all cardinalities of sets of pairwise disjoint elements of $b$. We say that $b$ satisfies the *countable chain condition* if $\mu(b) = \aleph_1$, i.e., if any set of pairwise disjoint elements of $b$ is at most countable. An element $a$ is an *atom* if $a \neq 0$ and if there is no $x \in b$, $0 < x < a$. We say that $b$ is *atomless* if it has no atoms. We say that a complete Boolean algebra $b$ is *rigid* if identity is the only automorphism of $b$.

## § 1. Boolean matrices.

DEFINITION. Let $b$ be a complete Boolean algebra, $\varkappa$, $\lambda$ infinite cardinals. A family

(1) $$\langle u_{\alpha\beta};\ \alpha<\varkappa,\ \beta<\lambda\rangle$$

of elements of $b$ is called $(\varkappa, \lambda)$-*Boolean matrix* provided that for any $\alpha<\varkappa$ the following conditions are satisfied

(i) $\bigvee_{\beta<\lambda} u_{\alpha\beta} = 1$,

(ii) $\beta_1 < \beta_2 < \lambda \rightarrow u_{\alpha\beta_1} \wedge u_{\alpha\beta_2} = 0$.

We say that (1) is $(\varkappa, \lambda)_s$-*matrix* if it satisfies (i), (ii) and

(iii) $\bigvee_{\alpha<\varkappa} u_{\alpha\beta} = 1$ for any $\beta<\lambda$.

We say that (1) is $(\varkappa, \lambda)_i$-*matrix* if it satisfies (i), (ii) and

(iv) $\alpha_1 < \alpha_2 < \varkappa \rightarrow u_{\alpha_1\beta} \wedge u_{\alpha_2\beta} = 0$ for any $\beta<\lambda$.

If (1) satisfies all four conditions we say that it is a $(\varkappa, \lambda)_b$-*matrix*.

It can be easily seen that if there is a generic ultrafilter on $b$ then $(\varkappa, \lambda)$-matrices determine mappings from $\varkappa$ to $\lambda$ in the Boolean extension of the universe. Particularly, $(\varkappa, \lambda)_s$-matrices determine surjective, $(\varkappa, \lambda)_i$-matrices injective and $(\varkappa, \lambda)_b$-matrices bijective mappings of $\varkappa$ to $\lambda$. Keeping this motivation in mind, we introduce an operation for matrices similar to multiplication of ordinary matrices.

If $\nu$ is a cardinal and

(2) $$\langle v_{\beta\gamma};\ \beta<\lambda,\ \gamma<\nu\rangle$$

is a Boolean $(\lambda, \nu)$-matrix on $b$, for any $\alpha<\varkappa$, $\gamma<\nu$ define

$$w_{\alpha\gamma} = \bigvee_{\beta<\lambda} (u_{\alpha\beta} \wedge v_{\beta\gamma})$$

and call the *family*

(3) $$\langle w_{\alpha\gamma}:\ \alpha<\varkappa,\ \gamma<\nu\rangle$$

the product of matrices (1) and (2). Given a generic ultrafilter on $b$, assume that (1) and (2) determine mappings $f$ from $\varkappa$ to $\lambda$ and $g$ from $\lambda$ to $\nu$ respectively. Then (3) corresponds to their composition $g \circ f$, in other words (3) satisfies (i), (ii). Moreover, if both (1) and (2) are $(\varkappa, \lambda)_s$-matrices ($(\varkappa, \lambda)_i$ or $(\varkappa, \lambda)_b$-matrices) then the same holds for (3). Algebraical proofs are straightforward.



Boolean matrices can be localised. If (1) is a $(\varkappa, \lambda)$-matrix on $b$ and $u \in b$ is non-empty, then $\langle u \wedge u_{\alpha\beta};\ \alpha<\varkappa,\ \beta<\lambda\rangle$ is a $(\varkappa, \lambda)$-matrix on the partial subalgebra $b|u$. Conversely, a $(\varkappa, \lambda)$-matrix on $b$ can be glued together from $(\varkappa, \lambda)$-matrices on partial subalgebras. More precisely, if the set of elements $u$ of $b$ with the property that there exists a $(\varkappa, \lambda)$-matrix on $b|u$ is dense in $b$, then there is a $(\varkappa, \lambda)$-matrix on $b$. Other types of matrices have the same local properties.

As to the existence of Boolean matrices on an algebra $b$, the following cases are trivial:

For $\varkappa \geqslant \lambda$ there exists always a $(\varkappa, \lambda)_s$-matrix and a $(\lambda, \varkappa)_i$-matrix on $b$. It suffices to put $u_{\alpha\alpha} = 1$ for $\alpha<\varkappa$ and $u_{\alpha\beta} = 0$ otherwise. On the other hand, if $\varkappa<\lambda$ and there is a $(\varkappa, \lambda)_s$ or $(\lambda, \varkappa)_i$-matrix on $b$ then $b$ is not $(\varkappa, \lambda)$-distributive and $\lambda<\mu(b)$ (see e.g. [12], [14]). Thus the existence of such matrices for $\varkappa<\lambda$ is not trivial.

LEMMA 1. *There is a $(\varkappa, \lambda)_s$-matrix on an algebra $b$ iff there is a $(\lambda, \varkappa)_i$-matrix on $b$.*

Proof. a) Assume that (1) is a $(\varkappa, \lambda)_s$-matrix on $b$. If we put

$$v_{\beta\alpha} = u_{\alpha\beta} - \bigvee_{\gamma<\beta} u_{\alpha\gamma}\quad \text{for any } \beta<\lambda,\ \alpha<\varkappa$$

then

(4) $$\langle v_{\beta\alpha};\ \beta<\lambda,\ \alpha<\varkappa\rangle$$

is a $(\lambda, \varkappa)_i$-matrix on $b$.

b) If (4) is a $(\lambda, \varkappa)_i$-matrix on $b$, put

$$u_{\alpha\beta} = \begin{cases} v_{\beta\alpha} & \text{for } \alpha<\varkappa,\ 0 \neq \beta<\lambda, \\ v_{0\alpha} \vee (-\bigvee_{\gamma<\lambda} v_{\gamma\alpha}) & \text{for } \alpha<\varkappa,\ \beta = 0. \end{cases}$$

LEMMA 2. *Let $\varkappa<\lambda$, let there be a $(\varkappa, \lambda)_s$-matrix on $b$. Then for any two cardinals $\varkappa_1, \lambda_1, \varkappa \leqslant \varkappa_1 \leqslant \lambda_1 \leqslant \lambda$ there is a $(\varkappa_1, \lambda_1)_s$-matrix on $b$.*

Proof. Assume that (1) is a $(\varkappa, \lambda)_s$-matrix. If we put

$$\bar{u}_{\alpha\beta} = \begin{cases} u_{\alpha\beta} & \text{if } \alpha<\varkappa,\ \beta \neq 0, \\ u_{\alpha 0} \vee \bigvee \{u_{\alpha\gamma};\ \lambda_1 \leqslant \gamma < \lambda\} & \text{if } \alpha<\varkappa,\ \beta = 0, \\ 1 & \text{if } \varkappa \leqslant \alpha < \varkappa_1,\ \beta = 0, \\ 0 & \text{if } \alpha_1 < \varkappa_1,\ 0 \neq \beta < \lambda_1 \end{cases}$$

then $\langle \bar{u}_{\alpha\beta};\ \alpha<\varkappa_1,\ \beta<\lambda_1\rangle$ has the desired property.

The following lemma is an analogue of Cantor-Bernstein theorem for Boolean matrices.

LEMMA 3. *Let $\varkappa>\lambda$ and let there exists a $(\varkappa, \lambda)_i$-matrix on $b$, then there is a $(\varkappa, \lambda)_b$-matrix on $b$.*

Proof. Let $\langle v_{\alpha\beta};\ \alpha<\varkappa,\ \beta<\lambda\rangle$ is a $(\varkappa,\lambda)_i$-matrix on $\boldsymbol{b}$. By induction on $n<\omega$, define $v^n_{\alpha\beta}$ for $\alpha<\varkappa$, $\beta<\lambda$ as follows

$$v^0_{\alpha\beta} = v_{\alpha\beta}, \qquad v^{n+1}_{\alpha\beta} = \bigvee_{\gamma<\lambda} (v^n_{\alpha\gamma} \wedge v_{\gamma\beta}) . \tag{5}$$

It follows by induction on $n$ from (5) and (iv) that for any $\gamma<\varkappa$, $\alpha<\lambda$ and $n<\omega$

$$v_{\alpha\alpha} \wedge v^n_{\gamma\alpha} = 0 \quad \text{whenever} \quad \alpha \neq \gamma . \tag{6}$$

Now define $u_{\alpha\beta}$, $\alpha<\varkappa$, $\beta<\lambda$ in the following way

$$u_{\alpha\beta} = \begin{cases} v_{\alpha\beta} & \text{if} \quad \alpha\geqslant\lambda, \\ -\bigvee\limits_{\lambda\leqslant\gamma<\varkappa}\bigvee\limits_{n<\omega} v^n_{\gamma\alpha} & \text{if} \quad \alpha=\beta<\lambda, \\ v_{\alpha\beta}\wedge \bigvee\limits_{\lambda\leqslant\gamma<\varkappa}\bigvee\limits_{n<\omega} v^n_{\gamma\alpha} & \text{if} \quad \alpha\neq\beta \text{ and } \alpha<\lambda. \end{cases} \tag{7}$$

We shall show that (1) is a $(\varkappa,\lambda)_b$-matrix. Conditions (i), (ii) are satisfied for $\alpha\geqslant\lambda$. If $\alpha<\lambda$, we have $v_{\alpha\alpha}\leqslant u_{\alpha\alpha}$ from (6) and (i), (ii) follows from (7).

To prove (iii), we start with the formula

$$\bigvee_{\alpha<\varkappa} u_{\alpha\beta} = u_{\beta\beta} \vee \Big(\bigvee_{\substack{\alpha<\lambda\\ \alpha\neq\beta}} u_{\alpha\beta} \vee \bigvee_{\lambda\leqslant\alpha<\varkappa} u_{\alpha\beta}\Big) . \tag{8}$$

It follows from (5), (6) and (7) that

$$\bigvee_{\substack{\alpha<\lambda\\ \alpha\neq\beta}} u_{\alpha\beta} = \bigvee_{\alpha<\lambda} \Big(v_{\alpha\beta}\wedge \bigvee_{\lambda\leqslant\gamma<\varkappa}\bigvee_{n<\omega} v^n_{\gamma\alpha}\Big) = \bigvee_{n<\omega}\bigvee_{\lambda\leqslant\gamma<\varkappa} v^{n+1}_{\gamma\beta} . \tag{9}$$

Since $u_{\gamma\beta} = v^0_{\alpha\beta}$ for any $\gamma$, $\lambda\leqslant\gamma<\varkappa$, we have

$$\bigvee_{n<\omega}\bigvee_{\lambda\leqslant\gamma<\varkappa} v^{n+1}_{\gamma\beta} \vee \bigvee_{\lambda\leqslant\gamma<\varkappa} u_{\gamma\beta} = \bigvee_{n<\omega}\bigvee_{\lambda\leqslant\gamma<\varkappa} v^n_{\gamma\beta} = -u_{\beta\beta}$$

and (iii) follows from (8) and (9).

The proof of (iv) must be considered for several cases. If $\alpha_1 \neq \beta$ and $\alpha_2 \neq \beta$ (iv) follows from the fact that $u_{\alpha_i\beta}\leqslant v_{\alpha_i\beta}$ for $i = 1, 2$ and from our assumption that $v_{\alpha\beta}$'s are elements of a $(\varkappa,\lambda)_i$-matrix. If $\alpha_1<\beta = \alpha_2<\lambda$ then

$$u_{\alpha_2\beta} \leqslant \bigvee_{\alpha_2<\lambda} \Big(v_{\alpha_2\beta}\wedge \bigvee_{\lambda\leqslant\gamma<\varkappa}\bigvee_{n<\omega} v^n_{\gamma\alpha_2}\Big) = \bigvee_{\lambda\leqslant\gamma<\varkappa}\bigvee_{n<\omega} v^{n+1}_{\gamma\beta} \leqslant -u_{\beta\beta}$$

according to (7) and (5). Similar argument applies to the case $\alpha_1 = \beta<\alpha_2<\lambda$. If $\alpha_1 = \beta<\lambda<\alpha_2$, (iv) follows directly from (7). This completes the proof of the lemma.

Combining Lemmas 1-3, we get

THEOREM 4. *Let $\varkappa$, $\lambda$ be cardinals $\varkappa\geqslant\lambda$. Let there exist a $(\varkappa,\lambda)_i$ or a $(\lambda,\varkappa)_s$-matrix on a complete Boolean algebra $\boldsymbol{b}$. Then for any two cardinals $\varkappa_1$, $\lambda_1$ such that $\varkappa\geqslant\varkappa_1\geqslant\lambda_1\geqslant\lambda$ there exists a $(\varkappa_1,\lambda_1)_b$-matrix on $\boldsymbol{b}$.*

Remark. A special case of the theorem was proved in [12] with the use of generic ultrafilters. Boolean matrices are motivated by Boolean valued models, but they have interesting applications to algebra and topology. With the use of Boolean matrices, the well-known Kripke's embedding theorem can be reformulated as follows.





THEOREM (Kripke [6]). *Let $\boldsymbol{b}$ be a complete Boolean algebra, let*

$$\langle u_{n\alpha};\ n<\omega,\ \alpha<\lambda\rangle \tag{10}$$

*be an $(\aleph_0,\lambda)_s$-matrix on $\boldsymbol{b}$ for $\lambda$ uncountable.*

*Then for any Boolean algebra $\boldsymbol{b}_1$ such that*

$$|\mathrm{Sp}(\boldsymbol{b}_1)|\leqslant\lambda \tag{11}$$

*there is an embedding of $\boldsymbol{b}_1$ into $\boldsymbol{b}$ preserving all the suprema. If in addition* (10) *is dense in $\boldsymbol{b}$, the conclusion of the theorem holds if* (11) *is replaced by $|\boldsymbol{b}_1|\leqslant\lambda$.*

Boolean matrices built up from regular open sets of a topological space were used in [12] to decompose the space into nowhere dense sets. There are also other types of Boolean matrices e.g. matrices corresponding to mappings that change cofinality in Boolean extensions. Properties of such matrices are investigated [14].

## § 2. Maximal subalgebras.

To motivate the concept of maximal subalgebras, we shall consider an equivalence relation on the set of all complete subalgebras of a complete Boolean algebra. We shall first introduce some notation and terminology. Let $\boldsymbol{b}$ be a complete Boolean algebra. For any set $a$ of elements of $\boldsymbol{b}$ and for any $u\in\boldsymbol{b}$, let $u\wedge\wedge a$ denote the set

$$u\wedge\wedge a = \{w\wedge u;\ w\in a\} .$$

For any two subalgebras $\boldsymbol{b}_1$, $\boldsymbol{b}_2$ of $\boldsymbol{b}$, let $I(\boldsymbol{b}_1,\boldsymbol{b}_2)$ is defined by

$$I(\boldsymbol{b}_1,\boldsymbol{b}_2) = \{u\in\boldsymbol{b};\ u\wedge\wedge\boldsymbol{b}_1 = u\wedge\wedge\boldsymbol{b}_2\} . \tag{12}$$

We shall write $\boldsymbol{b}_1\sim\boldsymbol{b}_2$ if $I(\boldsymbol{b}_1,\boldsymbol{b}_2)$ is a dense set in $\boldsymbol{b}$.

It follows from the definition that $\sim$ is an equivalence relation on the set of all subalgebras of $\boldsymbol{b}$. Reflexivity and symmetry are clear and transitivity follows from the fact that for any $v\in\boldsymbol{b}$, $v\leqslant u$ we have

$$u\wedge\wedge\boldsymbol{b}_1 = u\wedge\wedge\boldsymbol{b}_2 \rightarrow v\wedge\wedge\boldsymbol{b}_1 = v\wedge\wedge\boldsymbol{b}_2 .$$

Relation $\sim$ reflects some simple properties of subalgebras e.g. if $\boldsymbol{b}_1\sim\boldsymbol{b}_2$ and $\boldsymbol{b}_1$ is atomless then so is $\boldsymbol{b}_2$.

DEFINITION. A subalgebra $\boldsymbol{b}_1$ of a complete Boolean algebra $\boldsymbol{b}$ is called *maximal* if $\boldsymbol{b}_1\sim\boldsymbol{b}$.

It follows from the definition that $\boldsymbol{b}_1$ is a maximal subalgebra of $\boldsymbol{b}$ iff $\{u\in\boldsymbol{b};\ u\wedge\wedge\boldsymbol{b}_1 = \boldsymbol{b}|u\}$ is dense. Consequently, $\boldsymbol{b}_1$ has the following local maximality property: for any subalgebra $\boldsymbol{b}_2$ of $\boldsymbol{b}$ and any non-zero $v\in\boldsymbol{b}$, there is a non-zero $w$, $w\leqslant v$ such that $w\wedge\wedge\boldsymbol{b}_2\subseteq w\wedge\wedge\boldsymbol{b}_1$.

LEMMA 1. *Let $u\in I(\boldsymbol{b}_1,\boldsymbol{b}_2)$, $u_i = \pi_{\boldsymbol{b}_i}(u)$ for $i = 1, 2$. Then partial subalgebras $\boldsymbol{b}_1|u_1$, $\boldsymbol{b}_2|u_2$ are isomorphic.*

Proof. It suffices to put

$$f(v) = \pi_{\boldsymbol{b}_2}(v\wedge u)$$

for $v\in\boldsymbol{b}_1|u_1$. Then $f$ is the desired isomorphism.

LEMMA 2. (Mc Aloon [8]). *For any* $u \in I(b_1, b_2)$, *there is a* $v \in I(b_1, b_2)$, $v \geqslant u$ *and maximal with respect to* $\leqslant$.

Proof. It suffices to show that $I(b_1, b_2)$ satisfies the assumption of Kuratowski-Zorn Maximum principle. Let $a \subseteq I(b_1, b_2)$ be linearly ordered by $\leqslant$. Let $u = \bigvee a$. We shall show that $u \in I(b_1, b_2)$, i.e., that for any $v \in b_1$ there is $w \in b_2$ such that $u \wedge v = u \wedge w$. We can assume that $v \leqslant \pi_{b_1}(u)$ without any loss of generality. It follows from our assumptions on $a$ that

$$u \wedge v = \bigvee_{t \in a} (t \wedge v) = \bigvee_{t \in a} (t \wedge \pi_{b_2}(t \wedge v))$$
$$= \bigvee_{t \in a} t \wedge \bigvee_{t \in a} \pi_{b_2}(t \wedge v) = u \wedge \pi_{b_2}(u \wedge v).$$

This completes the proof.

LEMMA 3. *Let* $b_1 \sim b_2$. *For any maximal* $u$, $u \in I(b_1, b_2)$, *we have*

$$\pi_{b_1}(u) \vee \pi_{b_2}(u) = 1. \tag{13}$$

Proof. Suppose that (13) does not hold. There is a non-zero $v$, $v \in I(b_1, b_2)$ such that $v \wedge (\pi_{b_1}(u) \vee \pi_{b_2}(u)) = 0$, for $I(b_1, b_2)$ is dense in $b$. Since $u, v \in I(b_1, b_2)$, for any $w_1 \in b_1$, there exist $s, t \in b_2$ such that $w_1 \wedge u = s \wedge u$ and $w_1 \wedge v = t \wedge v$. If we put

$$w_2 = (s \wedge \pi_{b_2}(u)) \vee (t - \pi_{b_2}(u)), \quad \text{then} \quad w_2 \in b_2$$

and

$$w_1 \wedge (u \vee v) = w_2 \wedge (u \vee v).$$

Then $u \vee v \in I(b_1, b_2)$ which contradicts to the maximality of $u$.

COROLLARY 4. *Let* $b_1$ *be a maximal subalgebra of a complete Boolean algebra* $b$. *Let* $u$ *be a maximal element of* $I(b_1, b)$, *then* $\pi_{b_1}(u) = 1$ *and* $b_1$ *is isomorphic to partial subalgebra* $b|u$.

THEOREM 5. *A complete Boolean algebra* $b$ *is rigid iff there is no proper maximal subalgebra of* $b$.

Proof. a) Let $b_1$ be a proper maximal subalgebra of $b$. Take $u$ a maximal element of $I(b_1, b)$. According to Corollary 4, $b_1$ is isomorphic to partial subalgebra $b|u$. Since $b_1$ is a proper subalgebra of $b$, $u \neq 1$ and there exists a maximal element $v \in I(b_1, b)$, $v \neq u$. By the same argument as for $u$, we get $b_1$ isomorphic to $b|v$ and, consequently, $b|u$ isomorphic to $b|v$. The isomorphism of partial subalgebras $b|u$, $b|v$ can be extended to a non-trivial automorphism of $b$.

b) Suppose that $b$ is not rigid. Let $f$ be a non-trivial automorphism of $b$. Then there are two disjoint non-zero elements $u, v \in b$ such that $f(u) = v$. It can be easily verified that

$$b_1 = \{w_1 \vee w_2 \vee w_3 \colon w_1, w_2, w_3 \in b \;\&\; w_1 \wedge (u \vee v) = 0 \;\&\; w_2 \leqslant u \;\&\; w_3 = f(w_2)\}$$

is closed under infinite Boolean operations and under $-$. The corresponding subalgebra is maximal, since $u \wedge \wedge b_1 = b|u$ and $v \wedge \wedge b_1 = b|v$. It is a proper subalgebra, for $u \notin b_1$.



DEFINITION. (i) An atomless complete Boolean algebra $b$ is *minimal* if any atomless subalgebra of $b$ is maximal.

(ii) An atomless complete Boolean algebra is *simple* if it has no proper atomless subalgebra.

The concept of minimal Boolean algebras was introduced by McAloon in [8]. He showed that any complete Boolean algebra is simple iff it is rigid and minimal. The concept of maximal subalgebras was implicit both in the definition of minimality and in the proof of his theorem. Using the method of [8], we have proved Theorem 5 which is, in our opinion, the heart of McAllon's result. The characterization of simple Boolean algebras is an easy consequence of it. Our original version of the proof of Theorem 5 was motivated by the following result.

THEOREM (Jech [2]). *Let a complete Boolean algebra* $b$ *contain a proper complete subalgebra* $b_1$. *Then there is a subalgebra* $b_2$ *of* $b$ *which is not rigid. If* $b_1$ *is atomless, the same holds for* $b_2$.

We shall give here an alternative proof based on the notion of maximality. Let $u$ be an element of $b$ which does not belong to $b_1$. Put

$$b_1[u] = \{(x \wedge u) \vee (y \wedge -u);\ x, y \in b_1\}.$$

Then $b_1[u]$ is closed under Boolean operations. Let $b_2$ denote the corresponding complete subalgebra. Then $b_1$ is a maximal subalgebra of $b_2$ since $w \wedge \wedge b_1 = b_2|w$, $w = u, -u$. It follows from Theorem 5 that $b_2$ is not rigid. If we put

$$c^+ = \pi_{b_1}(-u) \wedge u, \quad c^- = \pi_{b_1}(u) \wedge -u, \quad c = c^+ \vee c^-.$$

it should be noted that $c \in b_1$ and $c^+, c^-$ are maximal elements of $I(b_1|c, b_2|c)$. Consequently, there is an automorphism $f$ of $b_2$ with $f(c^+) = c^-$.

DEFINITION. For a complete Boolean algebra $b$, let $b_{\mathrm{rig}}$ denote the subalgebra of all elements of $b$ that are left fixed by any automorphism of $b$.

PROPOSITION 6. *Assume that* $b_{\mathrm{rig}}$ *is a maximal subalgebra of a complete Boolean algebra* $b$. *Then*

(i) $\{u,\ b|u$ *is rigid*$\}$ *is dense in* $b$,

(ii) $b_{\mathrm{rig}}$ *is rigid.*

Proof. (i) We shall show that for arbitrary $u \in I(b, b_{\mathrm{rig}})$ the partial algebra $b|u$ is rigid. Suppose on the contrary that there is $u \in I(b, b_{\mathrm{rig}})$ and a non-trivial automorphism $f$ of $b|u$. Let $v, w \leqslant u$ be two non-zero disjoint elements such that $f(v) = w$. Extending $f$ by identity outside of $u$ we get a non-trivial automorphism of $b$. If we put $\bar{v} = \pi_{b_{\mathrm{rig}}}(v)$, we have $v \leqslant \bar{v} \in b_{\mathrm{rig}}$ and

$$w = f(v) \leqslant f(\bar{v}) = \bar{v}. \tag{14}$$

It follows from our assumption on $u$ and from $v \leqslant u$ that $v = u \wedge \bar{v}$. But according to (14), we have $v \vee w \leqslant u \vee \bar{v}$ — a contradiction.

(ii) It suffices to take a maximal $u \in I(b, b_{\mathrm{rig}})$ and $b_{\mathrm{rig}}$ is isomorphic to $b|u$ according to Corollary 4. The conclusion follows from (i).

The following example shows that the maximality of $b_{\rm rig}$ need not imply that $b$ is rigid.

Let $b_0$ be a rigid complete Boolean algebra. Let $b$ be the summ of two copies of $b_0$, i.e., the elements of $b$ are of the form $\langle v, w\rangle$, $v, w \in b_0$ and $\langle v_1, w_1\rangle \leqslant \langle v_2, w_2\rangle$ iff $v_1 \leqslant v_2$ and $w_1 \leqslant w_2$. Then $b_{\rm rig}$ consists of all elements $\langle v, v\rangle$, $v \in b_0$. It can be verified that $b_{\rm rig}$ is maximal in $b$. If we put $f(\langle v, w\rangle) = \langle w, v\rangle$ for any $\langle v, w\rangle \in b$, then $f$ is a non-trivial automorphism of $b$.

§ 3. **Rigid Boolean algebras of power $\aleph_1$ and CH.** The problem of powers of rigid (non-complete) Boolean algebras has been solved under Generalized Continuum hypothesis. Chronologically, Katětov [5] constructed a rigid Boolean algebra of power $2^{\aleph_0}$, Lozier [7] has shown that for any infinite cardinal $\varkappa$, there is a rigid Boolean algebra of power $2^{\varkappa}$. McKenzie and Monk [9] extended this result and proved that for any strong limit cardinal $\varkappa$ there is a rigid Boolean algebra of power $\varkappa$, so under GCH, there is a rigid Boolean algebra of power $\varkappa$ for any uncountable $\varkappa$.

The question remains open whether the same can be proved without GCH, or particularly, whether the existence of a rigid Boolean algebra of power $\aleph_1$ can be proved without the Continuum hypothesis. Our result gives a partial solution to the latter problem, namely we shall show that "$2^{\aleph_0} > \aleph_1$ + there is a Boolean algebra $b$ of power $\aleph_1$ such that both $b$ and the completion of $b$ are rigid" is consistent relatively to ZFC. Our consistency proof is based on a result of Jensen [4] who constructed a Suslin tree with particular properties in the constructible universe $L$.

Let us recall some terminology concerning trees. A tree is a partially ordered set $\langle t, \leqslant\rangle$ such that for any $x \in t$ the set $\{y;\ y < x\}$ is well-ordered. The order type of this set is denoted by $\|x\|$ and called the *order* of $x$. The set of all $x \in t$ of order $\alpha$ is called the *$\alpha$-th level* of $t$. A branch is a linearly ordered subset of $t$ containing with any $x \in t$ the set of all its predecessors. The length of a branch is the supremum of the orders of all its elements. An antichain is a set of pairwise incompatible elements of $t$. We say that $\langle t, \leqslant\rangle$ is a *normal $\omega_1$-tree* if $t$ has exactly $\omega_1$ levels and if

(i) every point has at least two immediate successors,

(ii) for each $x \in t$ and each $\alpha > \|x\|$, there is $y \in t$, $\|y\| = \alpha$ such that $y > x$,

(iii) every level of $t$ is at most countable.

A normal $\omega_1$-tree is called *Suslin tree* if in addition, every antichain is at most countable.

LEMMA 1. *Let $\langle t, \leqslant\rangle$ be a normal $\omega_1$-tree. Let $e$ be a mapping from $t$ to the set of real numbers with the following property*

(15) $$x < y \rightarrow e(x) < e(y) \quad \text{for any } x, y \in t.$$

*Then $\langle t, \leqslant\rangle$ is not a Suslin tree.*

Proof. We shall show that there is an uncountable antichain in $t$. For any $y \in t$ with a non-limit order $\|y\| = \alpha+1$, let $d(y)$ be a rational number such that $e(y^-) < d(y) < e(y)$, where $y^-$ is the only element such that $y^- < y$ and $\|y^-\| = \alpha$. Having defined $d$, we can assign a function $f_x$ to any $x \in t$ such that $f_x$ maps $\|x\|$ to



...t of rational numbers. For any $\beta < \|x\|$, put $f_x(\beta) = d(y)$, where $y$ is the only element of $t$ with $\|y\| = \beta+1$ and $y \leqslant x$. It follows from the definition and from (15) that all $f_x$ are strongly increasing functions and that $f_y \subseteq f_x$ whenever $x \leqslant y$.

We say that functions $f_x, f_y$, $x, y \in t$ are incompatible if neither $f_x \subseteq f_y$ nor $f_y \subseteq f_x$. Clearly, if $f_x, f_y$ are incompatible functions, then $y, x$ are incompatible elements of $t$. To complete the proof, it suffices to show that there exist $\aleph_1$ mutually incompatible functions in the set $\{f_x;\ x \in t\}$. We have

(16) $$\bigcup \{D(f_x);\ x \in t\} = \omega_1$$

since $t$ is an $\omega_1$-tree. For any rational number $q$, let

$$u_q = \{\alpha \in \omega_1;\ (\exists x \in t)(f_x(\alpha) = q)\}.$$

It follows from (16) that there exists a rational number $q_0$ such that $u_{q_0}$ is uncountable. For any $\alpha \in u_{q_0}$ take an $x_\alpha \in t$ such that $f_{x_\alpha}(\alpha) = q_0$. Then the set $\{f_{x_\alpha};\ \alpha \in u_{q_0}\}$ has the desired property.

Our consistency result is based on a construction due to Jensen [4] of a Suslin $\omega_1$-tree in the constructible universe $L$.

THEOREM (Jensen). *There exists a normal $\omega_1$-tree $t$ in $L$ with the following properties*

(i) *$V = L \rightarrow t$ is Suslin,*

(ii) *$\omega_1^L = \omega_1 \rightarrow t$ has at most one $\omega_1$-branch.*

We remember only some facts on the tree, the reader is referred to [4] for more detail. The elements of $t$ are certain countable sequences of positive rational numbers satisfying (among others) the following conditions:

for any two sequences $x = \langle x_\gamma;\ \gamma < \alpha\rangle$, $y = \langle y_\gamma;\ \gamma < \beta\rangle$, $\alpha, \beta < \omega_1$

(17) $$\sigma(x, y) = \sum_{\gamma < \alpha, \beta} |x_\gamma - y_\gamma| < \infty$$

(18) if $\alpha$ is limit, $\alpha \leqslant \beta$ and $x, y$ are incompatible then

$$\sum_{\delta < \gamma < \alpha} |x_\gamma - y_\gamma| > 0 \quad \text{for any } \delta < \alpha.$$

If $\omega_1^L = \omega_1$, it follows from (18) that the existence of two $\omega_1$-branches in $t$ would imply that there exists an uncountable strongly increasing sequence of real numbers.

In what follows (except for Lemma 3), let $t = \langle t, \leqslant_t\rangle$ denote the normal Suslin $\omega_1$-tree in $L$ from the proof of Jensen's theorem. Let $\leqslant_b$ be the inverse of the ordering $\leqslant_t$: $x \leqslant_b y$ iff $y \leqslant_t x$. The partially ordered set $\langle t, \leqslant_b\rangle$ can be embedded into a unique (up to isomorphism) complete Boolean algebra $b_t = \langle b_t;\ \leqslant_b\rangle$ such that $t$ is dense in $b_t$. Incompatible elements of $t$ are disjoint as elements of $b_t$, so the celularity of $b_t$ is determined by the cardinalities of antichains in $t$. Let $b_0$ be the least complete subalgebra of $b_t$ containing $t$; $b_0$ and $t$ have the same power. It is well-known that the existence of a non-trivial automorphism of $t$ implies non-trivial automorphisms of both $b_0$ and $b_t$ and that any automorphism of $b_0$ can be extended to an automorphism of $b_t$. Hence, to show that $b_0$ is rigid, it suffices to prove that $b_t$ has no non-trivial automorphism.

THEOREM 2. *Let $t = \langle t, \leqslant_t \rangle$ be as above. If $t$ is a Suslin $\omega_1$-tree (in the universe V), then the complete Boolean algebra $b_t$ is rigid.*

Proof. Instead of $t$, we shall deal with the subset $s \subseteq t$, consisting of all sequences of limit length. It is clear that $s$ is a cofinal subset of $t$ with respect to $\leqslant_t$, since $t$ is normal. Consequently, $s$ is dense in $b_t$. For any $x_1, x_2, y_1, y_2 \in s$ such that $x_1, y_1$ are incompatible, we have

(19) $$(x_1 <_t x_2 \;\&\; y_1 <_t y_2) \rightarrow 0 < \sigma(x_1, y_1) < \sigma(x_2, y_2)$$

according to (17) and (18).

Suppose that there is a non-trivial automorphism $h$ of $b_t$. Using Lemma 1 we shall show that this assumption is contradictory. Let $u_1, u_2$ be two disjoint non-zero elements of $b_t$ such that $h(u_1) = u_2$. Let $\leqslant$ denote the partial ordering of the complete Boolean algebra $b_t$. For $i = 1, 2$, let $s|u_i$ be the set of all $x \in s$, $x \leqslant u_i$. Then $s|u_i$ is dense in the partial algebra $b_t|u_i$ for $i = 1, 2$ and, consequently, $h^{-1}{}''(s|u_2)$ is a dense subset of $b_t|u_1$.

By induction on $\alpha < \omega_1$, we shall construct subtrees $t^1, t^2$ of $s|u_1$, $s|u_2$ respectively, $t^i_\alpha$ being the $\alpha$th level of $t^i$. To begin with, take arbitrary $x_0 \in s|u_1$ and $y_0 \in s|u_2$, $y_0 \leqslant h(x_0)$ and put

$$t^1_0 = \{x_0\}, \quad t^2_0 = \{y_0\}.$$

Having constructed $t^i_\beta$, $i = 1, 2$ for any $\beta < \alpha+1$, let $t^2_{\alpha+1} \subseteq s|u_2$ be a partition of $y_0$ strictly finer than $h'' t^1_\alpha$ and $t^1_{\alpha+1} \subseteq s|u_1$ be a partition strictly finer than $h^{-1}{}'' t^2_{\alpha+1}$.

For $\alpha$ limit, let $t^2_\alpha$ be a refinement of all partitions $t^2_\beta$, $\beta < \alpha$ (such a refinement exists, since $t$ is a normal $\omega_1$-tree). Let $t^1_\alpha$ be a refinement of the partition $h^{-1}{}'' t^2_\alpha$. For $i = 1, 2$ we set $t^i = \bigcup_{\alpha < \omega_1} t^i_\alpha$. We shall show that for $t^1$, there is a mapping $e$ which satisfies the assumptions of Lemma 1. For any $\alpha < \omega_1$, $x \in t^1_\alpha$, the partition $h'' t^1_\alpha$ of $y_0$ is finer than $t^2_\alpha$. Put $e(x) = \sigma(x, y(x))$, where $y(x)$ is the only element of $t^2_\alpha$ such that $y(x) \geqslant h(x)$.

Since $u_1, u_2$ are disjoint, $x, y(x)$ are incompatible in $t$. It follows from the construction of $t^1, t^2$ that for any $x_1, x_2 \in t^1$, we have

$$x_1 < x_2 \rightarrow y(x_1) < y(x_2).$$

But the restriction of the partial ordering $\leqslant$ to the set $t$ is the inverse of the ordering $\leqslant_t$. Thus according to (19), we have

$$x_1 < x_2 \rightarrow e(x_1) < e(x_2)$$

for any $x_1, x_2 \in t^1$. This completes the proof of the Theorem since from Lemma 1 we get that $t^1$ is not Suslin and this contradicts to our assumption on $t$.

To get the consistency result, we take a Boolean extension of $L$ in which the assumptions of Theorem 2 are satisfied but the Continuum hypothesis fails.



Let $\mathrm{Cant}(\aleph_0, \aleph_2)$ denote the complete Boolean algebra of all regular open sets of the product ${}^{\omega_2}\{0, 1\}$ in $L$ endowed with the normal product topology. For any mapping $h$ from a finite subset of $\omega_2$ to $\{0, 1\}$ the set $u_h = \{f \in {}^{\omega_2}\{0, 1\};\ h \subseteq f\}$ is regular open and the set $q$ of all subsets $u_h$, $h$ finite is dense in $\mathrm{Cant}(\aleph_0, \aleph_2)$. It is well-known that the just described complete Boolean algebra satisfies countable chain condition. Let $Z$ be a $L$-generic ultrafilter on $\mathrm{Cant}(\aleph_0, \aleph_2)$, then all cardinals are preserved and $2^{\aleph_0} = \aleph_2$ in the Boolean extension $L[Z]$. These facts are well-known (proof can be found e.g. in [10], [14]).

LEMMA 3. *Let $t = \langle t, \leqslant_t \rangle$ be a Suslin $\omega_1$-tree in L. Let Z be as above, then $t$ is a Suslin $\omega_1$-tree in $L[Z]$.*

Proof. Suppose that $\sigma \subseteq t$ is an uncountable antichain in $L[Z]$. Since $Z$ is a $L$-generic ultrafilter, we can assume (see [14], 4219) that there is a disjoint relation $r \in L$, $r \subseteq t \times \mathrm{Cant}(\aleph_0, \aleph_2)$ (i.e. $r^{-1}$ is a mapping from $t$ to $\mathrm{Cant}(\aleph_0, \aleph_2)$) such that $\sigma = r''Z$.

Put $f = r^{-1}$ and define

(20) $$f_1(x) = f(x) - \bigvee \{f(y);\ y \neq x,\ x, y \text{ are compatible}\}.$$

Then for any $x, y \in t$, $x \neq y$, we have

(21) $x, y$ are compatible or $f_1(x)$, $f_1(y)$ are disjoint.

If we put $r_1 = f_1^{-1}$, we have

(22) $$\sigma = r_1''Z = r''Z.$$

Since $f_1(x) \leqslant f(x)$ for any $x \in t$, we have $r_1''Z \subseteq r''Z$. The reverse inclusion is due to the fact that $\sigma$ is an antichain in $t$: if there were $y \in r''Z - r_1''Z$, then $f(y) \in Z$ and $f_1(y) \notin Z$. Then it would follow from (20) that

$$\bigvee \{f(w);\ w \neq y,\ w, y \text{ are compatible}\} \in Z$$

and, consequently, $f(w) \in Z$ for some $w$ compatible with $y$. So, there would be two compatible elements of $\sigma$ — a contradiction.

In the rest of the proof, we shall restrict ourselves to the dense subset $q$ described above. For any $x \in t$, let $v_x \subseteq q$ be a set of pairwise disjoint elements such that $\bigvee v_x = f_1(x)$.

Put

$$s = \{\langle x, u\rangle;\ u \in v_x \;\&\; x \in t\}.$$

Then $s \in L$ and has similar properties as $f_1$. For any $\langle x, u\rangle$, $\langle y, v\rangle \in s$, we have

(23) $\langle x, v\rangle \neq \langle y, v\rangle \rightarrow x, y$ are compatible or $u, v$ are disjoint.

But $Z$ is $L$-generic over $\mathrm{Cant}(\aleph_0, \aleph_2)$, thus $s''Z = r''Z = \sigma$. The set $q$ is naturally partitioned into subsets $q_n$, $n < \omega$ where $q_n = \{u_h;\ |h| = n\}$. Since $\sigma$ is uncountable in $L[Z]$, it follows that $|W(s)| \geqslant \aleph_1$. Then there is a natural number $n$ such that $|s'' q_n| = |s \upharpoonright q_n| \geqslant \aleph_1$. The rest of the proof goes entirely in $L$.

We shall apply the partition theorem (see [1])

(24) $$\aleph_1 \to (\aleph_1, \aleph_0)$$

to the set

$$[s \upharpoonright q_n]^2 = \{\{x, y\};\ \{x, y \in s \upharpoonright q_n \ \&\ x \neq y\}\}.$$

Let

$$a_1 = \{\{\langle x, u\rangle, \langle y, v\rangle\} \in [s \upharpoonright q_n]^2;\ x, y \text{ are incompatible in } \boldsymbol{t}\},$$

$$a_2 = [s \upharpoonright q_n]^2 - a_1.$$

According to (24) there exist either a subset $w_1 \subseteq s \upharpoonright q_n$, $|w_1| = \aleph_1$ and $[w_1]^2 \subseteq a_1$ or a subset $w_2 \subseteq s \upharpoonright q_n$, $|w_2| = \aleph_0$, $[w_2]^2 \subseteq a_2$. But both possibilities are contradictory: the existence of $w_1$ implies that $W(w_1) \subseteq t$ is an uncountable antichain in $L$ and the existence of $w_2$ and (23) implies that there is a countable set of pairwise disjoint elements of $q_n$. It can be shown by induction on $n$ that any subset of $q_n$ consisting of pairwise disjoint elements is finite (having cardinality $2^n$ at most). Thus we have shown that if there is an uncountable antichain in $L[Z]$, then $\boldsymbol{t}$ has an antichain of the same power in $L$. This completes the proof.

Let $\boldsymbol{t}$ be the normal Suslin $\omega_1$-tree in $L$ constructed by Jensen, let $Z$ be as above. It follows from Lemma 3 and the fact that all cardinals are preserved in $L[Z]$ that $\boldsymbol{t}$ remains the Suslin $\omega_1$-tree in $L[Z]$. According to Theorem 2, the complete Boolean algebra $\boldsymbol{b}_t$ constructed from $\boldsymbol{t}$ is rigid in $L[Z]$. The least subalgebra $\boldsymbol{b}_0$ of $\boldsymbol{b}_t$ containing $t$ has power $\aleph_1$ and is rigid, too. This proves the following

THEOREM 4. Con(ZFC)→Con(ZFC$+2^{\aleph_0} > \aleph_1 +$ "*there exists a rigid Boolean algebra of power* $\aleph_1$ *the completion of which is rigid as well*").

We shall finish with the following observation.

PROPOSITION. *Let* $\boldsymbol{b}_t$ *be as above, let* $\boldsymbol{b}$ *be the product of two copies of* $\boldsymbol{b}_t$ *(see* [11]). *If* $\omega_1^L = \omega_1$, *then there exists a* $(\aleph_0, \aleph_1)_b$*-matrix on* $\boldsymbol{b}$.

Proof. Let $Z$ be a generic ultrafilter on $b$. It follows from the well-known product lemma that $Z = Z_1 \times Z_2$, where $Z_1, Z_2$ are generic ultrafilters on $\boldsymbol{b}_t$, $Z_1 \neq Z_2$. Then $\sigma_i = Z_i \cap t$, $i = 1, 2$ are two different $\omega_1$-branches in $\boldsymbol{t}$ in the generic extension determined by $Z$. According to (ii) from Jensen's theorem, $\aleph_1$ must be collapsed in the generic extension, the last condition being equivalent to the conclusion of the proposition.

**Added in proof.** The consistency result of the last section was announced in Notices AMS 21 (1974), A500. Since then, S. Shelah has proved in ZFC that for every uncountable cardinal $\varkappa$ there is a rigid Boolean algebra of power $\varkappa$ with the rigid completion.

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