The shape of a map

by

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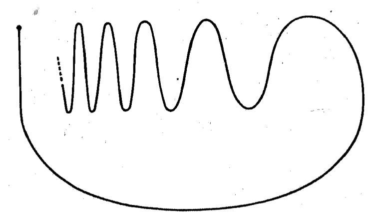
Abstract. The concept of the shape of a map is defined using an adaptation of a similar notion in etale homotopy theory. Čech and Vietoris constructions yield two functors $\mathcal{C}$ and $\mathcal{V}$ from $\mathrm{TOP}_{\mathrm{maps}}$ to pro-$\mathrm{CW}_{\mathrm{maps}}$ and pro-$\mathcal{H}$-$\mathrm{CW}_{\mathrm{maps}}$, respectively. Here $\mathcal{H}\,\mathrm{CW}_{\mathrm{maps}}$ denotes an appropriate homotopy category of maps of CW-complexes and, for any category $\mathcal{A}$, pro-$\mathcal{A}$ is an appropriate and well known category of inverse systems. Preliminaries on definitions and notation are given. An equivalence result of Dowker is extended to relate $\mathcal{C}$ and $\mathcal{V}$. Applications include the definitions of two functors from $(\mathrm{TOP}_0)_{\mathrm{maps}}$ (pointed maps) to LES (pro-$\mathcal{G}$), long exact sequences of pro-groups. They use a pro-adaptations of mapping cylinder and fiber resolution techniques and involve the pro-homotopy groups of the various spaces. The shape theoretic fiber of a map f is defined and an example shows that it is not, in general, shape equivalent to the shape of the homotopy theoretic fiber of f. Examples and applications to movable maps are given.

1. Introduction. The usual techniques and theorems of algebraic topology work well when applied to spaces having the homotopy type of a CW-complex. For more pathological spaces difficulties arise.

For example, consider the Warsaw circle S_W:



Globally S_W looks like the standard circle S but it has "local pathology" and, as is well known, this causes the homotopy and singular homology groups of S_W to vanish. In particular, $\pi_1(S_W) = 0$, while on the other hand, the Čech fundamental group $\check{\pi}_1(S_W)$ is equal to Z, the group of integers. In fact, $\check{F}(S_W) = F(S)$ for every functor F from $\mathscr{H}$, the homotopy category of CW-complexes, to the category of groups, where $\check{F}(S_W) = \varprojlim\{F(N)\}$, N denotes a nerve of an open covering of S_W and the inverse limit is taken over all such N. To see this, observe that the Čech tower (inverse system) of nerves of coverings of S_W has a cofinal subtower in which each element is a circle and the bonding maps have degree one.

Thus, in Čech theory one approximates a space by its "tower of nerves." Applying the homology and cohomology functors to this tower and then passing to the limit yields the Čech homology and cohomology groups. The inverse limit functor is not exact and it is known that Čech homology theory is not, in general, exact. One way to circumvent such problems is to work with the towers rather than just with the limits. One forms a category of towers in which cofinal towers are isomorphic. Such a construction has been discovered and rediscovered over the years: see, for example, Christie [5], Grothendieck [15], and Fox [11]. In Section II we give a brief description of this construction which associates with any category $\mathscr{A}$ a category pro-$\mathscr{A}$. The Čech construction yields a functor $\mathscr{C}$: TOP→pro-$\mathscr{H}$ and leads to Čech homotopy theory [21], [8] which is just one form of shape theory [3], [11], [8].

Just as a space X can be approximated by a tower of CW-complexes, a map $f\colon X\to Y$ can be approximated by a tower of maps between CW-complexes. The notion of the shape of a map is explained in Section III. It is adapted from a similar notion in etale homotopy theory [12] and gives a functor $\mathscr{C}\colon \mathrm{TOP}_{\mathrm{maps}}\to\text{pro-}\mathscr{H}\text{-}\mathrm{CW}_{\mathrm{maps}}$ where $\mathrm{TOP}_{\mathrm{maps}}$ and $\mathscr{H}\text{-}\mathrm{CW}_{\mathrm{maps}}$ are the appropriate categories of maps. We use both Čech and Vietoris constructions and relate these by extending an equivalence result of Dowker [6].

In Section IV we give applications: In IV.1 we show that any map induces a long exact sequence of pro-groups,

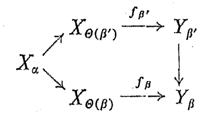
$$\ldots\to\text{pro-}\pi_i(X)\to\text{pro-}\pi_i(Y)\to\text{pro-}\pi_i(f)\to\ldots,$$

relating the pro-homotopy groups of X and Y and pro-groups pro-$\pi_i(f)$, which we define. In IV.2 we define the shape theoretic fiber of a map. An example shows that it is not shape equivalent to the shape of the homotopy theoretic fiber. We associate a second long exact sequence with a map. For certain maps $F\to X\to Y$ (shape quasi-fibrations) it yields an exact sequence,

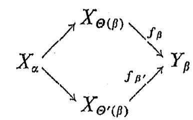
$$\ldots\to\text{pro-}\pi_i(F)\to\text{pro-}\pi_i(X)\to\text{pro-}\pi_i(Y)\to\ldots,$$

of pro-homotopy groups. Both parts of Section IV include applications to movable maps.

II. Pro-categories. If $\mathscr{Q}$ is any category, one can form a new category pro-$\mathscr{Q}$ whose objects are inverse systems $\{X_\alpha\}_{\alpha\in A}$ of objects of $\mathscr{Q}$ indexed by directed sets A. The set of morphisms in pro-$\mathscr{Q}$ from $\{X_\alpha\}_{\alpha\in A}$ to $\{Y_\beta\}_{\beta\in B}$ is denoted by pro-$\mathscr{Q}(\{X_\alpha\}_{\alpha\in A},\{Y_\beta\}_{\beta\in B})$ and equals $\varprojlim_{\beta\in B}(\varinjlim_{\alpha\in A}(\mathscr{Q}(X_\alpha, Y_\beta)))$ where $\mathscr{Q}(X_\alpha, Y_\beta)$ is the set of morphisms from X_α to Y_β in $\mathscr{Q}$. As this definition is somewhat opaque and does not yield a simple definition of composition, we shall make use of the following equivalent definition: A morphism in pro-$\mathscr{Q}$ from $\{X_\alpha\}_{\alpha\in A}$ to $\{Y_\beta\}_{\beta\in B}$ is an equivalence class of pairs $(\Theta,\{f_\beta\}_{\beta\in B})$ where (i) Θ is a function from B to A (not necessarily order preserving), (ii) for each $\beta\in B$, $f_\beta\colon X_{\Theta(\beta)}\to Y_\beta$ is a morphism in $\mathscr{Q}$ and (iii) for β,β' in B with $\beta'\geqslant\beta$ there is an $\alpha\in A$ for which the following diagram commutes:



The pairs $(\Theta,\{f_\beta\}_{\beta\in B})$ and $(\Theta',\{f'_\beta\}_{\beta\in B})$ are said to be *equivalent* if, for each $\beta\in B$, there is an $\alpha\in A$ with $\alpha\geqslant\Theta(\beta)$ and $\alpha\geqslant\Theta'(\beta)$ such that the following diagram commutes:



(In the above diagrams the unmarked maps are the bonding maps in the inverse systems.)

It is not hard to see that (i) $\mathscr{Q}$ can be embedded in pro-$\mathscr{Q}$ by regarding an object of $\mathscr{Q}$ as an inverse system over a one element directed set and (ii) any functor $T\colon\mathscr{Q}_1\to\mathscr{Q}_2$ induces a natural functor pro-T: pro-$\mathscr{Q}_1\to$pro-$\mathscr{Q}_2$. At times we use T for pro-T.

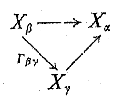
An account of pro-categories can be found in the Artin–Mazur Notes [1]. They allow A to be a "filtering category" (more general than a directed set). The definition of pro-$\mathscr{Q}$ given here is taken from [2].

The pro-object $\{X_\alpha\}_{\alpha\in A}\in$ pro-$\mathscr{Q}$ contains much more information about the inverse system than does the inverse limit $\varprojlim X_\alpha$, even if this inverse limit exists in $\mathscr{Q}$. The relationship between the pro-object $\{X_\alpha\}_{\alpha\in A}$ and $\varprojlim_{\alpha\in A} X$ is analogous to that between the germ of a function at a point p and the value of f at p. The following basic proposition show that $\{X_\alpha\}_{\alpha\in A}$ is "the germ at ∞" of the inverse system $\{X_\alpha\}_{\alpha\in A}$.

Proposition II.1. *Let A be a directed set and B a cofinal directed subset of A. If $\{X_\alpha\}_{\alpha\in A}$ is an object of* pro-$\mathscr{Q}$, *then it is isomorphic in* pro-$\mathscr{Q}$ *to the object $\{X_\alpha\}_{\alpha\in B}$.*

Proof. See [1], p. 150.

Let $\{X_\alpha\}_{\alpha\in A}$ be an object of pro-$\mathscr{Q}$ where $\mathscr{Q}$ is any category. The object is said to be *movable* if, for each $\alpha\in A$, there is a $\beta\in A$ with $\beta\geqslant\alpha$ such that for any $\gamma\geqslant\alpha$ there is a morphism $\Gamma_{\beta\gamma}\colon X_\beta\to X_\gamma$ making the following diagram commute:



(The unmarked maps are the bonding maps of $\{X_\alpha\}_{\alpha\in A}$).

If $\mathcal{Q}$ is the category of groups, then it is clear that a movable pro-group satisfies the Mittag–Leffler Condition (ML).

(ML): For each $\alpha \in A$, there is a $\beta \in A$ such that $\beta \geqslant \alpha$ and for all $\gamma \geqslant \beta$ the bonding homomorphisms $p_{\gamma\alpha}\colon X_\gamma \to X_\alpha$ have the same image.

The following properties of movability are obvious: (i) If B is confinal in A, then $\{X_\alpha\}_{\alpha\in A}$ is movable if and only if $\{X_\alpha\}_{\alpha\in B}$ is movable; (ii) If $T\colon \mathcal{Q}_1 \to \mathcal{Q}_2$ is a functor then pro-T maps movable objects in pro-$\mathcal{Q}_1$ to movable objects in pro-$\mathcal{Q}_2$; (iii) If, in (ii), $\mathcal{Q}_2$ is the category of groups, then pro-T maps movable objects in pro-$\mathcal{Q}_1$ to pro-groups satisfying (ML).

It is known ([4], p. 256) that the inverse limit functor is exact on countable pro-groups satisfying (ML). In Section IV.1 we give a precise statement of the version of this result which we need.

III. The shape of a map.

III.1. The basic categories. In this section we introduce the categories which we need. We start with the following:

TOP, the category of topological spaces and continuous maps,

CW, the category of CW-complexes and continuous maps,

$\mathcal{S}$, the category of simplicial sets and simplicial maps ([19]),

$\mathcal{H}$, the homotopy category associated with CW,

$\mathcal{K}$, the extended homotopy category associated with $\mathcal{S}$ ([19]).

Here, the objects of $\mathcal{H}$ are the objects of CW and the morphisms of $\mathcal{H}$ are the usual homotopy classes of morphisms in CW. The objects of $\mathcal{K}$ are the objects of $\mathcal{S}$ and in $\mathcal{K}$ the set of morphisms from X to Y is the set $[X, Y]$ which denotes the homotopy classes of morphisms from X to $\mathrm{Ex}^\infty Y$. In addition, we will let TOP_0, $\mathcal{H}_0$, etc. denote the obvious categories of pointed connected objects.

The above categories yield the *mapping categories* which *are our main objects of study*. In general, for any category $\mathcal{Q}$ we will use $\mathcal{Q}_{\mathrm{maps}}$ to denote the well known mapping category whose objects are the morphisms of $\mathcal{Q}$ and whose morphisms are given by commutative squares:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\alpha_1}\downarrow & & \downarrow{\scriptstyle\alpha_2} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

Here, $f\colon X \to Y$ and $f'\colon X' \to Y'$ are objects in $\mathcal{Q}_{\mathrm{maps}}$ and (α_1, α_2) is a morphism in $\mathcal{Q}_{\mathrm{maps}}$ from f to f'. Following this we have now $\mathrm{TOP}_{\mathrm{maps}}$, $\mathrm{CW}_{\mathrm{maps}}$, $\mathcal{S}_{\mathrm{maps}}$, $\mathcal{H}_{\mathrm{maps}}$ and $\mathcal{K}_{\mathrm{maps}}$. In addition, we have associated "homotopy" categories $\mathcal{H}$-$\mathrm{CW}_{\mathrm{maps}}$ and $\mathcal{K}$-$\mathcal{S}_{\mathrm{maps}}$. The objects of these are, respectively, the objects of $\mathrm{CW}_{\mathrm{maps}}$ (morphisms of CW) and the objects of $\mathcal{S}_{\mathrm{maps}}$ (morphisms of $\mathcal{S}$). The morphisms are "homotopy" classes of morphisms in $\mathrm{CW}_{\mathrm{maps}}$ and $\mathcal{S}_{\mathrm{maps}}$, respectively, where "homotopic" is appropriately defined. In $\mathrm{CW}_{\mathrm{maps}}$ two morphisms (α_1, α_2) and (β_1, β_2)

from $f\colon X \to Y$ to $f'\colon X' \to Y'$ are homotopic if there is a morphism (Θ_1, Θ_2) from $f \times 1_I\colon X \times I \to Y \times I$ to $f'\colon X' \to Y'$ such that Θ_1 is a homotopy from α_1 to β_1 and Θ_2 is a homotopy from α_2 to β_2. In $\mathcal{S}_{\mathrm{maps}}$, "homotopic" for morphisms is similarly defined using the notion of morphism in extended homotopy category $\mathcal{K}$.

We emphasize that the *objects* of $\mathcal{H}$-$\mathrm{CW}_{\mathrm{maps}}$ and $\mathcal{K}$-$\mathcal{S}_{\mathrm{maps}}$ *are actual maps* in CW and $\mathcal{S}$, respectively, *not homotopy classes of maps*, as is the case in $\mathcal{H}_{\mathrm{maps}}$ and $\mathcal{K}_{\mathrm{maps}}$. We have natural functors as follows:

$$\mathrm{CW}_{\mathrm{maps}} \to \mathcal{H}\text{-}\mathrm{CW}_{\mathrm{maps}} \to \mathcal{H}_{\mathrm{maps}},$$

$$\mathcal{S}_{\mathrm{maps}} \to \mathcal{K}\text{-}\mathcal{S}_{\mathrm{maps}} \to \mathcal{K}_{\mathrm{maps}}.$$

Applying pro yields categories pro-$\mathcal{S}$, pro-$\mathcal{S}_{\mathrm{maps}}$, pro-$\mathcal{K}$-$\mathcal{S}_{\mathrm{maps}}$, etc. and the induced pro functors.

We conclude this section by giving some remarks about morphisms in $\mathcal{S}$, $\mathcal{K}$, $\mathcal{S}_{\mathrm{maps}}$ and $\mathcal{K}$-$\mathcal{S}_{\mathrm{maps}}$. For a morphism α in $\mathcal{S}$, $[\alpha]$ will denote the image of α under the natural functor $\mathcal{S} \to \mathcal{K}$. For contiguous morphisms α and α' *in* $\mathcal{S}$, $[\alpha] = [\alpha']$. Similarly, for a morphism $\mu = (\mu_1, \mu_2)$ in $\mathcal{S}_{\mathrm{maps}}$, $[(\mu_1, \mu_2)]$ will denote the image of μ under the natural functor $\mathcal{S}_{\mathrm{maps}} \to \mathcal{K}$-$\mathcal{S}_{\mathrm{maps}}$. And, *if* $\mu' = (\mu_1', \mu_2')$ *is such that* μ_i and μ_i' *are contiguous* for $i = 1$ and 2, *then* $[(\mu_1, \mu_2)] = [(\mu_1', \mu_2')]$.

III.2. The functors $\mathcal{C}$ and $\mathcal{V}$. If X is a topological space, then the set $\mathrm{Cov}(X)$ of all open coverings of X forms a directed set partially ordered by refinement. If α is an open cover of X, then the Čech complex of X with respect to α, $C(X; \alpha)$ is that simplicial set with typical n-simplex an ordered $(n+1)$-tuple $\langle U_0, \ldots, U_n\rangle$ of open sets such that $\bigcap_{i=0}^{n} V_i \neq \emptyset$. If β is a refinement of α and $\nu\colon \beta \to \alpha$ is a refining map, then ν induces a simplicial map $\nu_*\colon C(X; \beta) \to C(X; \alpha)$ and any other refining map $\bar{\nu}\colon \beta \to \alpha$ induces $\bar{\nu}_*$ which is contiguous to ν_*. Hence, ν_* and $\bar{\nu}_*$ are "homotopic" and thus define the same element, $[\nu_*]$ in $[C(X; \beta), C(X; \alpha)]$. So we have $\{C(X; \alpha)\}_{\alpha\in \mathrm{Cov}(X)}$ is an inverse system in $\mathcal{K}$ or, equivalently, an object in pro-$\mathcal{K}$. We denote it by $\mathcal{C}(X)$. Next, for a map $f\colon X \to Y$ and $\alpha \in \mathrm{Cov}(Y)$, there is the natural simplicial map $f_\alpha\colon C(X; f^{-1}\alpha) \to C(Y; \alpha)$ and with $\Theta(\alpha) = f^{-1}\alpha$, $\{\Theta, \{f^{-1}\alpha\}_{\alpha\in\mathrm{Cov}(Y)}\}$ is the morphism in $\mathcal{K}$ induced by f. In this manner the usual Čech construction yields a functor $\mathcal{C}\colon \mathrm{TOP} \to \text{pro-}\mathcal{K}$.

Porter [21] has used the Vietoris construction to obtain a functor $\mathcal{V}\colon \mathrm{TOP} \to \text{pro-}\mathcal{S}$. If α is an open cover of X, then $V(X; \alpha)$ is that simplicial set with typical n-simplex an $(n+1)$-tuple $\langle x_0, x_1, \ldots, x_n\rangle$ of points of X all belonging to at least one $U \in \alpha$. If β refines α then there is a canonical simplicial map from $V(X; \beta)$ to $V(X; \alpha)$ given by $x \to x$ for a vertex of $V(X; \beta)$. This fact determines $\{V(X; \alpha)\}_{\alpha\in\mathrm{Cov}(X)} = \mathcal{V}(X)$ as an object in pro-$\mathcal{S}$ rather than pro-$\mathcal{K}$. Having $\mathcal{V}(X)$ in pro-$\mathcal{S}$ rather than pro-$\mathcal{K}$ has many advantages. By results of Dowker [6], $C(X)$ and $\mathcal{V}(X)$ are canonically isomorphic in pro-$\mathcal{K}$.

Now we proceed to extending $\mathcal{V}$ and $\mathcal{C}$ to mapping categories and to giving an extension of Dowker's result.

Definition of $\mathscr{V}$: $\mathrm{TOP}_{\mathrm{maps}} \to \text{pro-}\mathscr{S}_{\mathrm{maps}}$:

Let $f\colon X \to Y$ be a continuous map, α an open cover of Y and β an open covering of X with β refining $f^{-1}\alpha$. Then f induces a canonical simplicial map $f_{\alpha\beta}\colon V(X;\beta) \to V(Y;\alpha)$ given by $f_{\alpha\beta}(x) = f(x)$ for each vertex x of $V(X;\beta)$. The set of all such pairs (α,β) with $\beta \geqslant f^{-1}\alpha$ forms a directed set $\mathrm{Cov}(f)$ where $(\alpha',\beta') \geqslant (\alpha,\beta)$ if and only if $\beta' \geqslant \beta$ and $\alpha' \geqslant \alpha$. In this case, as before, $x \to x$ and $y \to y$ will yield a canonical morphism from $f_{\alpha'\beta'}$ to $f_{\alpha\beta}$ in $\mathscr{S}_{\mathrm{maps}}$. Thus we obtain an inverse system $\{f_{\alpha\beta}\colon V(X;\beta) \to V(Y;\alpha)\}_{\mathrm{Cov}(f)}$ in $\mathscr{S}_{\mathrm{maps}}$, or an element of pro-$\mathscr{S}_{\mathrm{maps}}$. We denote it by $\mathscr{V}(f)$. A morphism $g = (g_1, g_2)$ from f' to f in $\mathrm{TOP}_{\mathrm{maps}}$ yields a morphism $\mathscr{V}(g)$ from $\mathscr{V}(f')$ to $\mathscr{V}(f)$ in pro-$\mathscr{S}_{\mathrm{maps}}$ by the following definitions. Let $\Theta\colon \mathrm{Cov}(f) \to \mathrm{Cov}(f')$ be given by $\Theta(\alpha,\beta) = (g_2^{-1}\alpha, g_1^{-1}\beta)$, $(g_1)_{\alpha\beta}(x) = g_1(x)$ and $(g_2)_{\alpha\beta}(y) = g_2(y)$ for vertices x and y of $V(C;\beta)$ and $V(Y,\alpha)$, respectively, and let $g_{\alpha\beta} = ((g_1)_{\alpha\beta}, (g_2)_{\alpha\beta})$. Then $\mathscr{V}(f) = (\Theta, \{g_{\alpha\beta}\}_{(\alpha\beta)\in \mathrm{Cov}(f)}\}$.

Definition of $\mathscr{C}$: $\mathrm{TOP}_{\mathrm{maps}} \to \text{pro-}\mathscr{K}\text{-}\mathscr{S}_{\mathrm{maps}}$:

Let $\check{\mathrm{C}}\mathrm{ov}(f)$ be the set of all triples (α,β,ν) such that $(\alpha,\beta) \in \mathrm{Cov}(f)$ and $\nu\colon \beta \to f^{-1}\alpha$ is a refining map. The map ν determines a simplicial map $f_{\alpha\beta\nu}\colon C(X;\beta) \to C(Y;\alpha)$ given by $f_{\alpha\beta\nu}(W) = f\circ\nu(W)$ for $W \in \beta$, a vertex of $C(X;\beta)$. A refining map from (α',β',ν') to (α,β,ν) is a pair (μ_1,μ_2) with $\mu_1\colon \beta' \to \beta$, $\mu_2\colon \alpha' \to \alpha$, such that the following diagram commutes:

$$\begin{array}{ccc} \beta' & \xrightarrow{\nu'} & f^{-1}\alpha \\ {\scriptstyle \mu_1}\downarrow & & \downarrow{\scriptstyle \mu_2} \\ \beta & \xrightarrow{\nu} & f^{-1}\alpha \end{array}$$

A refining map (μ_1,μ_2) induces a map (μ_{1*},μ_{2*}) in $\mathscr{S}_{\mathrm{maps}}$ as follows:

$$\begin{array}{ccc} C(X;\beta') & \xrightarrow{f_{\alpha'\beta'\nu'}} & C(Y;\alpha') \\ {\scriptstyle \mu_{1*}}\downarrow & & \downarrow{\scriptstyle \mu_{2*}} \\ C(X;\beta) & \xrightarrow{f_{\alpha\beta\nu}} & C(Y;\alpha) \end{array}$$

Also, if (μ_1',μ_2') is another refining map from (α',β',ν') to (α,β,ν), then (μ_{1*}',μ_{2*}') and (μ_{1*},μ_{2*}) are contiguous in $\mathscr{S}_{\mathrm{maps}}$ and, hence, induce the same map $[(\mu_{1*},\mu_{2*})]$ in $\mathscr{K}\text{-}\mathscr{S}_{\mathrm{maps}}$. We partially order $\check{\mathrm{C}}\mathrm{ov}(f)$ by the possibility of refinement. With this ordering, $\check{\mathrm{C}}\mathrm{ov}(f)$ is a directed set and, hence,

$$\{C(X;\beta) \xrightarrow{f_{\alpha\beta\nu}} C(Y;\alpha)\}_{(\alpha,\beta,\nu)\in\check{\mathrm{C}}\mathrm{ov}(f)}$$

is an element of pro-$\mathscr{K}$-$\mathscr{S}_{\mathrm{maps}}$ which we denote by $\mathscr{C}(f)$. For a morphism $g = (g_1,g_2)\colon f' \to f$, the induced morphism $\mathscr{C}(g)$ is defined in the obvious way. A map $\Theta\colon \check{\mathrm{C}}\mathrm{ov}(f) \to \check{\mathrm{C}}\mathrm{ov}(f')$ is given by $\Theta(\alpha,\beta,\nu) = (g_2^{-1}\alpha, g_1^{-1}\beta, g_*(\nu))$ where $g_*(\nu)(W) = f'^{-1}g_2^{-1}f\nu g_1(W)$ for $W \in g_1^{-1}\beta$ and we set $\mathscr{C}(g) = \{\Theta, \{g_{\alpha\beta\nu}\}_{(\alpha,\beta,\nu)\in\check{\mathrm{C}}\mathrm{ov}(f)}\}$ with $g_{\alpha\beta\nu}\colon f_{\Theta(\alpha,\beta,\nu)} \to f_{\alpha\beta\nu}$ defined by $g_{\alpha\beta\nu} = ((g_1)_\beta, (g_2)_\alpha)$.

By composing $\mathscr{C}$ and $\mathscr{V}$ with the geometric realization functor $|\cdot|\colon \mathscr{S} \to \mathrm{CW}$ we can pass back to topology and might consider $\mathscr{V}$ and $\mathscr{C}$ as functors from $\mathrm{TOP}_{\mathrm{maps}}$ into pro-$\mathrm{CW}_{\mathrm{maps}}$, pro-$\mathscr{H}$-$\mathrm{CW}_{\mathrm{maps}}$, or pro-$\mathscr{H}_{\mathrm{maps}}$, as appropriate. We can also define functors

$$\mathscr{V}\colon (\mathrm{TOP}_0)_{\mathrm{maps}} \to \text{pro-}(\mathscr{S}_0)_{\mathrm{maps}} \quad\text{and}\quad \mathscr{C}\colon (\mathrm{TOP}_0)_{\mathrm{maps}} \to \text{pro-}\mathscr{K}\text{-}(\mathscr{S}_0)_{\mathrm{maps}},$$

using pointed coverings.

We conclude with the extension of Dowker's result.

PROPOSITION III.2.1. *The functors $\mathscr{C}$ and $\mathscr{V}$ from $\mathrm{TOP}_{\mathrm{maps}}$ to pro-$\mathscr{H}_{\mathrm{maps}}$ are naturally equivalent.*

(In this statement, the functors $\mathscr{C}$ and $\mathscr{V}$ are those actually defined above composed with the natural functors from pro-$\mathscr{S}_{\mathrm{maps}}$ and pro-$\mathscr{K}$-$\mathscr{S}_{\mathrm{maps}}$, respectively, into pro-$\mathscr{H}_{\mathrm{maps}}$.)

Proof. For an object K in $\mathscr{S}$, K^b will denote the first barycentric subdivision of K; for $g\colon K \to L$, $g^b\colon K^b \to L^b$ will be the usual simplicial map induced by g; and and $\varphi_K\colon K^b \to K$ will be given by $\varphi(x') = x$ where x is the first vertex of that simplex of which x' is the barycenter relative to some ordering of all of the vertices of K. Then $[\varphi_K]$ is an equivalence in $\mathscr{K}$ and $[g^b][\varphi_K]^{-1} = [\varphi_L]^{-1}[g]$ since $g\varphi_K \sim \varphi_L g^b$ with $\sim$ denoting contiguity.

Using this notation (with slight modification) yields $\varphi_\alpha\colon C(X;\alpha)^b \to C(X;\alpha)$ and $\bar{\varphi}_\alpha\colon V(X;\alpha)^b \to V(X;\alpha)$ for any $\alpha \in \check{\mathrm{C}}\mathrm{ov}(X)$. Also, if $f\colon X \to X'$ and $\alpha' \in \check{\mathrm{C}}\mathrm{ov}(X')$ with $\alpha > f^{-1}\alpha'$, then the φ's are natural in that $f_*\varphi_\alpha \sim \varphi_{\alpha'}f_*^b$ and $\bar{f}_*\bar{\varphi}_\alpha \sim \bar{\varphi}_{\alpha'}\bar{f}_*^b$.

Dowker's work yields maps $\psi_\alpha\colon C(X;\alpha)^b \to V(X;\alpha)$ and $\bar{\psi}_\alpha\colon V(X;\alpha)^b \to C(X;\alpha)$. And, he shows that they are natural in that $\bar{f}_*\psi_\alpha \sim \psi_{\alpha'}f^b$ and $f_*\bar{\psi}_\alpha \sim \bar{\psi}_{\alpha'}\bar{f}^b$ (Lemma 3 of [6]).

Next, following Dowker we define $\eta_\alpha = [\psi_\alpha][\varphi_\alpha]^{-1}$ and $\bar{\eta}_\alpha = [\bar{\psi}_\alpha][\bar{\psi}_\alpha]^{-1}$. The facts that $\eta_\alpha\bar{\eta}_\alpha = 1$ and $\bar{\eta}_\alpha\eta_\alpha = 1$ follow from the various contiguity statements given in Lemmas 5 and 6 of [6].

Finally, let $\Theta\colon \check{\mathrm{C}}\mathrm{ov}(f) \to \check{\mathrm{C}}\mathrm{ov}(f)$ be given by $\Theta(\alpha,\beta) = (\alpha,\beta,\nu_{\alpha\beta})$ for any choice of $\nu_{\alpha\beta}$ and $\bar{\Theta}\colon \check{\mathrm{C}}\mathrm{ov}(f) \to \mathrm{Cov}(f)$, by $\bar{\Theta}(\alpha,\beta,\nu) = (\alpha,\beta)$. The maps

$$\eta(f) = (\Theta, \{\eta_{\alpha\beta}\}_{(\alpha,\beta)\in\check{\mathrm{C}}\mathrm{ov}(f)})$$

from $\mathscr{C}(f)$ to $\mathscr{V}(f)$ and $\bar{\eta}(f) = (\bar{\Theta}, \{\bar{\eta}_{\alpha\beta\nu}\}_{(\alpha,\beta,\nu)\in\check{\mathrm{C}}\mathrm{ov}(f)})$ from $\mathscr{V}(f)$ to $\mathscr{C}(f)$ where $\eta_{\alpha\beta} = (\eta_\beta, \eta_\alpha)$ and $\bar{\eta}_{\alpha\beta\nu} = (\bar{\eta}_\beta, \bar{\eta}_\alpha)$ yield the required natural equivalence. That they are well defined and natural follows easily from the naturality conditions given above on the φ and ψ maps.

IV. Applications.

IV.1. The homotopy exact sequence of a map. For each i, π_i will denote the usual homotopy functor from the various appropriate categories to $\mathscr{G}$, the category of groups and homomorphisms. As indicated above, for $\pi_i\colon \mathscr{K}_0 \to \mathscr{G}$, pro-$\pi_i$ composed with $\mathscr{C}$ gives a functor from TOP_0 to pro-$\mathscr{G}$. Suppressing the $\mathscr{C}$, we have pro-$\pi_i(X, x_0)$, the ith pro-homotopy group of (X, x_0) ([21], [11]).

Let $f\colon (K,k)\to(L,l)$ be a map in $\mathscr{S}_0$ and M_f its mapping cylinder (see [19], p. 53). Then the diagram

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ & {\scriptstyle i}\searrow \quad \swarrow{\scriptstyle r} & \\ & M_f & \end{array}$$

commutes, where i is inclusion and r is a retraction such that L is a strong deformation retract of M_f. We define $\pi_i(f) = \pi_i(M_f, K)$ and obtain the usual long exact sequence

$$\ldots\to\pi_i(K)\to\pi_i(L)\to\pi_i(f)\to\ldots$$

The association of this sequence to f defines a functor $\pi\colon \mathscr{K}\text{-}(\mathscr{S}_0)_{\rm maps}\to \mathrm{LES}(\mathscr{G})$, the category of long exact sequences in $\mathscr{G}$. Applying pro yields a functor

$$\text{pro-}\pi\colon \text{pro-}\mathscr{K}\text{-}(\mathscr{S}_0)_{\rm maps}\to\text{pro-}(\mathrm{LES}(\mathscr{G}))\,.$$

Composing with $\mathscr{C}$ yields a functor

$$\text{pro-}\pi\circ\mathscr{C}\colon (\mathrm{TOP}_0)_{\rm maps}\to\text{pro-}(\mathrm{LES}(\mathscr{G}))\,.$$

In view of Proposition III.2.1, by factoring through $(\mathscr{K}_0)_{\rm maps}$ it is easy to show that composing with $\mathscr{V}$, instead of $\mathscr{C}$, yields an equivalent functor. Again we omit $\mathscr{C}$ and write

$$\text{pro-}\pi\colon (\mathrm{TOP}_0)_{\rm maps}\to\text{pro-}(\mathrm{LES}(\mathscr{G}))\,.$$

For each map $f\colon (X,x_0)\to(Y,y_0)$ we wish to obtain a long exact sequence of pro-groups. To do this we need the category LES(pro-$\mathscr{G}$) and a natural functor $\lambda\colon$ pro-(LES($\mathscr{G}$))$\to$LES(pro($\mathscr{G}$)). This requires the appropriate notion of exactness in pro-$\mathscr{G}$. Here we follow Mardešić as in [18]: In any category with zero-objects and kernels one can define exactness. The sequence $G\xrightarrow{f}H\xrightarrow{g}K$ at H is said to be *exact at* H if and only if (1) $g\circ f=0$ and (2) if $i\colon N\to H$ is the kernel of g and $f'\colon G\to N$ is that unique morphism with $i\circ f'=f$, then f' is an epimorphism. Since pro-$\mathscr{G}$ is a category with zero-objects and Kernels (see, e.g., [18], [20]), exactness is defined in the manner just described. The obvious definition of long exact sequence in pro-$\mathscr{G}$ can now be made. Thus, one obtains the category LES(pro-$\mathscr{G}$).

Certain simple morphisms play an important role in pro-categories and are involved in our consideration here. Specifically, in pro-$\mathscr{Q}$ for any $\mathscr{Q}$, $f\colon \{X_\alpha\}_{\alpha\in A}\to\{Y_\beta\}_{\beta\in B}$ is said to be *special morphism*, [18], provided that $A=B$, and $f=(1_A,\{f_\alpha\}_{\alpha\in A})$ where 1_A is the identity on A, and whenever $\alpha\leqslant\alpha'$, the following diagram commutes:

$$\begin{array}{ccc} X_\alpha & \longleftarrow & X_{\alpha'} \\ {\scriptstyle f_\alpha}\downarrow & & \downarrow{\scriptstyle f_{\alpha'}} \\ Y_\alpha & \longleftarrow & Y_{\alpha'} \end{array}$$

(The unmarked maps are the appropriate bonding maps.) These maps are important in that every map in pro-$\mathscr{Q}$ can be "replaced up to isomorphism" by a *special morphism* [18]. This is useful in showing that pro-$\mathscr{G}$ is a category with kernels.

Returning now to pro-$\mathscr{G}$ we state without proof a fact given and proved by Mardešić in [18]:

PROPOSITION IV.1.1. *If* $f\colon \{G_\alpha\}_{\alpha\in A}\to\{H_\beta\}_{\beta\in B}$ *and* $g\colon \{H_\beta\}_{\beta\in B}\to\{K_\gamma\}_{\gamma\in C}$ *are special morphisms in pro-*$\mathscr{G}$, *then* $\{G_\alpha\}_{\alpha\in A}\xrightarrow{f}\{H_\alpha\}_{\alpha\in A}\xrightarrow{g}\{K_\alpha\}_{\alpha\in A}$ *is exact in pro-*$\mathscr{G}$ *at* $\{H_\alpha\}_{\alpha\in A}$ *provided that for each* $\alpha\in A$, $G_\alpha\xrightarrow{f_\alpha}H_\alpha\xrightarrow{g_\alpha}K_\alpha$ *is exact at* H_α *in* $\mathscr{G}$.

We now have our desired result as a corollary:

COROLLARY IV.1.1. *There is a natural functor*

$$\gamma\colon \text{pro-}(\mathrm{LES}(\mathscr{G}))\to\mathrm{LES}(\text{pro-}\mathscr{G})\,.$$

We indicate the definition of γ as follows: Let $\{s_\alpha\}_{\alpha\in A}$ be an object of pro-(LES($\mathscr{G}$)) with s_α denoting the sequence

$$\ldots\to G_\alpha^{i+1}\to G_\alpha^i\xrightarrow{h_\alpha^i}G_\alpha^{i-1}\to\ldots$$

for $i\in Z$, the set of integers. Then $\gamma(\{s_\alpha\}_{\alpha\in A})$ is the sequence

$$\ldots\to\{G_\alpha^{i+1}\}_{\alpha\in A}\to\{G_\alpha^i\}_{\alpha\in A}\xrightarrow{h^i}\{G_\alpha^{i-1}\}_{\alpha\in A}\to\ldots$$

where $h^i=(1_A,\{h_\alpha^i\}_{\alpha\in A})$. The exactness of this sequence in pro-$\mathscr{G}$ is an immediate consequence of Proposition IV.1.1. Other details in defining γ are left to the reader.

An object in LES(pro-$\mathscr{G}$) with the property that all of the morphisms of the sequence are special morphisms will be called a *special exact sequence of pro-groups*. Note that the images under γ of objects in pro-(LES($\mathscr{G}$)) are such sequences. For completeness, we give here a "special" version of the known exactness result mentioned in Section II.

PROPOSITION IV.1.2. *If*

$$\ldots\to\{G_\alpha^{i+1}\}_{\alpha\in A}\xrightarrow{h^{i+1}}\{G_\alpha^i\}_{\alpha\in A}\xrightarrow{h^i}\{G_\alpha^{i-1}\}_{\alpha\in A}\to\ldots$$

is a special exact sequence of pro-groups each satisfying (ML) *and* A *is countable, then the limit sequence,*

$$\ldots\to\varprojlim\{G_\alpha^{i+1}\}_{\alpha\in A}\xrightarrow{\lim h^{i+1}}\varprojlim\{G_\alpha^i\}_{\alpha\in A}\xrightarrow{\lim h^i}\to\ldots$$

is exact.

Indication of proof. It is well known that in this situation the desired exactness will hold provided that

$$\varprojlim\{h_\alpha^{i+1}(G_\alpha^{i+1})\}\subset(\varprojlim h^{i+1})(\varprojlim\{G_\alpha^{i+1}\})$$

holds for each i. That is, that "the limit of the images is contained in the image of the limit". For this one can restrict attention to a finite portion of the sequence and show that there, without loss of generality, it may be assumed that A is the set of natural numbers with the usual order and (from the (ML) condition) that each of the bonding maps is onto. The result then follows easily using induction and standard diagram chasing to obtain an appropriate element in $\varprojlim\{G_\alpha^{i+1}\}$.

Now, returning to the functor pro-π and composing with γ we get

$$(\text{pro-}\pi)\circ\gamma\colon (\mathrm{TOP}_0)_{\mathrm{maps}}\to \mathrm{LES}(\text{pro-}\mathscr{G}).$$

We omit the γ and for each $f\colon (X, x_0)\to(Y, y_0)$ have pro-$\pi(f)$, a long exact sequence of pro-groups, where it is easy to see that those pro-groups appearing are the pro-homotopy-groups of (X, x_0) and (Y, y_0) and a third set of pro-groups which we will define to be the pro-homotopy-groups of f. That is, we let

$$\text{pro-}\pi_i(f) = \{\pi_i(f_{\alpha\beta\nu})\}_{(\alpha,\beta,\nu)\in\check{\mathrm{C}}\mathrm{ov}(f)}$$

and then summarize our results as follows:

THEOREM IV.1.1. *There is a functor* pro-π *from* $(\mathrm{TOP}_0)_{\mathrm{maps}}\to\mathrm{LES}(\text{pro-}\mathscr{G})$ *which associates the sequence*

$$\ldots\to\text{pro-}\pi_i(X, x_0)\to\text{pro-}\pi_i(Y, y_0)\to\text{pro-}\pi_i(f)\to\ldots$$

to the map $f\colon (X, x_0)\to(Y, y_0)$.

Remark. The same techniques could be used to obtain pro-analogs of other familiar long exact and spectral sequences.

A map $f\colon (X, x_0)\to(Y, y_0)$ is said to be *movable* if and only if $\mathscr{C}(f)$ is movable in pro-$\mathscr{K}$-$(\mathscr{S}_0)_{\mathrm{maps}}$. For certain of these we get a limit exact sequence as follows:

THEOREM IV.1.2. *If* $f\colon (X, x_0)\to(Y, y_0)$ *is a movable map in* TOP_0 *and* X *and* Y *are compact metric spaces, then there is a long exact sequence*

$$\ldots\to\check{\pi}_i(X, x_0)\to\check{\pi}_i(Y, y_0)\to\check{\pi}_i(f)\to\ldots$$

where $\check{\pi}_i(X, x_0) = \varprojlim\text{pro-}\pi_i(X, x_0)$, *etc.*

Proof. Since X and Y are compact metric spaces, $\check{\mathrm{C}}\mathrm{ov}(f)$ has a countable cofinal subset. Using this we may assume that each pro-group in the special long exact sequence

$$\ldots\to\text{pro-}\pi_i(X, x_0)\to\text{pro-}\pi_i(Y, y_0)\to\text{pro-}\pi_i(f)\to\ldots$$

is indexed by a countable set. Since $\mathscr{C}(f)$ is movable pro-$\pi(\mathscr{C}(f))$ is movable in pro-$(\mathrm{LES}(\mathscr{G}))$. It is easy to see that γ takes a movable object in pro-$(\mathrm{LES}(\mathscr{G}))$ to a sequence of movable pro-groups. Thus each term of our sequence satisfies (ML) and Proposition IV.1.2 applies to show that the limit sequence is exact.

Mardešić has proved the following Whitehead Theorem in shape theory:

THEOREM IV.1.3. *If* $f\colon (X, x_0)\to(Y, y_0)$ *is a map in* TOP_0, X *and* Y *are finite dimensional and*

$$f_*\colon \text{pro-}\pi_i(X, x_0)\to\text{pro-}\pi_i(Y, y_0)$$

the appropriate induced map is an isomorphism for each i, *then* f *is a shape equivalence* [18].

This theorem, combined with Theorem IV.1.2 yields:

THEOREM IV.1.4. *If* $f\colon (X, x_0)\to(Y, y_0)$ *is a movable map in* TOP_0, X *and* Y *are compact finite dimensional metric spaces, and* $\check{f} = \varprojlim f_*\colon \check{\pi}_i(X, x_0)\to\check{\pi}_i(Y, y_0)$ *is an isomorphism for each* i, *then* f *is a shape equivalence.*

Proof. From the exact sequence of Theorem IV.1.2, $\check{\pi}_i(f) = 0$ for each i. Furthermore, as in the proof of Theorem IV.1.2, pro-$\pi_i(f)$ satisfies (ML) and has a countable cofinal subset whose limit is 0. It follows that pro-$\pi_i(f)$ is isomorphic to the trivial pro-group. Now, from the exactness of the pro-sequence, f_* is an isomorphism for each i and Mardešić's Theorem applies to show that f is a shape equivalence.

Remark. Keesling [17] has shown that one can weaken the hypothesis of Theorem IV.1.4 to only assuming that X and Y are movable. There are non-movable maps between movable spaces. We give an example of one such map using the construction described by Draper and Keesling in [7] but starting with simpler spaces. Our example will show that the condition of "movable map" in Theorem IV.1.2 cannot be replaced with the assumption that X and Y are movable.

EXAMPLE IV.1.1. Let $S_1\leftarrow\ldots S_{n-1}\xleftarrow{\mu_n}S_n\leftarrow\ldots$ be the non-movable inverse system of circles which has the dyadic solenoid D as its limit. That is, let each $S_n = S$, the unit circle in the complex plane and $\mu_n(z) = z^2$. Apply the construction of Draper and Keesling to obtain $X_1\leftarrow\ldots X_{n-1}\xleftarrow{\mu_n}X_n\leftarrow\ldots$ where $X_n = S_1\vee S_2\vee\ldots\vee S_n$, the wedge of n-circles at the point s, $\nu_n(z) = z$ for $z\in S_i$ with $1\leqslant i\leqslant n-1$ and $\nu_n(z) = z^2$ for $z\in S_n$. This system is movable. The space $X = \varprojlim X_n$ is $B\cup D$ where B is a countable bouquet of circles, $B\cap D$ is the wedge point of the bouquet, and the circles converge to D. Continuing the construction, consider the trivial system

$$S_1\leftarrow\ldots S_{n-1}\xleftarrow{\sigma_n}S_n\leftarrow\ldots$$

where $\sigma_n(z) = s$ for each z and augment as before to obtain $Y_1\leftarrow\ldots Y_{n-1}\xleftarrow{\tau_n}Y_n\leftarrow\ldots$ with $Y_n = X_n$ and $\tau_n(z) = z$ for $z\notin S_n$ and $\tau_n(z) = s$ for $z\in S_n$. Next, for each $n\in N$, define $f_n\colon (X_{n+1}, s)\to(Y_n, s)$ by $f_n(z) = z$ for $z\notin S_n$ and $f_n(z) = s$ for $z\in S_n$. We will consider the object $\{f_n\}_{n\in N}$ in pro-$\mathscr{K}$-$(\mathscr{S}_0)_{\mathrm{maps}}$ where the bonding maps are the obvious pairs (ν_{n+1}, τ_n).

The space $Y = \varprojlim Y_n$ is a countable bouquet of circles converging to a point and with $f = \varprojlim f_n$ we have $f\colon (X, x_0)\to(Y, y_0)$ in $(\mathrm{TOP}_0)_{\mathrm{maps}}$ with s yielding the points x_0 and y_0 and $\mathscr{C}(f) = \{f_n\}_{n\in N}$. It remains to show that (1) $\mathscr{C}(f)$ is not movable and (2) for this f, the long limit sequence of Theorem IV.1.2 is not exact.

For both (1) and (2) we have $F(x_1x_2\ldots x_n) = \pi_1(X_n, s) = \pi_1(Y_n, s)$, the free group with n generator, with x_j representing a generator of $\pi_1(S_j)$ and consider the following diagram which we will call "d".

$$\begin{array}{ccccc}
F(x_1x_2) & \xrightarrow{\nu_{2*}} & F(x_1x_2x_3) & \xrightarrow{\nu_{3*}} & \ldots F(x_1x_2\ldots x_{n+1})\ldots \\
\downarrow{\scriptstyle f_{1*}} & & \downarrow{\scriptstyle f_{2*}} & & \downarrow{\scriptstyle f_{n*}} \\
F(x_1) & \xrightarrow{\tau_{1*}} & F(x_1x_2) & \xrightarrow{\tau_{2*}} & \ldots F(x_1x_2\ldots x_n)\ldots
\end{array}$$

For (1), careful consideration of the "d" will show that it represents a non-movable object in pro-$(\mathscr{G})_{\text{maps}}$. But, then $\mathscr{C}(f)$ is not movable, for otherwise, the functor pro-π_1 would take a movable object pro-$\mathscr{K}$-$(\mathscr{S}_0)_{\text{maps}}$ to a non-movable one in pro-$(\mathscr{G})_{\text{maps}}$.

For (2), note that the exactness in Theorem IV.1.2 can and *should be interpreted to include exactness of*

$$\to\check{\pi}_1(X, x_0)\xrightarrow{\check{f}}\check{\pi}_1(Y, y_0)\to\check{\pi}_1(f)\to 0\,.$$

This is the case because we restrict ourselves to connected spaces and have exactness

$$\to\pi_1(C(X;\beta))\xrightarrow{f_{\alpha\beta\nu*}}\pi_1(C(Y,\alpha))\to\pi_1(f_{\alpha\beta\nu}) = \pi_1(M_{f_{\alpha\beta\nu}}, C(X;\alpha))\to 0$$

for each $(\alpha, \beta, \nu)\in \check{\mathrm{C}}\mathrm{ov}(f)$.

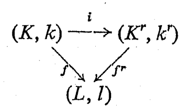
Now, for the f of this example, $\pi_1(X_{n+1}, s)\to\pi_1(Y_n, s)$ is surjective and hence $\pi_1(f_n) = 0$. It follows that $\check{\pi}_1(f) = 0$. However, consideration of "d" shows easily that $\check{f} = \underleftarrow{\lim}\{f_{n*}\}$ is not a surjection.

IV.2. The shape theoretic fiber of a map. For each map f in $(\mathrm{TOP}_0)_{\text{maps}}$ we will determine, functorially, three objects in pro-$\mathscr{S}_0$ each of which is, in some sense, a shape fiber of f. We will use the Vietoris functor together with the functors fib and $\mathscr{F}$ described as follows:

The *functor* fib: $(\mathscr{S}_0)_{\text{maps}}\to\mathscr{S}_0$ (or $(\mathrm{TOP}_0)_{\text{maps}}\to\mathrm{TOP}_0$): Here for $f\colon (X, x)\to(Y, y)$, $\mathrm{fib}(f) = (f^{-1}(y), x)$. As usual we denote fib$(f)$ by f^{-1}.

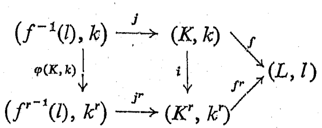
The *functor* $\mathscr{F}\colon (\mathscr{S}_0)_{\text{maps}}\to(\mathscr{S}_0)_{\text{maps}}$ called *fiber resolution* and described in [13], VI. Here, for $f\colon (K, k)\to(L, l)$ one can associate to f in a functorial way the commutative diagram:

(1)



where i is a weak homotopy equivalence and f^r is a Kan fibration. Then $\mathscr{F}(f) = f^r$ and $(\mathscr{F}(f))^{-1}$ is called the *homotopy theoretic fiber of* f. One has also the commutative diagram:

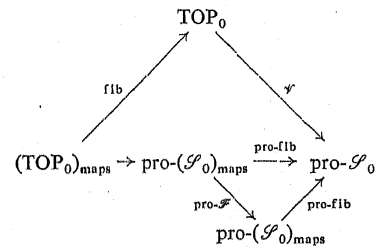
(2)



It is easy to check that the maps $\varphi_{(K,k)}\colon f^{-1}\to(\mathscr{F}(f))^{-1}$ define a natural transformation from fib to fib $\circ\,\mathscr{F}$. We apply pro to obtain functors pro-fib and pro-$\mathscr{F}$ and a natural transformation φ: pro-fib$\to$pro-(fib $\circ\,\mathscr{F}$) = pro-fib $\circ$ pro-$\mathscr{F}$.

Next, we consider the following diagram of categories and functors:

(3)



The three paths from $(\mathrm{TOP}_0)_{\text{maps}}$ to pro-$\mathscr{S}_0$ define the three "fibers" mentioned above. For $f\colon (X, x)\to(Y, y)$ in $(\mathrm{TOP}_0)_{\text{maps}}$ we have the following, supressing the pro- an using -1 for fib:

1. $\mathscr{V}(f^{-1})$, the Vietoris type of the fiber of f.
2. $(\mathscr{V}(f))^{-1}$, the fiber of the Vietoris type of f.
3. $\big(\mathscr{F}(\mathscr{V}(f))\big)^{-1}$, the Vietoris theoretic fiber of f.

The first two of the above functors are naturally equivalent, that is, the "shape of fiber" and the "fiber of the shape" coincide. There is a natural transformation from the second functor to the third, but as one would expect, they do not coincide. We have the following theorems:

THEOREM IV.2.1. *There is a canonical map* $\psi\colon \mathscr{V}(f^{-1})\to(\mathscr{V}(f))^{-1}$ *which is an isomorphism if* Y *is Hausdorff.*

THEOREM IV.2.2. *There is a canonical map* $\varphi\colon (\mathscr{V}(f))^{-1}\to\big(F(\mathscr{V}(f))\big)^{-1}$.

Proofs. For Theorem IV.2.2, we simply use the natural transformation φ mentioned above after first applying $\mathscr{V}$.

For Theorem IV.2.1, we define morphisms

$$\psi_f = (m, \psi_{\alpha\beta})_{(\alpha,\beta)\in\mathrm{Cov}(f)}\colon \mathscr{V}(f^{-1})\to(\mathscr{V}(f))^{-1}$$

and

$$\Theta_f = (n, \varphi_\gamma)_{\gamma\in\mathrm{Cov}(f^{-1}(y))}\colon (\mathscr{V}(f))^{-1}\to\mathscr{V}(f^{-1})\,.$$

The maps $m\colon \mathrm{Cov}(f)\to\mathrm{Cov}(f^{-1}(y))$ is defined by $m(\alpha, \beta) = (j^{-1}(\beta))$ where $j\colon f^{-1}(y)\to X$ is inclusion. The map $n\colon \mathrm{Cov}(f^{-1}(y))\to\mathrm{Cov}(f)$ is given by $n(\gamma) = (\{Y\}, \{X-f^{-1}(y)\}\cup\beta_\gamma)$ where for each $U\in\gamma$, U^* is some open set in X with $U^*\cap f^{-1}(y) = U$ and $\beta_\gamma = \{U^*|\ U\in\gamma\}$.

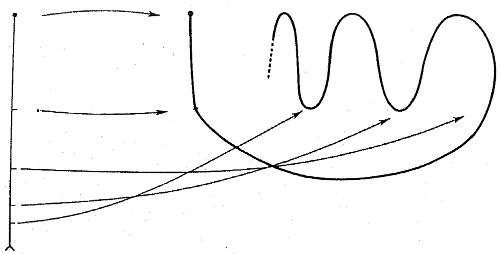
It is important to note that for any $(\alpha, \beta)\in\mathrm{Cov}(f)$ and any $\gamma\in\mathrm{Cov}(f^{-1}(y))$, the set of vertices for each of the simplicial sets $f_{\alpha\beta}^{-1}(y)$ and $(f^{-1}(y);\gamma)$ is just the set of points in $f^{-1}(y)$. Also, the bonding maps in the inverse systems $\{f_{\alpha\beta}^{-1}(y), x\}_{\alpha\beta\in\mathrm{Cov}(f)}$ and $\{(f^{-1}(y);\gamma)x)\}_{\gamma\in\mathrm{Cov}(f^{-1}(y))}$ are defined by the identities on vertices.

We define $\Theta_\gamma(w) = w$ and $\psi_{\alpha\beta}(w) = w$ for each vertex w. It is easy to check that these extend to the simplicial sets involved since $f_{\alpha\beta}^{-1}(y)$ and $(f^{-1}(y); j^{-1}(\beta))$ have the same simplexes and $(f^{-1}(y); y)$ and $f^{-1}_{n(y)}(y)$ have the same simplexes. Furthermore, the nature of the bonding maps makes it easy to verify that each composition $\psi \circ \Theta$ and $\Theta \circ \psi$ is equivalent to the appropriate identity morphism.

Remark. Analogous constructions can be carried out with the Čech functor $\mathscr{C}$ replacing $\mathscr{V}$ and the $\mathscr{F}$ induced functor from $(\mathscr{K}_0)_{\text{maps}} \to (\mathscr{K}_0)_{\text{maps}}$ replacing $\mathscr{F}$. Furthermore, it will follow from diagram (3) and Proposition III.2.1, that the resulting functors are equivalent in pro-$\mathscr{K}_0$ to those given here using $\mathscr{V}$.

EXAMPLE IV.2.1 *The fibers* $(\mathscr{V}(f))^{-1}$ *and* $\big(F(\mathscr{V}(f))\big)^{-1}$ *are not necessarily isomorphic in* pro-K_0, *even when* f *is assumed to be a Hurewicz fibration.*

Let f: $[0, 1) \to S_W$ be the Hurewicz fibration pictured below: Since f is 1-1,



$(\mathscr{V}(f))^{-1} = \mathscr{V}(f^{-1})$ is the trivial element of pro-K_0. However, $\mathscr{V}(f)$ has a cofinal subsystem which is equivalent in pro-$(K_0)_{\text{maps}}$ to that system in which each term is the map g: $[0, 1) \to S_1$ defined by $g(t) = e^{2\pi it}$ and the bonding maps are the identities. Since $\mathscr{F}(g)$ is the universal covering of S_1 by the real line, we see that $\big(\mathscr{F}(\mathscr{V}(f))\big)^{-1}$ is isomorphic to Z.

Now, the functor pro-$f \circ \mathscr{V}$ maps $(\text{TOP}_0)_{\text{maps}}$ into FIB, the full subcategory of pro-$(\mathscr{S}_0)_{\text{maps}}$ whose objects are Kan fibrations. We compose with the usual long exact sequence functor from FIB to pro-LES$(\mathscr{G})$ and λ from pro-LES$(\mathscr{G})$ to LES(pro-$\mathscr{G}$). Thus, we obtain a second functor from $(\text{TOP}_0)_{\text{maps}}$ into LES(pro-$\mathscr{G}$).

THEOREM IV.2.3. *There is a functor from* $(\text{TOP}_0)_{\text{maps}}$ *to* LES(pro-$\mathscr{G}$) *which associates to the map* f: $(X, x) \to (Y, y)$ *the special long exact sequence of* pro-groups

$$\ldots \to \text{pro-}\pi_i\Big(\big(\mathscr{F}(\mathscr{V}(f))\big)^{-1}\Big) \to \text{pro-}\pi_i(X, x) \xrightarrow{f_*} \text{pro-}\pi_i(Y, y) \to \ldots$$

where the unmarked maps are appropriately induced from those of the exact sequence for the fibrations in the system $\mathscr{F}(\mathscr{V}(f))$.

We shall say that f: $X \to Y$ is a *shape-quasi-fibration* if and only if, for each $x \in X$ and $f_x = f$: $(X, x) \to (Y, f(x))$, it holds that the canonical map

$$\Lambda_f^x = \varphi_{f_x} \circ \Theta_{f_x}\colon \mathscr{V}(f_x^{-1}) \to \big(\mathscr{F}(\mathscr{V}(f_x))\big)^{-1}$$

is an isomorphism. That is, Λ_f^x induces isomorphisms on the pro-homotopy-groups. Then, we have the following trivial corollaries:

COROLLARY IV.2.1. *If* f: $(X, x) \to (Y, y)$ *is a shape-quasi-fibration, then there is a special long exact sequence of pro-homotopy-groups*

$$\ldots \to \text{pro-}\pi_i(f^{-1}(y), x) \xrightarrow{i_*} \text{pro-}\pi_i(X, x) \xrightarrow{f_*} \text{pro-}\pi_i(Y, y) \xrightarrow{\Delta_*} \ldots$$

where Δ_* *is appropriately induced by* $(\Lambda_f^x)_*$ *and* i *denotes inclusion.*

COROLLARY IV.2.2. *If* f: $(X, x) \to (Y, y)$ *is a movable shape-quasi-fibration, then there is a long exact limit sequence*

$$\ldots \to \check{\pi}_i(f^{-1}(y), x) \to \check{\pi}_i(X, x) \to \check{\pi}_i(Y, y) \to \ldots,$$

obtained by taking the limit of the sequence of Corollary IV.2.1.

Thus, it remains to identify those fibrations (or maps) f for which Λ_f^x is an isomorphism. We have noted in Example IV.2.1, that Λ_f^x is not necessarily an isomorphism in pro-$\mathscr{K}_0$ even when f is a fibration. This is not surprising since in that case the maps in the system $\mathscr{C}(f)$ fail to be fibrations. We might say such an f is *not a shape-fibration.*

We conclude by raising the following natural questions for various sorts of fibration:

(1) Which fibrations are shape-fibrations?

(2) Which fibrations are shape-quasi-fibrations?

A version of (1) has been partially answered by Scharlemann [22]. Generalizing work of Fox [11], he proves an extension theorem which shows that certain fiber bundles with ANR structure group are shape-fiber bundles.

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Boolean matrices, subalgebras and automorphisms of complete Boolean algebras

by

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Abstract. We shall prove several theorems on complete Boolean algebras motivated by the theory of Boolean valued models. Section 1 deals with Boolean matrices that correspond to collapsing mappings in Boolean extensions of the universe. An analogue of Cantor–Bernstein theorem for Boolean matrices is proved. The notion of maximal subalgebras is introduced in Section 2. It is shown that a complete Boolean algebra is rigid iff it does not contain any proper maximal subalgebra. The last Section is devoted to the problem of existence of rigid (non-complete) Boolean algebras of power $\aleph_1$. It is shown that such algebras can exist independently on the Continuum hypothesis (CH). Namely, the statement "$\neg$CH+there is a rigid Boolean algebra of power $\aleph_1$ the completion of which is rigid as well" is consistent relatively to ZFC. Only the proof of Lemma 3 Section 3 makes use of Boolean valued models explicitly, the other proofs are algebraical.

§ 0. **Preliminaries.** Standard set theoretical notation and terminology is used through the paper. Ordinal numbers are denoted $\alpha, \beta, \gamma, \ldots$ and an ordinal coincides with the set of all smaller ordinals. Infinite cardinals are denoted by $\varkappa, \lambda, \ldots$ and are identified with initial ordinals. The cardinality of a set x is denoted by $|x|$. A Boolean algebra $\boldsymbol{b}$ is the structure $\langle b, \vee, \wedge, -, 0, 1\rangle$ satisfying the usual axioms. We use bold face letters to distinguish Boolean algebras from their universes. Every Boolean algebra is partially ordered by $\leqslant$ and 1 is the greatest and 0 the least element. It should be noted that the operations are definable in terms of $\leqslant$ and vice versa. We say that $\boldsymbol{b}$ is a complete Boolean algebra if the operations $\vee$ and $\wedge$ corresponding to supremum and infimum with respect to $\leqslant$ can be extended to any subset of b. As customary, these infinite operations are then denoted by $\bigvee, \bigwedge$. For any Boolean algebra $\boldsymbol{b}$, let $\mathrm{Sp}\,\boldsymbol{b}$ denote the *set of all subsets* a of b such that $\sup a$ exists. Thus $\boldsymbol{b}$ is a *complete Boolean algebra* iff $\mathrm{Sp}\,\boldsymbol{b} = P(b)$ (the power set of b).

Let $\boldsymbol{b}$ be a complete Boolean algebra. We say that $\boldsymbol{b}_1$ is a (*complete*) *subalgebra* of $\boldsymbol{b}$ if $\boldsymbol{b}_1$ is closed under infinite operations and under —. For any $u \in b$ define $\pi_{b_1}(u)$ as follows

$$\pi_{b_1}(u) = \bigwedge \{v \in b_1;\, v \geqslant u\}\,.$$

For any $u \in b$, let $\boldsymbol{b}|u$ denote the partial algebra with the universe $b|u = \{v \in b;\, v \leqslant u\}$ and operations $-_u v = u - v$, $\bigvee_u a = u \wedge \bigvee a$, $\bigwedge_u a = \bigwedge a$ for any $v \in b|u$ and $a \subseteq b|u$. Clearly, $\boldsymbol{b}|u$ is a complete Boolean algebra.