Baire category in spaces of probability measures

by

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Abstract. Only separable metric spaces X are considered here, so that the space $M(X)$ of probability measures on X (endowed with the weak-* topology) is separable metric. Let TC, PC, and BC be abbreviations for "topologically complete", "pseudo-complete" (i.e. contains a dense TC subspace), and "Baire complete", respectively. It is well known that (X is TC)$\Rightarrow$(X is PC) $\Rightarrow$(X is BC) and that the implications are irreversible. Prohorov [7] proved that (X is TC) $\Leftrightarrow$($M(X)$ is TC). It is the purpose of this note to show that (X is PC)$\Rightarrow$($M(X)$ is PC)$\Rightarrow$($M(X)$ is BC) $\Rightarrow$(X is BC) and that the implications are irreversible. The Continuum Hypothesis is assumed where needed.

The notation, definitions and theorems in Parthasarathy's book [6] will be assumed here. Only metric spaces X will be considered. If d is a metric for X, X^d will denote the d-completion of X. $B(X)$ denotes the Borel σ-field generated by the open subsets of X. Let X be a subspace of Y. For each $\mu \in M(X)$, let μ^Y denote the element of $M(Y)$ such that $\mu^Y(E) = \mu(E \cap X)$ for each $E \in B(Y)$. Then, if $\upsilon \in M(Y)$, then υ will be said to have *restriction* to X if there is a $\mu \in M(X)$ (necessarily unique) such that $\upsilon = \mu^Y$ (this will happen if and only if $\upsilon^*(X) = 1$). $M(X)$ may be considered to be topologically imbedded in $M(Y)$ as the set $M_1 = \{\mu^Y | \ \mu \in M(X)\}$. Properties TC, PC [5], and BC have been extensively investigated in [1].

THEOREM 1. *Let X be separable metric. If X is* PC, *then $M(X)$ is* PC.

Proof. Let X_1 be a dense G_δ subset of X such that X_1 is a G_δ subset of X^d. Let e be the restriction of d to $X_1 \times X_1$. X_1^e and X^d are isometric. Let $E \subseteq X_1$ be a countable dense subset of X. Let $M_1 = \{\mu \in M(X) | \ \mu(X_1) = 1\}$. M_1 is dense in $M(X)$ because the set of measures with finite support from E is dense in $M(X)$([6], p. 44). M_1 is also topologically equivalent to $\{\mu \in M(X_1^e) | \ \mu(X_1) = 1\}$, which is a G_δ in the complete space $M(X_1^e)$ (see the proof of Theorem 6.5, [6], p. 46). Therefore M_1 is topologically complete, and $M(X)$ is PC.

THEOREM 2. *There exists a subspace X of the reals such that $M(X)$ is* PC *but X is not* PC.

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Proof. Let X be a subset of $Y = [0, 1]$ such that X and $Y-X$ both intersect every Cantor subset of Y ([3], p. 514). Assume the metric is the relative Euclidean metric so that Y is isometric to X^d. X is not PC because $Y-X$ is second category in Y. Let $M_1 = \{\mu \in M(X) | \ \mu \text{ is non-atomic}\}$ and $M_2 = \{\mu \in M(Y) | \ \mu \text{ is non-atomic}\}$. Since X intersects every Cantor set in Y, it follows that $\mu^*(X) = 1$ for every $\mu \in M_2$. So associating each element of M_2 with its unique restriction to X will show that M_1 and M_2 are homeomorphic. But M_2 is a dense G_δ in the complete space $M(Y)$. Theorefore M_1 is a dense subset of $M(X)$ which is TC, so $M(X)$ is PC.

LEMMA 1. *Let X be separable metric. If $N \in B(X)$ is nowhere dense in X and $\varepsilon > 0$, then $L = \{\mu \in M(X) | \ \mu(N) > \varepsilon\}$ is nowhere dense in $M(X)$.*

Proof. Let Q be open in $M(X)$ and E be a countable subset of X-cl(N) which is dense in X. There exists $\mu_1 \in Q$ with support a finite subset $(x_1, x_2, \ldots, x_n)$ of E. For each $q = 1, 2, \ldots, n$, let O_q be a neighborhood of x_q which does not intersect N. Then the open set $Q \cap \{\mu \in M(X) | \ \mu(X - O_1 \cup O_2 \cup \ldots \cup O_n) < \varepsilon\}$, which contains μ_1, does not intersect L.

THEOREM 3. *Let X be separable metric. If $M(X)$ is BC, then X is BC.*

Proof. Let $\{G_n\}$ be a sequence of open, dense in X subsets of X. For each n, set $M_n = \{\mu \in M(X) | \ \mu(G_n) > 1 - (\frac{1}{2})^{n+1}\}$. Each M_n is open in $M(X)$ and (from Lemma 1) dense in $M(X)$. Since $M(X)$ is BC, $\bigcap_{i=1}^{\infty} M_i$ is dense in $M(X)$. For every $x \in X$ and open neighborhood N of x, there exists a sequence $\{\mu_i\}$ from $\bigcap_{i=1}^{\infty} M_i$ converging to p_x. For some n, $\mu_n(N) > \frac{1}{2}$. Since

$$\mu_n(G_1 \cap G_2 \cap \ldots) > \tfrac{1}{2}, \quad N \cap G_1 \cap G_2 \cap \ldots \neq \emptyset,$$

so X is BC.

THEOREM 4. *The Continuum Hypothesis implies the existence of a subspace X of the reals such that X is* BC *but $M(X)$ is not* BC.

Proof. N. Lusin showed that the Continuum Hypothesis implies the existence of an uncountable subset L of $[0, 1]$ each nowhere dense in $[0, 1]$ subset of which is countable ([3], p. 525). It follows that L is second category in the reals, so there is some interval Y in which L is uncountably dense. Assume $Y = [0, 1]$, and let $X = L$. Every nowhere dense in X subset of X is countable, and X is uncountably dense in itself, so X is BC. Every element of $M(X)$ has an atom ([3], p. 532), so every element of $M(X)$ is totally atomic. Thus, $M(X)$ is homeomorphic to $M_1 = \{\mu \in M(Y) | \ \mu(C) = 1$ for some countable $C \subset X\}$. Let $M_2 = \{\mu \in M(Y) | \ \mu \text{ is non-atomic}\}$. Since X is dense in Y, M_1 is dense in $M(Y)$ ([6], p. 44). M_2 is a dense G_δ subset of $M(Y)$, and $M_1 \subseteq M(Y) - M_2$. It follows that M_1 is not BC when considered as space (in fact, every open in M_1 set is first category in M_1). Thus, $M(X)$ is not BC.

Remark 1. Theorem 4 was proved by utilizing a space X which is dense in itself and BC (therefore uncountably dense in itself) and also a so-called *β space* [8] (i.e. every element of $M(X)$ is totally atomic). It is natural to ask if the existence of such a space can be established without the aid of the Continuum Hypothesis. The existence of an *uncountable β*-space has been so established [2, 8], but it can be shown that the technique used could not possibly yield a space which is also BC.

Theorems 1, 2, and 3 still leave open the possibility that, within the context of spaces $M(X)$ of probability measures on metric spaces X, properties PC and BC are equivalent. It is known that compactness and local compactness are equivalent within this setting. Indeed, Luther [4] showed that "... if X is *any* topological space, then $P_\sigma(X)$ is locally compact if and only if it is compact", where $P_\sigma(X)$ is the appropriate generalization to the topological setting of the notion of the "space of probability measures". The following lemma and Theorem 5 supply evidence that the analogous situation does not hold for properties PC and BC.

LEMMA 2. *Let $Y = [0, 1]$, C be F_σ first category in Y, Q be open in $M(Y)$, and L be F_σ first category in $M(Y)$. Then there exists a dense F_σ first category subset D of $Y-C$ such that $\{\mu \in M(Y) | \ \mu(D) = 1\}$ intersects $Q-L$.*

Proof. Let $C = \bigcup_{i=1}^{\infty} N_i$, where each N_i is closed nowhere dense. The set

$$\{\mu \in M(Y) | \ \mu(C) > 0\} = \bigcup_{i,j=1}^{\infty} \{\mu \in M(Y) | \ \mu(N_i) \geqslant 1/j\},$$

and each set in the union on the right is closed nowhere dense in $M(Y)$ (see argument for Theorem 3). Therefore, $L' = L \cup \{\mu \in M(Y) | \ \mu(C) > 0\}$ is first category in $M(Y)$. The set of all non-atomic $\mu \in M(Y)$ is residual in $M(Y)$, so there is a non-atomic measure υ in $Q - L'$. Then, $\upsilon(C) = 0$, so since υ is regular, there exists a sequence $\{D_n\}$ of closed subsets of $Y-C$ such that for each n, $\upsilon(D_n) > 1 - 1/n$ and D_n is nowhere dense. Let E be a countable dense subset of $Y-C$. $D = E \cup D_1 \cup D_2 \cup \ldots$ is the desired set.

THEOREM 5. *The Continuum Hypothesis implies the existence of a subspace X of $Y = [0, 1]$ such that $M(X)$ is* BC *but not* PC.

Proof. X will be constructed as the union of the sets in a transfinite sequence $\{A_\alpha\}$ of disjoint dense F_σ first category subsets of Y. Assume the Continuum Hypothesis is true and let $\{C_\alpha\}$, $\{Q_\alpha\}$, and $\{L_\alpha\}$ be well ordered sequences consisting of the dense F_σ first category subsets of Y, the open subsets of $M(Y)$, and the F_σ first category subsets of $M(Y)$, respectively, each transfinite sequence indexed by the countable ordinals. For convenience, set $A_0 = B_0 = C_0 = L_0 =$ the empty set, and proceed inductively as follows. Assume α is a countable ordinal such that A_β and B_β have been defined for every $\beta < \alpha$. (1) Set $C_A =$ the union of all C_γ which precede (in the sequence $\{C_\delta\}$) or equal B_β for some $\beta < \alpha$. Using Lemma 2 with $C = C_A$, $Q = Q_\alpha$, and $L = \bigcup_{\beta \leqslant \alpha} L_\beta$, let A_α be the first set D (in the sequence $\{C_\delta\}$) satisfying the conclusion of the Lemma. (2) Set $C_B =$ the union of all C_γ which precede or equal A_α. Using Lemma 2 with $C = C_B$, $Q = Q_\alpha$, and $L = \bigcup_{\beta \leqslant \alpha} L_\beta$, let B_α be the first set D satisfying the conclusion of the lemma. This process can be completed for each countable ordinal, because the union of countably many F_σ first category sets (e.g. $C_A, C_B, \bigcup_{\beta\leqslant\alpha} L_\beta$) is still F_σ first category. Now, let $X = \bigcup_\alpha A_\alpha$, and $X' = \bigcup_\alpha B_\alpha$, both with the relative Euclidean topology.

$M(X)$ can be considered to be imbedded in $M(Y)$ as the set

$$M_1 = \{\mu^Y |\ \mu \in M(X)\},$$

so it will be shown that M_1, considered as space, is BC but not PC.

M_1 is dense in $M(Y)$, and $M(Y)$ is TC, so there is a metrization of $M(Y)$ (and M_1, relatively) such that $M(Y)$ is the completion of M_1. Now, suppose there is an open set Q in $M(Y)$ such that $M_1 \cap Q$ is first category in $M(Y)$. Then, $M_1 \cap Q$ is a subset of some L_β. There is an $\alpha > \beta$ such that $Q_\alpha \subseteq Q$. Now, A_α was chosen so that there is some $\mu \in Q_\alpha - \bigcup_{\gamma\leqslant\alpha} L_\gamma$ such that $\mu(A_\alpha) = 1$. Since $A_\alpha \in B(X)$ and $A_\alpha \in B(Y)$, μ is in $M_1 \cap Q$, and this is a contradiction. Therefore, M_1 is dense and second category in every open subset of $M(Y)$. It follows that M_1 as space (therefore $M(X)$) is BC.

$M(X')$ can be considered to be imbedded in $M(Y)$ as the set

$$M_2 = \{\mu^Y |\ \mu \in M(X')\},$$

and it can similarly be shown that M_2 is dense and second category in every open subset of $M(Y)$. But M_1 and M_2 are disjoint, for suppose there exist $\mu \in M(X)$ and $\upsilon \in M(X')$ such that $\mu^Y = \upsilon^Y$. There will exist some countable ordinal α such that $\mu^Y(C_\alpha) = \upsilon^Y(C_\alpha) = 1$. Note that for every $\beta > \alpha$, C_α intersects neither A_β nor B_β, so

$$\mu\Big(\bigcup_{\beta\leqslant\alpha} A_\beta\Big) = \upsilon\Big(\bigcup_{\beta\leqslant\alpha} B_\beta\Big) = 1.$$

But $\bigcup_{\beta\leqslant\alpha} A_\beta$ and $\bigcup_{\beta\leqslant\alpha} B_\beta$ are in $B(Y)$ and disjoint, so

$$\mu^Y\Big(\bigcup_{\beta\leqslant\alpha} A_\beta\Big) = \upsilon^Y\Big(\bigcup_{\beta\leqslant\alpha} B_\beta\Big) = 1;$$

and

$$\mu^Y\Big[\bigcup_{\beta\leqslant\alpha} (A_\beta \cup B_\beta)\Big] = 2,$$

which is a contradiction. It follows that M_1 and $M(Y) - M_1$ are both second category in $M(Y)$, so that M_1 (therefore $M(X)$) cannot be PC.

Remark 2. There should exist completeness conditions C_1 and C_2 *for* X such that $M(X)$ is PC if and only if X is C_1 and $M(X)$ is BC if and only if X is C_2. The author is at present unable to supply intrinsic topological characterizations of C_1 and C_2.

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