The Chinese remainder theorem and sheaf representations

by

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Abstract. Let $\mathfrak{A}$ be a universal algebra whose congruence lattice contains a distributive upper regular sublattice $\mathscr{L}$ which is compactly generated, consists of permuting congruences and such that the intersection of two compact members of $\mathscr{L}$ is compact. Then $\mathfrak{A}$ is representable as all the global sections in a sheaf of algebras in which the base space is the set of prime members of $\mathscr{L}$ endowed with the dual hull-kernel topology. This representation includes Comer's in which Boolean lattices of factor congruences are employed and is particularly applicable to algebras which are "modules" over semilattice-ordered semigroups and distributive lattices.

1. Introduction. Basically this paper is concerned with universal algebras which have a bounded distributive lattice associated with them in such a way that the filters of this lattice give rise to a sublattice of the lattice of congruences of the algebra for which the Chinese remainder theorem holds, and thus enables the algebra to be represented in a natural way as the algebra of all global sections in a certain sheaf. Thus, in Section 2, we consider the general case which was described in the accompanying abstract. After discussing some examples, we move onto Section 3, wherein we consider algebras which are acted on by a commutative semilattice-ordered semigroup. We then consider special cases of this situation in the remaining two sections.

In general, our notation and terminology for universal algebra and lattice theory follows that of [8] and [9]. For background on sheaf representations of universal algebras, we refer to [3].

2. The general sheaf representation. Let $\mathfrak{A}$ be a non-trivial universal algebra with congruence lattice $C(\mathfrak{A})$. Let $\mathscr{L}$ be a subset of $C(\mathfrak{A})$ such that the following conditions hold.

(2.1) *The bounds ω and ι of $C(\mathfrak{A})$ are in $\mathscr{L}$.*

(2.2) *For any $\Theta, \Phi \in \mathscr{L}$, $\Theta \cap \Phi \in \mathscr{L}$.*

(2.3) *$\mathscr{L}$ is upper regular in $C(\mathfrak{A})$. That is, for any $\{\Theta_i\} \subset \mathscr{L}$, $\bigvee_{\mathscr{L}} \Theta_i$ exists in $\mathscr{L}$ and is equal to $\bigvee_{C(\mathfrak{A})} \Theta_i$.*

(2.4) *The lattice $\mathscr{L}$ is distributive.*

(2.5) *$\mathscr{L}$ consists of permuting congruences.*

(2.6) *The lattice $\mathscr{L}$ is compactly generated in its own right.*

(2.7) *The intersection of two compact members of $\mathscr{L}$ is compact.*

A member π of $\mathscr{L}$ is called *prime* if, for any $\Theta, \Phi \in \mathscr{L}$, $\Theta \cap \Phi \subseteq \pi$ implies $\Theta \subseteq \pi$ or $\Phi \subseteq \pi$.

Before continuing, let us consider some of the connections and implications of these conditions on $\mathscr{L}$.

Of course, (2.1), (2.2) and (2.3) ensure that $\mathscr{L}$ is a complete lattice. However, we emphasize that we are not assuming $\mathscr{L}$ to be a lower regular sublattice of $C(\mathfrak{A})$, i.e. $\mathscr{L}$ need not be closed under arbitrary set-theoretic intersections. Conditions (2.3) and (2.7) ensure that the compact members of $\mathscr{L}$ form a sublattice of $C(\mathfrak{A})$. It is not hard to see that (2.6), (2.7), (2.3) together with (2.4) and (2.5), postulated only for the lattice of compact members of $\mathscr{L}$, ensure (2.2), (2.4) and (2.5) for $\mathscr{L}$. In view of (2.4), the prime members of $\mathscr{L}$ are nothing more than its meet-irreducible elements. The following proposition sheds further light on (2.2); we will need part of it subsequently.

PROPOSITION 2.1. *Suppose $\mathscr{L}$ satisfies* (2.1), (2.3) *and* (2.4) *and yet the infimum $\Theta \wedge \Phi$ of two members Θ and Φ of $\mathscr{L}$ may not be their set-theoretic intersection. Then the following are equivalent.*

(i) *For each $\Theta, \Phi \in \mathscr{L}$, $\Theta \wedge \Phi = \Theta \cap \Phi$.*

(ii) *For each $\Theta \in \mathscr{L}$, $\Theta = \bigcap \{\pi \in \mathscr{L}\colon \pi$ is prime; $\Theta \subseteq \pi\}$.*

Proof. (i)$\Rightarrow$(ii). Suppose $x, y \in A$ and $x \not\equiv y(\Theta)$. Then,

$$\mathscr{K} = \{\Phi \in \mathscr{L}\colon \Theta \subseteq \Phi,\ x \not\equiv y(\Phi)\}$$

is non-empty. Order $\mathscr{K}$ by set inclusion and let $\mathscr{H}$ be a chain in $\mathscr{K}$. Then,

$$\Psi = \bigcup \{\Phi \in \mathscr{K}\colon \Phi \in \mathscr{H}\}$$

is in $\mathscr{L}$ by (2.3), and clearly it is the least upper bound of $\mathscr{H}$ in $\mathscr{K}$. By Zorn's lemma, there exist maximal elements in $\mathscr{K}$. Let π be such an element. As $x \not\equiv y(\Theta)$, $\pi \neq \iota$. Let $\Theta', \Phi \in \mathscr{L}$ be such that $\Theta' \not\subseteq \pi$ and $\Phi \not\subseteq \pi$. Then, $x \equiv y((\pi \vee \Theta') \cap (\pi \vee \Phi))$. By (i) and the other conditions, we conclude that $x \equiv y(\pi \vee (\Theta' \cap \Phi))$ and consequently $\Theta' \cap \Phi \not\subseteq \pi$. Thus, π is prime. This and (2.1) immediately imply (ii), as required.

(ii)$\Rightarrow$(i) Of course $\Theta \cap \Phi$ is contained in $\Theta \wedge \Phi$. Assume that (ii) holds. Using (ii) it easily follows that $\Theta \wedge \Phi \subset \Theta$, Φ and hence $\Theta \cap \Phi = \Theta \wedge \Phi$.

LEMMA 2.2. *A sublattice $\mathscr{L}$ of $C(\mathfrak{A})$ satisfies* (2.4) *and* (2.5) *if and only if it satisfies the Chinese remainder theorem, i.e., for any $x_1, x_2, \ldots, x_n \in A$ and any $\Theta_1, \Theta_2, \ldots, \Theta_n \in \mathscr{L}$, $x_i \equiv x_j(\Theta_i \vee \Theta_j)$ for all $i, j = 1, 2, \ldots, n$ implies that there exists $x \in A$ such that $x \equiv x_j(\Theta_j)$ for each $j = 1, 2, \ldots, n$.*

Proof. This is a generalization of [8], Chapter 5, Exercice 68 and has already been explicitly observed in [7], Corollary 3.3.

Let $\mathscr{K}$ be the distributive lattice of compact elements in $\mathscr{L}$. Of course (2.6) ensures that $\mathscr{L}$ is isomorphic to the lattice of ideals of $\mathscr{K}$. Note that we have not assumed ι to be in $\mathscr{K}$.

Let $\mathscr{P}(\mathscr{L}, \mathfrak{A})$ be the set of prime elements in $\mathscr{L}$. Endow it with the dual hull-kernel topology. That is, declare the sets $h(K) = \{\pi \in \mathscr{P}(\mathscr{L}, \mathfrak{A})\colon K \subseteq \pi\}$ $(K \in \mathscr{K})$ to be open. Since $h(K_1 \vee K_2) = h(K_1) \cap h(K_2)$ and $\mathscr{P}(\mathscr{L}, \mathfrak{A}) = \bigcup \{h(K)\colon K \in \mathscr{K}\}$, these "hulls" are a base for the open sets.

LEMMA 2.3. *The space $\mathscr{P}(\mathscr{L}, \mathfrak{A})$ is compact.*

Proof. The set of prime ideals of any distributive lattice with 0, endowed with the dual hull-kernel topology, is homeomorphic in a natural way to the set of prime filters endowed with the hull-kernel topology. Hence, the lemma follows from the dual of [9], Lemma 4, p. 119.

Let $\mathscr{S}(\mathscr{L}, \mathfrak{A}) = \dot{\bigcup} \{\mathfrak{A}/\pi\colon \pi \in \mathscr{P}(\mathscr{L}, \mathfrak{A})\}$, and for any $x \in A$, let $\hat{x}$ be the Gelfand transform of x, i.e. $\hat{x}$ is the function $\hat{x}\colon \mathscr{P}(\mathscr{L}, \mathfrak{A}) \to \mathscr{S}(\mathscr{L}, \mathfrak{A})$, defined by $\hat{x}(\pi) = [x]\pi$, for each $\pi \in \mathscr{P}(\mathscr{L}, \mathfrak{A})$.

LEMMA 2.4. *For any $x, y \in A$, the set $\mathfrak{E}(x, y) = \{\pi \in \mathscr{P}(\mathscr{L}, \mathfrak{A})\colon \hat{x}(\pi) = \hat{y}(\pi)\}$ is open in $\mathscr{P}(\mathscr{L}, \mathfrak{A})$.*

Proof. Let $\Xi \in \mathfrak{E}(x, y)$. By (2.3) and (2.6), $\Theta(x, y) \subseteq \Xi = \{K \in \mathscr{K}\colon K \subseteq \Xi\}$. Hence, $\Theta(x, y) \subseteq H$ for some $H \in \mathscr{K}$ such that $H \subseteq \Xi$. Thus, if $\Sigma \in \mathscr{P}(\mathscr{L}, \mathfrak{A})$ is such that $\Sigma \in h(H)$ then $\Theta(x, y) \subseteq \Sigma$, i.e. $\hat{x}(\Sigma) = \hat{y}(\Sigma)$. Hence, $\Xi \in h(H) \subseteq \mathfrak{E}(x, y)$, and $\mathfrak{E}(x, y)$ is open.

From general considerations ([3], Lemma 3.3) Lemma 2.4 implies that if $\mathscr{S}(\mathscr{L}, \mathfrak{A})$ is endowed with the finest topology making the Gelfand transforms continuous (i.e. $\mathscr{S}(\mathscr{L}, \mathfrak{A})$ is given the topology whose base for the opens is $\{\hat{x}(h(K))\colon x \in A$ and $K \in \mathscr{K}\}$ then $(\mathscr{S}(\mathscr{L}, \mathfrak{A}), \mathscr{P}(\mathscr{L}, \mathfrak{A}))$ is a sheaf of algebras of the same type as that of $\mathfrak{A}$. Let $\Gamma(\mathscr{S}(\mathscr{L}, \mathfrak{A}), \mathscr{P}(\mathscr{L}, \mathfrak{A}))$ denote the algebra of all global section of this sheaf.

THEOREM 2.5. *The map $x \mapsto \hat{x}$ is an isomorphism of $\mathfrak{A}$ onto the algebra*

$$\Gamma(\mathscr{S}(\mathscr{L}, \mathfrak{A}), \mathscr{P}(\mathscr{L}, \mathfrak{A})).$$

Proof. By Proposition 2.1, $\omega = \bigcap \{\pi \in \mathscr{L}\colon \pi \in \mathscr{P}(\mathscr{L}, \mathfrak{A})\}$. Hence the map is an embedding of $\mathfrak{A}$ onto the algebra of global sections. Let σ be any global section. Because of the compactness of $\mathscr{P}(\mathscr{L}, \mathfrak{A})$ and the fact that global sections agree on an open neighbourhood of a point where they agree, there exist $x_1, x_2, \ldots, x_n \in A$ and $K_1, K_2, \ldots, K_n \in \mathscr{K}$ such that $\sigma(\pi) = \hat{x}_i(\pi)$ for all $\pi \in h(K_i)$, when $i = 1, 2, \ldots, n$. Then, $\hat{x}_i(\pi) = \hat{x}_j(\pi)$ for all $\pi \in h(K_i) \cap h(K_j) = h(K_i \vee K_j)$, whenever $i, j = 1, 2, \ldots, n$. By Proposition 2.1 once more, $x_i \equiv x_j(K_i \vee K_j)$ for any $i, j = 1, 2, \ldots, n$. By Lemma 2.2, there exists $x \in A$ such that $x \equiv x_j(K_j)$ for $j = 1, 2, \ldots, n$. That is, $\hat{x}(\pi) = \hat{x}_j(\pi)$ for all $\pi \in h(K_j)$, whenever $j = 1, 2, \ldots, n$. Thus, $\sigma(\pi) = \hat{x}(\pi)$ for all $\pi \in \mathscr{P}(\mathscr{L}, \mathfrak{A})$, and the proof is completed.

EXAMPLE 2.6. Suppose algebra $\mathfrak{A}$ is such that its congruences permute and form a distributive lattice in which the intersection of two compact congruences is compact.

Then, we may take $\mathscr{L}$ to be $C(\mathfrak{A})$ to obtain a sheaf representation for $\mathfrak{A}$. Of course, this applies to Boolean rings. Due to a recent result of Shores and Wiegand [13], this last instance is capable of immense generalization. Indeed, in [13], Corollary 1.11, it is established that if the lattice of ideals of a commutative ring with identity is distributive (i.e. we are dealing with the so-called arithmetical rings) then the intersection of two finitely generated ideals is finitely generated. Since ideals and congruences of a ring may be identified, and in doing so finitely generated ideals correspond to compact congruences, we have an example of our situation.

EXAMPLE 2.7. In [3], Comer generalized R. S. Pierce's sheaf representation to rings using central idempotents to a representation of a wide class of universal algebras by using a well-behaved Boolean sublattice of all factor congruences. Our situation can be applied to obtain Comer's theorem ([3] Theorem 3.7). For, Comer's condition I and the equivalent of condition II given by [3], Proposition 2.3 implies that the ideals of his Boolean lattice of factor congruences satisfies (2.1) through to (2.7) inclusive.

3. Semilattice-ordered semigroup modules. By a *semilattice-ordered semigroup* S we mean an algebra $(S; \vee, \cdot)$ such that $(S; \cdot)$ is a *commutative* semigroup, $(S; \vee)$ is a semilattice and the equation $x(y\vee z) = xy\vee xz$ is identically satisfied. A semilattice-ordered semigroup, or more briefly a $\vee$-semigroup, is *integral* if $(S; \cdot)$ has an identity which is also the largest element of $(S; \vee)$, while it is *negatively-ordered* if $xy\leqslant x$ for all $x, y\in S$ (here $x\leqslant y$ if and only if $x\vee y = y$). Of course, S is negatively-ordered if it is integral.

A *filter* F of a $\vee$-semigroup S is a (non-empty) subsemigroup of $(S; \cdot)$ such that $x\in S$, $y\in F$ and $y\leqslant x$ imply $x\in F$. Ordered by set-inclusion the filters on S form a lattice $\mathscr{F}(S)$, where the infimum is set-theoretic intersection and the join of a collection $\{F_i\}$ of filters is given by

$$\vee F_i = \{a\in S\colon\ a\geqslant f_1 f_2 \dots f_n,\ f_j\in F_j \text{ and } F_j\in\{F_i\} \text{ for } i = 1, 2, \dots, n\}.$$

Note also that $F_1\cap F_2 = \{f_1\vee f_2\colon f_1\in F_1 \text{ and } f_2\in F_2\}$ for any $F_1, F_2\in\mathscr{F}(S)$. For $x\in S$, the smallest filter containing x is given by

$$[x) = \{y\in S\colon\ x^k\leqslant y \text{ for some } k\geqslant 1\}.$$

PROPOSITION 3.1. *If S is a negatively ordered $\vee$-semigroup then the lattice $\mathscr{F}(S)$ of its filters is distributive.*

Proof. Let $x\in F_1\cap(F_2\vee F_3)$, where $F_i\in\mathscr{F}(S)$ for $i = 1, 2, 3$. Then $x\in F_1$ and $f_2 f_3\leqslant x$ for suitable $f_2\in F_2$, $f_3\in F_3$. Now

$$(x\vee f_2)(x\vee f_3) = x^2\vee x f_2\vee x f_3\vee f_2 f_3\leqslant x\vee f_2 f_3 = x,$$

since S is negatively ordered. In addition, $x\vee f_2\in F_1\cap F_2$ and $x\vee f_3\in F_1\cap F_3$. Hence, $x\in(F_1\cap F_2)\vee(F_1\cap F_3)$ and the proposition follows.

Since we are going to make use of $\mathscr{F}(S)$, Proposition 3.1 explains why *we shall henceforth assume that each $\vee$-semigroup is negatively ordered.*

An algebra $(M; \vee, f_\gamma)_{\gamma<\alpha}$ is a *module over a semilattice-ordered semigroup* S if the following conditions hold.

(3.1) $(M; \vee)$ is a semilattice.

(3.2) For each $a\in S$ and $m\in M$, there is a unique element, denoted by ma, in M.

(3.3) For each $a\in S$ and $m, n\in M$, $(m\vee n)a = ma\vee na$.

(3.4) For each $a, b\in S$ and $m\in M$, $m(a\vee b) = ma\vee mb$.

(3.5) For each $a, b\in S$ and $m\in M$, $m(ab) = (ma)b$.

(3.6) For each $a\in S$ and $m\in M$, $ma\vee m = m$, i.e. $ma\leqslant m$.

(3.7) Each $f_\gamma(\gamma<\alpha)$ is *admissible* in the sense that either f_γ is a nullary operation or if not then the following property holds:

if the arity of f_γ is k_γ and $m_1, m_2, \dots, m_{k_\gamma}\in M$, $n_1, n_2, \dots, n_{k_\gamma}\in M$ and $a\in S$ satisfy $m_i a\leqslant n_i$ and $n_i a\leqslant m_i$ for $i = 1, 2, \dots, k_\gamma$, then

$$f_\gamma(m_1, m_2, \dots, m_{k_\gamma})a\leqslant f_\gamma(n_1, n_2, \dots, n_{k_\gamma})$$

and

$$f_\gamma(n_1, n_2, \dots, n_{k_\gamma})a\leqslant f_\gamma(m_1, m_2, \dots, m_{k_\gamma}).$$

It will be convenient to regard a $\vee$-semigroup module M as an algebra $\mathscr{M}_S = (M; \vee, f_\gamma, g_\delta)_{\gamma<\alpha,\delta<\beta}$ of type $\langle 2; \dots, k_\gamma, \dots; 1, 1, \dots\rangle$, where k_γ is the arity of the admissible operation f_γ and g_δ is the unary operation on M induced by multiplication by $a_\delta\in S = \{a_\delta\colon \delta<\beta\}$. We emphasize that any sheaf representation of $\mathscr{M}_S = (M; \vee, f_\gamma, g_\delta)_{\gamma<\alpha,\delta<\beta}$ in a sheaf of modules of type $\langle 2; \dots, k_\gamma, \dots; 1, 1, \dots\rangle$ induces a representation of the reduct $(M; \vee, f_\gamma)_{\gamma<\alpha}$ in a sheaf of algebras of type $\langle 2; \dots, k_\gamma, \dots\rangle$. In examples, we often wish to consider these reducts and there will be no ambiguity if we omit to mention the type.

Let F be a filter in S. Define a binary relation on M by: $m\equiv n(\Theta(F))$ $(m, n\in M)$ if and only if $mf\leqslant n$ and $nf\leqslant m$ for some $f\in F$.

PROPOSITION 3.2. *The following statements hold.*

(i) *For each $F\in\mathscr{F}(S)$, $\Theta(F)$ is a congruence on the algebra $\mathscr{M}_S$.*

(ii) *For any two filters $F_1, F_2\in\mathscr{F}(S)$. $\Theta(F_1)$ permutes with $\Theta(F_2)$.*

(iii) *For any two filters $F_1, F_2\in\mathscr{F}(S)$, $\Theta(F_1\cap F_2) = \Theta(F_1)\cap\Theta(F_2)$.*

(iv) *For any set $\{F_i\}\subseteq\mathscr{F}(S)$, $\Theta(\vee F_i) = \vee\Theta(F_i)$.*

Proof. (i) $\Theta(F)$ is reflexive because of (3.6). It is symmetric by its definition and transitive due to (3.3) and (3.5). It has the substitution property for the operation $\vee$ because of (3.3). By (3.7), it has the substitution property for each f_γ — of course, (3.7) was postulated to ensure this. The substitution property holds for each g_δ due to (3.3) and (3.5), and hence $\Theta(F)$ is a congruence.

(ii) Suppose $m\equiv n(\Theta(F_1)\circ\Theta(F_2))$ for given $m, n\in M$. Then there is $p\in M$ such that $m\equiv p(\Theta(F_1))$ and $p\equiv n(\Theta(F_2))$. There are $f_1\in F_1$ and $f_2\in F_2$ such that $mf_1 \leqslant p$, $pf_1 \leqslant m$ and $pf_2 \leqslant n$, $nf_2 \leqslant p$. By (3.3) and (3.5), $mf_1f_2 \leqslant pf_2 \leqslant n$ and $nf_1f_2 \leqslant m$. Hence, $m = m \vee nf_1f_2$ and $n = n \vee mf_1f_2$. Let $w = mf_2 \vee nf_1$. Then,

$$wf_2 = mf_2^2 \vee nf_1f_2 \leqslant m \vee nf_1f_2 = m,$$

while

$$mf_2 = (m \vee nf_1f_2)f_2 = mf_2 \vee nf_1f_2^2 \leqslant mf_2 \vee nf_1 = w.$$

Similarly, $wf_1 \leqslant n$ and $nf_1 \leqslant w$. Thus, $m \equiv n(\Theta(F_2) \circ \Theta(F_1))$, and $\Theta(F_1)$ permutes with $\Theta(F_2)$.

(iii) Let $m, n \in M$ be such that $m \equiv n(\Theta(F_1) \cap \Theta(F_2))$, so that $mf_1 \leqslant n$, $nf_1 \leqslant m$, $mf_2 \leqslant n$, and $nf_2 \leqslant m$ for suitable $f_1 \in F_1$ and $f_2 \in F_2$. By (3.4), $m(f_1 \vee f_2) \leqslant n$ and $n(f_1 \vee f_2) \leqslant m$. Hence, $m \equiv n(\Theta(F_1 \cap F_2))$ and (iii) follows easily.

(iv) If $m \equiv n(\Theta(VF_i))$ then (3.4) implies that

$$mf_{i_1}f_{i_2} \ldots f_{i_k} \leqslant n \quad \text{and} \quad nf_{i_1}f_{i_2} \ldots f_{i_k} \leqslant m$$

for some $f_{i_r} \in F_{i_r} \in \{F_i\}$, for each $r = 1, 2, \ldots, k$. Now for any filter F and $m \in M$, $mf = m(\Theta(F))$ since $(mf)f \leqslant m$ and $mf \leqslant mf$. Hence,

$$m \equiv (\Theta(F_{i_1}))mf_{i_1} \equiv (\Theta(F_{i_2}))mf_{i_1}f_{i_2} \equiv \ldots \equiv mf_{i_1}f_{i_2} \ldots f_{i_k}(\Theta(F_{i_k})).$$

Thus $m \equiv mf_{i_1}f_{i_2} \ldots f_{i_k}(V\Theta(F_i))$. Similarly, $n \equiv nf_{i_1}f_{i_2} \ldots f_{i_k}(V\Theta(F_i))$. But,

$$(mf_{i_1}f_{i_2} \ldots f_{i_k})f_{i_1} \ldots f_{i_k} \leqslant nf_{i_1}f_{i_2} \ldots f_{i_k}$$

and

$$(nf_{i_1}f_{i_2} \ldots f_{i_k})f_{i_1} \ldots f_{i_k} \leqslant mf_{i_1}f_{i_2} \ldots f_{i_k}.$$

Hence,

$$mf_{i_1}f_{i_2} \ldots f_{i_k} \equiv nf_{i_1}f_{i_2} \ldots f_{i_k}(V\Theta(F_i))$$

and so $m \equiv n(V\Theta(F_i))$, and (iv) follows.

We say that S has a zero, 0, if $(S; \vee)$ has a smallest element 0. Since we are assuming that S is negatively ordered 0 is the zero of the semigroup $(S; \cdot)$.

We will not be able to get much further unless we ensure that the map $F \mapsto \Theta(F)$ of $\mathcal{F}(S)$ into $C(\mathcal{M}_S)$ is an embedding. We also need to ensure that $\Theta([a))$, $a \in S$, is a compact member of $\{\Theta(F): F \in \mathcal{F}(S)\}$. Of course, we will also require $\omega = \Theta(F)$ and $\iota = \Theta(G)$ for suitable filters F and G on S. Therefore, a module $\mathcal{M}_S$ is said to be *well-behaved* if the following conditions are satisfied.

(3.8) S has 0 and 1.

(3.9) $(M, \vee)$ has 0 and 1, and $(S, \vee)$ is a subsemilattice of $(M, \vee)$ such that the zeros and identities respectively coincide.

(3.10) The 0 and 1 of M are considered as nullary admissible operations in $\mathcal{M}_S$.

(3.11) $\mathcal{M}_S$ is unitary in the sense that $m1 = m$ for all $m \in M$.

Note that (3.11) implies (3.6). For, if $m \in M$ and $a \in S$ then $ma \vee m = ma \vee m1 = m(a \vee 1) = m$. It is clear that $\Theta([1)) = \omega$, $m0 = 0$ for all $m \in M$, and $\Theta([0)) = \iota$. In addition, (3.11) ensures that the map $F \mapsto \Theta(F)$ is one-to-one. Since

the compact members of $\mathcal{F}(S)$ are the joins of a finite number of principal filters and $[a) \cap [b) = [a \vee b)$ for any $a, b \in S$, Proposition 3.1 implies that the intersection of two compact members of $\mathcal{F}(S)$ is compact. These remarks and Proposition 3.2, ensure that $\mathcal{L}(\mathcal{M}_S) = \{\Theta(F): F \in \mathcal{F}(S)\}$ satisfies each of the conditions (2.1), (2.2), ..., (2.7).

A filter F is *prime* if $a \vee b \in F (a, b \in S)$ implies $a \in F$ or $b \in F$, and $F \neq S$. Of course, filter F is prime if and only if $\Theta(F)$ is prime in $\mathcal{L}(\mathcal{M}_S)$. Let $\mathcal{P}(S)$ be the set of prime filters of S endowed with the dual hull-kernel topology. Let

$$\mathcal{S}(\mathcal{M}_S) = \dot{\bigcup} \{\mathcal{M}_S/\Theta(F): F \in \mathcal{P}(S)\},$$

and endow it with the finest topology which makes the Gelfand transforms $\hat{m} (m \in M$ and $\hat{m}: \mathcal{P}(S) \mapsto \mathcal{P}(\mathcal{M}_S))$ continuous. Clearly, if we take $\mathcal{L} = \mathcal{L}(\mathcal{M}_S)$ and $\mathfrak{A} = \mathcal{M}_S$ we can identify the sheaf $(\mathcal{S}(\mathcal{L}, \mathfrak{A}), \mathcal{P}(\mathcal{L}, \mathfrak{A}))$ of Section 2, with the sheaf $(\mathcal{S}(\mathcal{M}_S), \mathcal{P}(S))$ and hence Theorem 2.5 implies.

THEOREM 3.3. *Let $\mathcal{M}_S$ be a well-behaved module over a $\vee$-semigroup S with 0 and 1. Then $\mathcal{M}_S$ is canonically isomorphic to $\Gamma(\mathcal{S}(\mathcal{M}_S), \mathcal{P}(S))$.*

There is a slightly different version of Theorem 3.3 which we will now construct. A non-empty subset J of $\vee$-semigroup S is called an *ideal* if it is a subsemigroup of $(S; \cdot)$ such that $a, b \in J$, $c \in S$ and $c \leqslant a \vee b$ imply $c \in J$. An ideal J is called *prime* if $J \neq S$ and $ab \in J$ implies either $a \in J$ or $b \in J$. Of course, an ideal J is prime if and only if $S \setminus J$ is a prime filter, and this ensures that the map $J \mapsto S \setminus J$ is a homeomorphism of $\mathcal{Q}(S)$, the set of prime ideals of S endowed with the hull-kernel topology, onto $\mathcal{P}(S)$. Let $\mathcal{T}(\mathcal{M}_S) = \dot{\bigcup} \{\mathcal{M}_S/\Theta(S \setminus Q): Q \in \mathcal{Q}(S)\}$ and give it the usual topology. Then, without difficulty, we obtain the following alternative form of Theorem 3.3.

THEOREM 3.4. *Let $\mathcal{M}_S$ be a well-behaved module over a $\vee$-semigroup S with 0 and 1. Then, $\mathcal{M}_S$ is canonically isomorphic to $\Gamma(\mathcal{T}(\mathcal{M}_S), \mathcal{Q}(S))$.*

Before leaving this general situation we would like to briefly discuss another phenomenon which will also be specialized in the subsequent sections of this paper.

The reduct $\mathcal{M} = (M; \vee, f_\gamma)_{\gamma < \alpha}$ of a "well-behaved" module $\mathcal{M}_S$ over a $\vee$-semigroup S with 1 (i.e. we suppose S is integral but need not have a zero) is said to have *filter-determined congruences* if $\mathcal{L}(\mathcal{M}_S) = C(\mathcal{M})$, that is, if the map $F \to \Theta(F)$ is onto. Using Propositions 3.1 and 3.2, we obtain

PROPOSITION 3.5. *If the reduct $\mathcal{M}$ of a well-behaved module $\mathcal{M}_S$ over an integral $\vee$-semigroup has filter-determined congruences then $C(\mathcal{M})$ is distributive and consists of permuting congruences.*

4. Applications to $\vee$-semigroups and distributive lattices. A $\vee$-semigroup S with 0 and 1 is said to be *residuated* (*pseudoresiduated*) if for any $a, b \in S$ ($a \in S$) there exists a necessarily unique element denoted by $a: b$ (a^*) such that for any $c \in S$, $cb \leqslant a$ if and only if $c \leqslant a: b$ ($ca = 0$ if and only if $c \leqslant a^*$). While S is an *l-semigroup* or *lattice ordered semigroup* if $(S; \vee)$ is in fact a lattice.

We will regard a $\vee$-semigroup S with 0 and 1 which is (i) residuated, or (ii) pseudoresiduated, or (iii) an l-semigroup as an algebra (a) $(S; \vee, \cdot, :, 0, 1)$ of type

$\langle 2, 2, 2, 0, 0\rangle$ or (b) $(S; \vee, \cdot, *, 0, 1)$ of type $\langle 2, 2, 1, 0, 0\rangle$, or (c) $(S; \vee, \cdot, \wedge, 0, 1)$ of type $\langle 2, 2, 2, 0, 0\rangle$, respectively. We also regard pseudoresiduated l-semigroups and residuated l-semigroups as algebras of the appropriate type. Of course, we regard S as a reduct of the module S_S which is certainly well-behaved. In addition, a distributive lattice $(S; \vee, \wedge, 0, 1)$ will be regarded as a $\vee$-semigroup with $\wedge = \cdot$, and the class of all such lattices is a subvariety of the variety of $\vee$-semigroups. There are similar interpretations for distributive pseudocomplemented lattices in the variety of pseudoresiduated $\vee$-semigroups, and pseudo-Boolean algebras (in the sense of [12]; Heyting algebras, implicative lattices, relatively pseudocomplemented lattices are equivalent terminologies) in the variety of residuated $\vee$-semigroups. This was the viewpoint of [5]. The following facts are mentioned and exploited in [5].

LEMMA 4.1. *The operations*: (*residuation*), $*$(*pseudoresiduation*), *and* $\wedge$ (*the infimum in a lattice-order*) *are admissible in the sense of* (3.7).

S is *reduced* (*dense*) if $a \vee b = 1$ ($ab = 0$) implies $a = 1$ or $b = 1$ ($a = 0$ or $b = 0$) for any $a, b \in S$. We say that S is *local* if it has a unique maximal ideal. It is easy to see that S is reduced if it is local. Reduced $\vee$-semigroups arose in [5]. We can now specialize Theorem 3.4 to obtain

THEOREM 4.2. *Let S be a $\vee$-semigroup with* 0 *and* 1. *Then,* $(\mathcal{T}(S), \mathcal{Q}(S))$ *is a sheaf of local $\vee$-semigroups with* 0 *and* 1, *which is a sheaf of residuated* (*pseudoresiduated, lattice-ordered, etc.*) *$\vee$-semigroups if S is residuated* (*pseudoresiduated, lattice-ordered, etc.*). *In all cases, S is isomorphic to* $\Gamma(\mathcal{T}(S), \mathcal{Q}(S))$.

The specialization of Theorem 4.2, which gives a sheaf representation of bounded distributive lattices as the lattice of global sections in a sheaf of local lattices, has already been established by Brezuleanu and Diaconescu in [2]; their approach is different from ours.

From Theorem 4.2 and [5], Theorem 4.4, we obtain

COROLLARY 4.3. *A residuated $\vee$-semigroup* (*l-semigroup*) *satisfies the equation* $(a:b) \vee (b:a) = 1$ *if and only if the sheaf* $(\mathcal{T}(S), \mathcal{Q}(S))$ *of Theorem* 4.2 *is a sheaf of totally ordered l-semigroups.*

Using [5], Theorem 4.2, we obtain in a similar manner

COROLLARY 4.4. *A pseudoresiduated $\vee$-semigroup* (*l-semigroup*) *satisfies the equation* $a^* \vee a^{**} = 1$ *if and only if the sheaf* $(\mathcal{T}(S), \mathcal{Q}(S))$ *is a sheaf of reduced, dense $\vee$-semigroups* (*l-semigroups*).

Of course, Corollary 4.3 and Corollary 4.4 have specializations to relative Stone lattices and Stone lattices, respectively. They also give pleasant sheaf-theoretic interpretations of the sub-direct representations of [5]. Corollary 4.3 can also be specialized to describe the nth variety $\mathcal{L}_n (1 \leqslant n < \omega)$ of relative Stone lattices considered in the variety of pseudo-Boolean algebras (see [11]); for $1 < n < \omega$, the stalks are m-chains with $1 \leqslant m \leqslant n$ ([11], Theorems 2, 3 and Section 2). We omit the details. The specialization of Corollary 4.4 to distributive lattices can also be generalized to describe the associated stalks when S is in $\mathcal{B}_n (1 \leqslant n < \omega)$, the nth variety of distributive pseudocomplemented lattices. In this case the stalks are m-dense for $1 \leqslant m \leqslant n$, where lattice S with 0 is m-dense if $(0]$ is the intersection of m prime ideals (of course S is dense if and only if it is 1-dense). Again we omit details; for some information on congruences in this case see [6].

We now consider another type of specialization of Theorem 3.4. Let S be a $\vee$-semigroup with 0 and 1 and let $E(S) = \{a \in S\colon a^2 = a\}$.

PROPOSITION 4.5. *Order $E(S)$ by:* $a \leqslant b$ $(a, b \in E(S))$ *if and only if* $a = ab$. *Then $E(S)$ is a distributive lattice and subsemilattice of* $(S; \vee)$. *Moreover, S is a well-behaved $E(S)$-module.*

Proof. We omit the easy computations.

Thus, Proposition 4.5 yields a sheaf representation of $\vee$-semigroup S which might be fruitful in certain cases.

Let $D(S) = \{a \in S\colon ab = 0$ and $a \vee b = 1$ for some (necessarily unique) b in $S\}$. Regarding S as a special case of a semiring, $D(S)$ is nothing more than the set of (central) complemented elements of S, in the sense of [4]. Thus, the following result is clear.

PROPOSITION 4.6. *$D(S)$ is a Boolean sublattice of $E(S)$ and S is a well-behaved $E(S)$-module.*

Of course, the sheaf representation which is yielded by Proposition 4.6 is a special case of Comer's theorem (as in Section 2). This follows from the relationship between direct summands and central complemented elements as explored in [4].

5. Lattices which are modules over a distributive sublattice. Let L be a lattice with 0 and 1. Recall that an element $s \in L$ is called *standard* (*neutral*) if, for any $x, y \in L$, $x \wedge (s \vee y) = (x \wedge s) \vee (x \wedge y)$ (the sublattice of L generated by $\{s, x, y\}$ is distributive). An element is *central* if it is neutral and complemented. *Dual-standard elements* are defined in a dual manner. Let $\mathrm{Sd}(L)$, $\mathrm{N}(L)$ and $\mathrm{Z}(L)$, respectively denote the set of all dual-standard elements, neutral elements, and central elements. $\mathrm{Z}(L)$ is the *centre* of L and each is a distributive sublattice of the lattice. Also, $\mathrm{Z}(L) \subseteq \mathrm{N}(L) \subseteq \mathrm{Sd}(L)$, and $\mathrm{Z}(L)$ is Boolean. For information, the reader can do no better than refer to Grätzer and Schmidt [10].

PROPOSITION 5.1. *A lattice L with* 0 *and* 1 *is a well-behaved module over each of the distributive sublattices* $\mathrm{Sd}(L)$, $\mathrm{N}(L)$ *and* $\mathrm{Z}(L)$.

Proof. Here the main point that must be checked is the satisfaction of (3.3) and (3.4) for $\mathrm{Sd}(L)$. But these follow from the duals of Theorem 2 ($\delta'(i)$) and Theorem 5 of [10].

Thus Proposition 5.1 yields sheaf-representations of an arbitrary bounded lattice. The $\mathrm{Z}(L)$-module case is once more an instance of Comer's Theorem.

There is one final specialization which we wish to describe briefly. Let S be a bounded distributive lattice and let $J(S)$ be its lattice of ideals. Of course $J(S)$ can be regarded as a well-behaved module over S. Let F be a filter in S and let $\bar{F} = \{J \in J(S)\colon J \supseteq (f]$ for some $f \in F\}$. Thus, $\bar{F}$ is the filter in $J(S)$, generated by F.

Here, $\Theta(\bar{F})$, as described in Section 2, is $\Psi(\bar{F})$, the smallest congruence on $J(S)$ which has $\bar{F}$ as a congruence class (this notation is in line with that of [14]). Also, for a prime ideal P in S, it is not hard to show that $[J]\Theta(L\setminus P) = J^{e_P}$, where J^{e_P} is "the extension of J to $S/\Psi(S\setminus P)$" so that $J^{e_P} = \{[s]\Psi(S\setminus P) : s \in J\}$ (again we use the notation of [14] and $\Psi(S\setminus P)$ is the smallest congruence on S having the prime filter $S\setminus P$ as a congruence class). Hence, the sheaf representation of the S-module $J(S)$ yields

THEOREM 5.2 (The principle of localization). *Let S be a distributive lattice with* 0 *and* 1 *and let J and K be ideals in S. Then, $J = K$ if and only if, for each prime ideal $P \in \mathcal{Q}(S)$, the extensions J^{e_P} and K^{e_P} to the local lattice $S/\Psi(S\setminus P)$ are equal.*

Proof. Two global sections are equal if and only if they agree at each point of the base space.

The "localization" $S/\Psi(S\setminus P)$ has been studied in [2] and [14]. Theorem 5.2 has never been formally stated in the literature. Of course, it is the analogue of the familiar technique used in the ideal theory of commutative rings.

Recently, Swamy [15] has given a different type of sheaf representation for universal algebras $\mathfrak{A}$ such that $C(\mathfrak{A})$ is distributive and consists of permuting congruences. Theorem 5.2 shows in forcible manner that the type of representation arising from Theorem 2.5 is of a different nature.

6. Modules with filter-determined congruences. In conclusion, we would like to point out that many interesting modules with filter-determined congruences exist. In [5], Theorem 3.1, the author showed that a $\vee$-semigroup (l-semigroup) with 0 and 1 has filter-determined congruences if and only if it is a Boolean lattice. This generalized a well-known theorem of J. Hashimoto in the theory of distributive lattices.

From Theorem 8 of Blyth's paper [1], it follows that residuated $\vee$-semigroups and l-semigroups have filter-determined congruences. This result had already been established for pseudo-Boolean algebras by Rasiowa and Sikorski [12]. Thus, in this situation, Proposition 3.5 applies and the sheaf representation of Example 2.6 is thus illustrated once more.

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Accepté par la Rédaction le 30. 6. 1975