The Chinese remainder theorem  
and sheaf representations

by

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Abstract. Let \( \mathfrak{A} \) be a universal algebra whose congruence lattice contains a distributive upper regular sublattice \( \mathcal{L} \) which is compactly generated, consists of permuting congruences and such that the intersection of two compact members of \( \mathcal{L} \) is compact. Then \( \mathfrak{A} \) is representable as all the global sections in a sheaf of algebras in which the base space is the set of prime members of \( \mathcal{L} \) endowed with the dual null-regular topology. This representation includes Cornish's in which Boolean lattices of factor congruences are employed and is particularly applicable to algebras which are "modules" over semilattice-ordered semigroups and distributice lattices.

1. Introduction. Basically this paper is concerned with universal algebras which have a bounded distributive lattice associated with them in such a way that the filters of this lattice give rise to a sublattice of the lattice of congruences of the algebra for which the Chinese remainder theorem holds, and thus enables the algebra to be represented in a natural way as the algebra of all global sections in a certain sheaf. Thus, in Section 2, we consider the general case which was described in the accompanying abstract. After discussing some examples, we move onto Section 3, wherein we consider algebras which are acted on by a commutative semilattice-ordered semigroup. We then consider special cases of this situation in the remaining two sections.

In general, our notation and terminology for universal algebra and lattice theory follows that of [8] and [9]. For background on sheaf representations of universal algebras, we refer to [3].

2. The general sheaf representation. Let \( \mathfrak{A} \) be a non-trivial universal algebra with congruence lattice \( C(\mathfrak{A}) \). Let \( \mathcal{L} \) be a subset of \( C(\mathfrak{A}) \) such that the following conditions hold.

2.1 The bounds \( \alpha \) and \( \beta \) of \( C(\mathfrak{A}) \) are in \( \mathcal{L} \).
2.2 For any \( \Theta, \Phi \in \mathcal{L} \), \( \Theta \cap \Phi \in \mathcal{L} \).
2.3 \( \mathcal{L} \) is upper regular in \( C(\mathfrak{A}) \). That is, for any \( \{ \Theta_i \} \subseteq \mathcal{L} \), \( V_{\mathcal{L}} \Theta_i \) exists in \( \mathcal{L} \) and is equal to \( V_{C(\mathfrak{A})} \Theta_i \).
2.4 The lattice \( \mathcal{L} \) is distributive.
2.5 \( \mathcal{L} \) consists of permuting congruences.
(2.6) The lattice $\mathcal{L}$ is compactly generated in its own right.

(2.7) The interaction of two compact members of $\mathcal{L}$ is compact.

A member $\pi$ of $\mathcal{L}$ is called prime if, for any $\theta, \phi \in \mathcal{L}$, $\theta \cap \phi \leq \pi$ implies $\theta \leq \pi$ or $\phi \leq \pi$.

Before continuing, let us consider some of the connections and implications of these conditions on $\mathcal{L}$.

Of course, (2.1)(i), (2.2) and (2.3) imply that $\mathcal{L}$ is a complete lattice. However, we emphasize that we are not assuming $\mathcal{L}$ to be a lower regular sublattice of $\mathcal{C}(\Omega)$, i.e., $\mathcal{L}$ need not be closed under arbitrary set-theoretic intersections. Conditions (2.3) and (2.7) ensure that the compact members of $\mathcal{L}$ form a sublattice of $\mathcal{C}(\Omega)$. It is not hard to see that (2.6), (2.7), (2.3) together with (2.4) and (2.5), postulated only for the lattice of compact members of $\mathcal{L}$, ensure (2.2), (2.4) and (2.5) for $\mathcal{L}$. In view of (2.4), the prime members of $\mathcal{L}$ are nothing more than their meet-irreducible elements. The following proposition sheds further light on (2.2); we will need part of it subsequently.

**PROPOSITION 2.1.** Suppose $\mathcal{L}$ satisfies (2.1), (2.3) and (2.4) and yet the infimum $\theta \land \phi$ of two members $\theta$ and $\phi$ of $\mathcal{L}$ may not be their set-theoretic intersection. Then the following are equivalent.

(i) For each $\theta, \phi \in \mathcal{L}$, $\theta \land \phi = \theta \cap \phi$.

(ii) For each $\theta \in \mathcal{L}$, $\theta = \bigcap \{ \pi \in \mathcal{L} : \pi \text{ is prime}; \theta \leq \pi \}$. 

**Proof.** (i) $\Rightarrow$ (ii). Suppose $x, y \in A$ and $x \neq y(\theta)$. Then,

$\mathcal{X} = \{ \phi \in \mathcal{L} : \theta \leq \phi \Rightarrow \phi \neq \theta \}$

is non-empty. Order $\mathcal{X}$ by set inclusion and let $\mathcal{X}$ be a chain in $\mathcal{X}$. Then,

$\mathcal{Y} = \{ \phi \in \mathcal{X} : \theta \leq \phi \} \neq \emptyset$

is in $\mathcal{L}$ by (2.3), and clearly it is the least upper bound of $\mathcal{X}$ in $\mathcal{X}$. By Zorn's lemma, there exist maximal elements in $\mathcal{X}$. Let $\pi$ be such an element. As $x \neq y(\theta)$, $\pi$ is not $\phi$. Let $\theta', \phi \in \mathcal{L}$ be such that $\theta' \neq \pi$ and $\phi \neq \pi$. Then, $x = y(\theta' \land \phi)$.

By (i) and the other conditions, we conclude that $x = y(\theta' \land \phi)$ and consequently $\theta' \land \phi \neq \pi$. Thus, $\pi$ is prime. This and (2.1) immediately imply (ii), as required.

(ii) $\Rightarrow$ (i). Of course $\theta \land \phi$ is contained in $\theta \cap \phi$. Assume that (ii) holds. Using (i) easily follows that $\theta \land (\phi \land \theta) = \theta \land \phi$. 

**LEMMA 2.2.** A sublattice $\mathcal{L}$ of $\mathcal{C}(\Omega)$ satisfies (2.4) and (2.5) if and only if it satisfies the Chinese remainder theorem, i.e., for any $x_1, x_2, \ldots, x_n \in A$ and any $\theta_1, \theta_2, \ldots, \theta_n \in \mathcal{L}$, $x = x_1(\theta_1 \lor \theta_2) \lor \ldots \lor x_n(\theta_n \lor \theta_1)$ for all $i, j = 1, 2, \ldots, n$ implies that there exists $x \in A$ such that $\pi = x(\theta_i)$ for each $j = 1, 2, \ldots, n$.

**Proof.** This is a generalization of (8), Chapter 5, Exercise 6b and has already been explicitly observed in [7], Corollary 3.3.

Let $\mathcal{X}$ be the distributive lattice of compact elements in $\mathcal{L}$. Of course (2.6) ensures that $\mathcal{X}$ is isomorphic to the lattice of ideals of $\mathcal{X}$. Note that we have not assumed $\theta$ to be in $\mathcal{X}$.

Let $\mathcal{I}(\mathcal{L}, \Omega)$ be the set of prime ideals in $\mathcal{L}$. Endow it with the dual hull-kernel topology. That is, declare the sets $h(K) = \{ \pi \in \mathcal{I}(\mathcal{L}, \Omega) : K \leq \pi(\Omega) \}$ to be open. Since $h(K, \Omega) = h(K_1) \cap h(K_2)$ and $\mathcal{I}(\mathcal{L}, \Omega) = \bigcup \{ h(K) : K \in \mathcal{X} \}$, these "hulls" are a base for the open sets.

**LEMMA 2.3.** The space $\mathcal{I}(\mathcal{L}, \Omega)$ is compact.

**Proof.** The set of prime ideals of any distributive lattice with 0, endowed with the dual hull-kernel topology, is homeomorphic in a natural way to the set of prime filters endowed with the hull-kernel topology. Hence, the lemma follows from the dual of [9], Lemma 4, p. 119.

Let $\mathcal{S}(\mathcal{L}, \Omega) = \bigcup \{ \pi(\Omega) : \pi \in \mathcal{I}(\mathcal{L}, \Omega) \}$, and for any $x \in A$, let $\delta$ be the Gelfand transform of $x$, i.e. $x$ is the function $x : \mathcal{I}(\mathcal{L}, \Omega) \to \mathcal{S}(\mathcal{L}, \Omega)$, defined by $x(\pi) = \pi(x)$, for each $\pi \in \mathcal{I}(\mathcal{L}, \Omega)$.

**LEMMA 2.4.** For any $x, y \in A$, the set $\{ \pi \in \mathcal{I}(\mathcal{L}, \Omega) : x \neq y(\pi) \}$ is open in $\mathcal{I}(\mathcal{L}, \Omega)$.

**Proof.** Let $\pi \in \mathcal{I}(\mathcal{L}, \Omega)$. By (2.3) and (2.6), $\theta \land \phi = \theta \lor \phi$. Hence, $\{ \pi \in \mathcal{I}(\mathcal{L}, \Omega) : x \neq y(\pi) \}$ is open in $\mathcal{I}(\mathcal{L}, \Omega)$. 

From general considerations ([3], Lemma 3.3) Lemma 2.4 implies that if $\mathcal{I}(\mathcal{L}, \Omega)$ is endowed with the finest topology making the Gelfand transforms continuous (i.e. $\mathcal{I}(\mathcal{L}, \Omega)$ is given the topology whose base consists in $\{ \pi \in \mathcal{I}(\mathcal{L}, \Omega) : x \in A \}$ and $x \in K \in \mathcal{X}$) then $\mathcal{I}(\mathcal{L}, \Omega)$ is a sheaf of algebras of the same type as that of $\mathcal{L}$. Let $I(\mathcal{L}, \Omega)$ denote the algebra of all global sections of this sheaf.

**THEOREM 2.5.** The map $\omega : x \to x \circ \omega$ is an isomorphism of $\mathcal{L}$ onto the algebra $I(\mathcal{L}, \Omega)$.

**Proof.** By Proposition 2.1, $\omega = \bigcap \{ \pi :\pi \in \mathcal{I}(\mathcal{L}, \Omega) \}$. Hence the map is an embedding of $\mathcal{L}$ onto the algebra of global sections. Let $\sigma$ be any global section. Because of the compactness of $\mathcal{I}(\mathcal{L}, \Omega)$ and the fact that global sections agree on an open neighbourhood of a point where they agree, there exist $x_1, x_2, \ldots, x_n \in A$ and $K_1, K_2, \ldots, K_n \in \mathcal{X}$ such that $\sigma(x) = \frac{1}{n} \sum x_i$, for all $\pi \in h(K_i)$, whenever $i = 1, 2, \ldots, n$. Then, $\sigma(x) = \frac{1}{n} \sum x_i$. For all $\pi \in h(K_i) \cap h(K_j)$, whenever $i = 1, 2, \ldots, n$. By Proposition 2.1, once more, $x_i = x_j(K_i \lor K_j)$ for any $i, j = 1, 2, \ldots, n$. Then, $\sigma(x) = \frac{1}{n} \sum x_i$ for all $\pi \in h(K_i) \cap h(K_j)$, whenever $i = 1, 2, \ldots, n$. That is, $\sigma(x) = \frac{1}{n} \sum x_i$. For all $\pi \in h(K_i)$ whenever $j = 1, 2, \ldots, n$. Thus, $\sigma(x) = \frac{1}{n} \sum x_i$ for all $\pi \in h(K_i)$, and the proof is completed.

**EXAMPLE 2.6.** Suppose algebra $\mathcal{L}$ is such that its congruences permute and form a distributive lattice in which the intersection of two compact congruences is compact.
Then, we may take \( \mathcal{L} \) to be \( C(\mathbb{N}) \) to obtain a sheaf representation for \( \mathcal{Y} \). Of course, this applies to Boolean rings. Due to a recent result of Shore and Wiegand [13], this last instance is capable of immense generalization. Indeed, in [13], Corollary 1.11, it is established that if the lattice of ideals of a commutative ring with identity is distributive (i.e., we are dealing with the so-called arithmetical rings) then the intersection of two finitely generated ideals is finitely generated. Since ideals and congruences of a ring may be identified, and in doing so finitely generated ideals correspond to compact congruences, we have an example of our situation.

**Example 2.7.** In [3], Comer generalized R. S. Pierce's sheaf representation to rings using central idempotents to a representation of a wide class of universal algebras by using a well-behaved Boolean sublattice of all factor congruences. Our situation can be applied to obtain Comer's theorem (3) Theorem 3.7. For, Comer's condition (I) and the equivalence of condition II given by [3], Proposition 2.3 implies that the ideals of his Boolean lattice of factor congruences satisfies (2.1) through to (2.7) inclusive.

3. Semilattice-ordered semigroup modules. By a semilattice-ordered semigroup \( S \) we mean an algebra \((S; \vee, \cdot)\) such that \((S; \cdot)\) is a commutative semigroup, \((S; \vee)\) is a semilattice and the equation \( x(y \vee z) = xy \vee xz \) is identically satisfied. A semilattice-ordered semigroup, or more briefly a \( \vee \)-semigroup, is integral if \((S; \cdot)\) has an identity which is also the largest element of \((S; \vee)\), while it is negatively-ordered if \( xy \leq xz \) for all \( x, y \in S \) (here \( x \leq y \) if and only if \( xy = y \)). Of course, \( S \) is negatively-ordered if it is integral.

A filter \( F \) of a \( \vee \)-semigroup \( S \) is a (non-empty) subsemigroup of \((S; \cdot)\) such that \( x \cdot y \in F \) and \( y \cdot x \cdot y \) imply \( x \cdot y \cdot y \in F \). Ordered by set-inclusion the filters on \( S \) form a lattice \( \mathcal{F}(S) \), where the infimum is set-theoretic intersection and the join of a collection \( \{F_i\} \) of filters is given by

\[
V F_i = \{x \in S : a \cdot f_i \cdot f_j \leq f_i, f_j \in F_i \text{ and } f_k \in F_j \text{ for } i = 1, 2, ..., n\}.
\]

Note also that \( F_1 \cap F_2 \) is \( \{f_1 \vee f_2 : f_1 \in F_1 \text{ and } f_2 \in F_2 \} \) for any \( F_1, F_2 \in \mathcal{F}(S) \). For \( x \in S \), the smallest filter containing \( x \) is given by

\[
[S] = \{y \in S : x \leq y \text{ for some } k \geq 1\}.
\]

**Proposition 3.1.** If \( S \) is a negatively-ordered \( \vee \)-semigroup then the lattice \( \mathcal{F}(S) \) of its filters is distributive.

**Proof.** Let \( x \in F_1 \cap (F_2 \vee F_3) \), where \( F_1 \in \mathcal{F}(S) \) for \( i = 1, 2, 3 \). Then \( x \in F_i \) and \( f_2 \leq f_2 \leq x \) for suitable \( f_2 \in F_2 \). Now

\[
(x \cdot f_2)(x \cdot f_2) = x^2 \cdot x \cdot f_2 \cdot f_2 \cdot f_2 \cdot f_2 \cdot f_2 \cdot f_2 = x,
\]

since \( S \) is negatively ordered. In addition, \( x \cdot f_2 \in F_1 \cap F_3 \) and \( x \cdot f_2 \in F_1 \cap F_3 \). Hence, \( x \in (F_1 \cap F_2) \vee (F_1 \cap F_3) \) and the proposition follows.

Since we are going to make use of \( \mathcal{F}(S) \), Proposition 3.1 explains why we shall henceforth assume that each \( \vee \)-semigroup is negatively ordered.

An algebra \((M; \vee, f_p)_{p \leq m} \) is a module over a semilattice-ordered semigroup \( S \) if the following conditions hold.

3.1. \((M; \vee)\) is a semilattice.

3.2. For each \( a \in S \) and \( m \in M \), there is a unique element, denoted by \( ma \), in \( M \).

3.3. For each \( a \in S \) and \( m, n \in M \), \( (m \vee n)a = ma \vee na \).

3.4. For each \( a, b \in S \) and \( m \in M \), \( m(a \vee b) = ma \vee mb \).

3.5. For each \( a, b \in S \) and \( m \in M \), \( m(ab) = (ma)b \).

3.6. For each \( a \in S \) and \( m \in M \), \( ma \cdot m = ma \cdot m \).

3.7. Each \( f_p(\cdot \cdot \cdot \cdot \cdot) \) is admissible in the sense that either \( f_p \) is a nullary operation or if not then the following property holds:

- if the arity of \( f_p \) is \( k_p \) and \( n_1, n_2, ..., n_{k_p} \in M \), \( n_1, n_2, ..., n_{k_p} \in M \) and \( a \in S \) satisfies \( m \cdot n_1 \leq m, m \cdot n_2 \leq m, ..., m \cdot n_{k_p} \leq m \), then

\[
f_p(m, n_1, n_2, ..., n_{k_p}) a \leq f_p(m, n_1, n_2, ..., n_{k_p}).
\]

It will be convenient to regard a \( \vee \)-semigroup module \( M \) as an algebra \( \mathcal{M} = (M; \vee, f_p, \theta_b)_{p \leq m} \) of type \( (2; 2, ..., 2; 1, 1, ...) \), where \( k_p \) is the arity of the admissible operation \( f_p \) and \( \theta_b \) is the unary operation on \( M \) induced by multiplication with \( b \in S = \{a : a \cdot b \leq b\} \). We emphasize that any sheaf representation of \( \mathcal{M} = (M; \vee, f_p, \theta_b)_{p \leq m} \) in a sheaf of modules of type \( (2; 2, ..., 2; 1, 1, ...) \) induces a representation of the reduct \((M; \vee, f_p)_{p \leq m} \) in a sheaf of algebras of type \( (2; 2, ..., 2; 1, 1, ...) \). In examples, we often wish to consider these reducts and there will be no ambiguity if we omit to mention the type.

Let \( F \) be a filter in \( S \). Define a binary relation on \( M \) by: \( m = n(\Theta(F)) \) if and only if \( m \leq n \) and \( n \leq m \) for some \( f_p \in F \).

**Proposition 3.2.** The following statements hold.

(i) For each \( F \in \mathcal{F}(S) \), \( \Theta(F) \) is a congruence on the algebra \( \mathcal{M} \).

(ii) For any two filters \( F_1, F_2 \in \mathcal{F}(S) \), \( \Theta(F_1) \cap \Theta(F_2) \) permutes with \( \Theta(F_2) \).

(iii) For any two filters \( F_1, F_2 \in \mathcal{F}(S) \), \( \Theta(F_1 \cap F_2) = \Theta(F_1) \cap \Theta(F_2) \).

(iv) For any set \( \{F_i\} \subseteq \mathcal{F}(S) \), \( \Theta(V F_i) = \vee \Theta(F_i) \).

**Proof.** (i) \( \Theta(F) \) is reflexive because of (3.6). It is symmetric by its definition and transitive due to (3.3) and (3.5). It has the substitution property for the operation \( \vee \) because of (3.7). It has the substitution property for each \( f_p \) of course, (3.7) was postulated to ensure this. The substitution property holds for each \( \theta_b \) due to (3.3) and (3.5), and hence \( \Theta(F) \) is a congruence.

(ii) Suppose \( m = n(\Theta(F_1) \cap \Theta(F_2)) \) for given \( m, n \in M \). Then there is \( p \in M \) such that \( m = p(\Theta(F_1) \cap \Theta(F_2)) \). There are \( f_1 \in F_1 \) and \( f_2 \in F_2 \) such
that $m_f \leq p_f \leq m$ and $n_f \leq n$. By (3.3) and (3.5), $m_f \leq m$ and $n_f \leq m$. Hence, $m = m \vee m_f \leq m$ and $n = n \vee m_f \leq n$. Let $w = m_f \leq n_f$. Then,

$$w_f = m_f \vee n_f \leq m \vee n_f = m,$$

while

$$m_f = (m \vee n_f) \leq m \vee n_f \leq m \vee n_f = w.$$

Similarly, $w_f \leq n$ and $n_f \leq w$. Thus, $m = n(\Theta(F_1) \wedge \Theta(F_2))$, and $\Theta(F_1)$ permutes with $\Theta(F_2)$.

(iii) Let $m, n \in M$ be such that $m = n(\Theta(F_1) \wedge \Theta(F_2))$, so that $m_f \leq n$ and $n_f \leq m$ for suitable $f_1 \in F_1$ and $f_2 \in F_2$. By (3.4), $m_f \vee n_f \leq m$ and $n_f \vee m_f \leq n$. Hence, $m = n(\Theta(F_1) \wedge \Theta(F_2))$ and (iii) follows similarly.

(iv) If $m = n(\Theta(F_1))$ then (3.4) implies that

$$m_f \leq n_f \leq m_f \wedge n_f \leq m \quad \text{and} \quad n_f \leq m_f \wedge n_f \leq m_f$$

for some $f_1 \in F_1, f_2 \in F_2$, for each $r = 1, 2, \ldots, k$. Now for any filter $F$ and $m \in M$,

$$m_f = m(\Theta(F)) \quad \text{since} \quad (m_f \wedge m_f \leq m) \quad \text{and} \quad m_f \leq m_f$$

Hence,

$$m_f = m(\Theta(F)) = (\Theta(F_1)) = m_f \leq n_f \leq m_f \leq m_f.$$

Thus $m = m_f \vee n_f \leq m_f \vee n_f \leq m_f \vee n_f \leq m_f$. Similarly, $n \leq n_f \leq n_f \leq n_f \leq n_f \leq n_f$.

But,

$$m_f \leq n_f \leq m_f \wedge n_f \leq m_f \quad \text{and} \quad n_f \leq m_f \wedge n_f \leq m_f.$$

Hence, $m_f \leq n_f \leq m_f \wedge n_f \leq m_f \wedge n_f \leq m_f \wedge n_f \leq m_f$. And hence,

$$m_f \leq n_f \leq m_f \wedge n_f \leq m_f \wedge n_f \leq m_f \wedge n_f \leq m_f.$$

And so $m = n(\Theta(F_1))$, and (iv) follows.

We say that $S$ has a zero, $0$, if $(S; \vee)$ has a smallest element $0$. Since we are assuming that $S$ is negatively ordered $0$ is the zero of the semigroup $(S; \vee)$.

We will not be able to get much further unless we ensure that the map $F \mapsto \Theta(F)$ of $\mathcal{F}(S)$ into $C(\mathcal{M})$ is an embedding. We also need to ensure that $\Theta(a)$, $a \in S$, is a compact member of $(\Theta(F) : F \in \mathcal{F}(S))$. Of course, we will also require $a = \Theta(1)$ for suitable filters $F$ and $G$ on $S$. Therefore, a module $\mathcal{M}$ is said to be well-behaved if the following conditions are satisfied.

(3.8) $S$ has 0 and 1.

(3.9) $M = (\mathfrak{M}; \vee, 0, 1)$ has 0 and 1, and $(S; \vee)$ is a subsemilattice of $(M; \vee, 0, 1)$ such that the zeros and identities respectively coincide.

(3.10) The 0 and 1 of $M$ are considered as nullary admissible operations in $\mathcal{M}$.

(3.11) $\mathcal{M}$ is unitary in the sense that $m_0 = m$ for all $m \in M$.

Note that (3.11) implies (3.6). For, if $m \in M$ and $a \in S$ then $m(\vee a) = m \vee a = m \vee m = m$. It is clear that $\Theta(1) = 0$, $m_0 = 0$ for all $m \in M$, and $\Theta(0) = 1$. In addition, (3.11) ensures that the map $F \mapsto \Theta(F)$ is one-to-one. Since

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A filter $F$ is prime if $a \vee b \in F(a, b) \in S$ implies $a \in F$ or $b \in F$, and $F \neq S$. Of course, filter $F$ is prime if and only if $\Theta(F)$ is prime in $\mathcal{M}$. Let $\mathcal{P}(S)$ be the set of prime filters of $S$ endowed with the dual hull-kernel topology. Let

$$\mathcal{P}(S) = \{ P : P \in \mathcal{P}(S) \},$$

and endow it with the finest topology which makes the Gelfand transforms $\theta_m : \text{Hom}(\mathfrak{M}, \mathcal{P}(S))$ continuous. Clearly, if we take $\mathcal{M} = (\mathfrak{M}; \vee, 0, 1)$ and $\mathfrak{M} = \mathcal{M}$, we can identify the sheaf $(\mathcal{P}(\mathfrak{M}), \mathcal{P}(\mathfrak{M}), \emptyset)$ of Section 2 with the sheaf $(\mathcal{P}(\mathcal{M}), \mathcal{P}(\mathcal{M}), \emptyset)$ and hence Theorem 2.5 implies.

Theorem 3.3. Let $\mathcal{M}$ be a well-behaved module over a $\vee$-semigroup $S$ with 0 and 1. Then $\mathcal{M}$ is canonically isomorphic to $(\mathcal{P}(\mathcal{M}), \mathcal{P}(\mathcal{M}), \emptyset)$.

There is a slightly different version of Theorem 3.3 which we will now construct.

A non-empty subset $J$ of $\vee$-semigroup $S$ is called an ideal if it is a subsemigroup of $(S; \vee)$ such that $a, b \in J, c \in S$ and $c \vee a \vee b$ imply $c \in J$. An ideal $J$ is called prime if $J \neq S$ and whenever $a \vee b \in J$ implies either $a \in J$ or $b \in J$. Of course, an ideal $J$ is prime if and only if $J \mapsto J$ is a prime filter, and this ensures that the map $J \mapsto J$ is a homomorphism of $\mathfrak{M}$, the set of prime ideals of $S$ endowed with the hull-kernel topology, onto $\mathcal{P}(S)$. Let $\mathcal{M}(S) = \{ \mathcal{M}(S) : Q \in \mathcal{P}(S) \}$ and give it the usual topology. Then, without difficulty, we obtain the following alternative form of Theorem 3.3.

Theorem 3.4. Let $\mathcal{M}$ be a well-behaved module over a $\vee$-semigroup $S$ with 0 and 1. Then $\mathcal{M}$ is canonically isomorphic to $(\mathcal{P}(\mathcal{M}), \mathcal{P}(\mathcal{M}), \emptyset)$.

Before leaving this general situation we would like to briefly discuss a well-behaved module $\mathcal{M}$ over a $\vee$-semigroup $S$ with 1 (i.e. we suppose $S$ is integral but need not have a zero) is said to have filter-determined congruences if $\mathcal{M}(S) = \mathcal{M}(S)$, that is, if the map $F \mapsto \Theta(F)$ is onto. Using Propositions 3.1 and 3.2, we obtain

Proposition 3.5. If the reduced $\mathcal{M}$ of a well-behaved module $\mathcal{M}$ over an integral $\vee$-semigroup $S$ has filter-determined congruences then $\mathcal{M}(S)$ is distributive and consists of permuting congruences.

4. Applications to $\vee$-semigroups and distributive lattices. A $\vee$-semigroup $S$ with 0 and 1 is said to be residuated (pseudoresiduated) if for any $a, b \in S$ (or $a \in S$) there exists a uniquely defined element denoted by $a : b$ (or $a : b$) such that for any $c \in S$, $c \vee a : b$ if and only if $c \vee b : a$. While $S$ is an $\vee$-semigroup or lattice ordered semigroup if $(S; \vee)$ is in fact a lattice.

We will regard a $\vee$-semigroup $S$ with 0 and 1 which is (i) residuated, or (ii) residuated, or (iii) residuated as an algebra $(a) \in S$; $\vee, 0, 1$ of type
psuedocomplemented lattices. In this case the stalks are m-dense for \(1 \leq m \leq n\), where lattice \(S\) with \(0\) is \(m\)-dense if \(0\) is the intersection of \(m\) prime ideals (of course \(S\) is dense if and only if it is \(1\)-dense). Again we omit details; for some information on congruences in this case see [6].

We now consider another type of specialization of Theorem 3.4. Let \(S\) be a \(v\)-semigroup with \(0\) and \(1\) and let \(E(S) = \{ a \in S : a^2 = a \}\).

**Proposition 4.5.** Order \(E(S)\) by: \(a \leq b\) if \((a, b \in E(S))\) and only if \(a = ab\). Then \(E(S)\) is a distributive lattice and sublattice of \((S, \lor, \land)\). Moreover, \(S\) is a well-behaved \(E(S)\)-module.

Proof. We omit the easy computations.

Thus, Proposition 4.5 yields a sheaf representation of \(v\)-semigroup \(S\) which might be fruitful in certain cases.

Let \(D(S) = \{ a \in S : ab = 0 \text{ and } a \lor b = 1 \text{ for some (necessarily unique) } b \in S \}\). Regarding \(S\) as a special case of a semiring, \(D(S)\) is nothing more than the set of (central) complemented elements of \(S\), in the sense of [4]. Thus, the following result is clear.

**Proposition 4.6.** \(D(S)\) is a Boolean sublattice of \(E(S)\) and \(S\) is a well-behaved \((E(S))\)-module.

Of course, the sheaf representation which is yielded by Proposition 4.6 is a special case of Comer's theorem (as in Section 2). This follows from the relationship between direct summands and central complemented elements as explored in [4].

### 5. Lattices which are modules over a distributive sublattice.

Let \(L\) be a lattice with \(0\) and \(1\). Recall that an element \(s \in L\) is called standard (neutral) if, for any \(x, y \in L\), \((x \lor y) = (x \lor s) \lor (y \lor s)\) (the sublattice of \(L\) generated by \((s, s, y)\) is distributive). An element is central if it is neutral and complemented. Dual-standard elements are defined in a dual manner. Let \(\text{Std}(L), \text{N}(L)\) and \(Z(L)\), respectively denote the set of all dual-standard elements, neutral elements, and central elements. \(Z(L)\) is the centre of \(L\) and each is a sublattice of the lattice. Also, \((Z(L) = \text{Std}(L)\) and \(Z(L)\) is Boolean. For information, the reader can do no better than refer to Grätzer and Schmidt [10].

**Proposition 5.1.** A lattice \(L\) with \(0\) and \(1\) is a well-behaved module over each of the distributive sublattices \(\text{Std}(L), \text{N}(L)\) and \(Z(L)\).

Proof. Here the main point that must be checked is the satisfaction of (3.3) and (3.4) for \(\text{Std}(L)\). But these follow from the duals of Theorem 2 (\(\lor\)) and Theorem 5 of [10].

Thus Proposition 5.1 yields sheaf-representations of an arbitrary bounded lattice. The \(Z(L)\)-module case is once more an instance of Comer's Theorem.

There is one final specialization which we wish to describe briefly. Let \(S\) be a bounded distributive lattice and let \(J(S)\) be its lattice of ideals. Of course \(J(S)\) can be regarded as a well-behaved module over \(S\). Let \(F\) be a filter in \(S\) and let \(F = \{ J \in J(S) : J \not\subseteq f \}\) for some \(f \in F\). Thus, \(F\) is the filter in \(J(S)\), generated by \(F\).
Here, $\theta(F)$, as described in Section 2, is $\varphi(F)$, the smallest congruence on $J(S)$ which has $F$ as a congruence class (this notation is in line with that of [14]). Also, for a prime ideal $P$ in $S$, it is not hard to show that $[J/F] = J^*$, where $J^*$ is "the extension of $J$ to $\varphi(S,F)$" so that $J^* = \{x \varphi(S,F) : x \in J\}$. (again we use the notation of [14] and $\varphi(S,F)$ is the smallest congruence on $S$ having the prime filter $S$ as a congruence class). Hence, the sheaf representation of the $S$-module $J(S)$ yields

**Theorem 5.2** (The principle of localization). Let $S$ be a distributive lattice with 0 and 1 and let $J$ and $K$ be ideals in $S$. Then, $J = K$ if and only if, for each prime ideal $P$ in $S$, the extensions $J*$ and $K*$ to the local lattice $\varphi(S,F)$ are equal.

**Proof.** Two global sections are equal if and only if they agree at each point of the base space.

The "localization" $\varphi(S,F)$ has been studied in [2] and [14]. Theorem 5.2 has never been formally stated in the literature. Of course, it is the analogue of the familiar technique used in the ideal theory of commutative rings.

Recently, Swamy [15] has given a different type of sheaf representation for universal algebras such that $\varphi(S)$ is distributive and consists of permuting congruences. Theorem 5.2 shows in forcible manner that the type of representation arising from Theorem 2.5 is of a different nature.

6. **Modules with filter-determined congruences.** In conclusion, we would like to point out that many interesting modules with filter-determined congruences exist. In [5], Theorem 3.1, the author showed that a $\vee$-semigroup (a $\vee$-semigroup) with 0 and 1 has filter-determined congruences if and only if it is a Boolean lattice. This generalized a well-known theorem of J. Hashimoto in the theory of distributive lattices.

From Theorem 8 of Blyth's paper [1], it follows that residuated $\vee$-semigroups and $\vee$-semigroups have filter-determined congruences. This result has already been established for pseudo-Boolean algebras by Rasiowa and Sikorski [12]. Thus, in this situation, Proposition 3.5 applies and the sheaf representation of Example 2.6 is thus illustrated once more.

**References**