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A geometric filtration of $\mathfrak{N}_*^{Z_2}$

by

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Abstract. If α is the cobordism class of a manifold with involution, $e(\alpha)$ is defined to be the smallest integer n such that a representative of α can be Z_2-equivariantly embedded in R^{n+s}, for some s, where Z_2 acts on R^{n+s} by multiplying the first n coordinates by -1. Z_2-cobordism classes α_k, β_k are exhibited such that $e(\alpha_k) = e(\beta_k)$ but $e(\alpha_k+\beta_k) = e(\alpha_k)-k$, for arbitrarily large integers k.

Let $\mathfrak{N}_*^{Z_2}$ be the cobordism ring of manifolds with involutions. Let z_n denote RP^n with the Z_2-action given in homogeneous coordinates by $[x_0, x_1, \ldots, x_n] \to [-x_0, x_1, \ldots, x_n]$. Define $\Gamma\colon \mathfrak{N}_*^{Z_2} \to \mathfrak{N}_{*+1}^{Z_2}$ by

$$\Gamma[M, T] = \left[\frac{M\times S^1}{(m, z)\sim(Tm, -z)},\ [m, z]\to[m, \bar{z}]\right].$$

Then $\{1\} \cup \{(\Gamma^i z_{n_1}) z_{n_2} \ldots z_{n_k} |\ i\geqslant 0, k\geqslant 1, n_1\geqslant n_2\geqslant \ldots \geqslant n_k > 1\}$ is a set of generators for $\mathfrak{N}_*^{Z_2}$ as an $\mathfrak{N}_*$-module (Alexander [1], Stong [7]), where Γ^0 denotes the identity map.

To define a geometric filtration of $\mathfrak{N}_{i+j}^{Z_2}$, let $F(i, j)$ be the set of Z_2-cobordism classes having representatives which, for some s, can be Z_2-equivariantly embedded in R^{i+s}, furnished with the Z_2-action $(x_1, \ldots, x_{i+s}) \to (-x_1, \ldots, -x_i, x_{i+1}, \ldots, x_{i+s})$. It is known that $(\Gamma^i z_{n_1}) z_{n_2} \ldots z_{n_k} \in F(i+n_1+n_2+\ldots+n_k, 0)$ and $\notin F(i+n_1+n_2+\ldots+n_k-1, 1)$ (Bix [2]). But even if $a, b \in F(i, j)$ and $\notin F(i-1, j+1)$, it is possible that $a+b \in F(i-k, j+k)$ for some $k>0$. The main result of this paper is that such drops in dimension occur with k arbitrarily large.

THEOREM. *$z_n^2 + z_{n+1} z_{n-1} + \Gamma^{n+1} z_{n-1} + \Gamma^{n-1} z_{n+1} \in F(n+1, n-1)$ and $\notin F(n, n)$, while $z_n^2 + z_{n+1} z_{n-1}$ and $\Gamma^{n+1} z_{n-1} + \Gamma^{n-1} z_{n+1} \in F(2n, 0)$ and $\notin F(2n-1, 1)$, for all $n \geqslant 3$.*

Proof. The classifying map of the normal bundle to the fixed-point set of a manifold with involution defines a monomorphism $i\colon \mathfrak{N}_*^{Z_2} \to \bigoplus_{k=0}^{*} \mathfrak{N}_{*-k}(\mathrm{BO}(k)) \cong \mathfrak{N}_*[x_0, x_1, \ldots]$, where x_n is the bordism class of the canonical line bundle over RP^n (Boardman [3], [4], Conner and Floyd [5]). We identify a manifold with involution with the image under i of its Z_2-cobordism class. So $z_n = x_{n-1} + x_0^n$. And

$$\Gamma^i z_n = x_{n-1} x_0^i + x_0^{n+i} + [z_n]_2 x_0^i + [\Gamma z_n]_2 x_0^{i-1} + [\Gamma^2 z_n]_2 x_0^{i-2} + \ldots + [\Gamma^{i-1} z_n]_2 x_0,$$

where $[\ ]_2$ denotes the class of a manifold in $\mathfrak{N}_*$. Therefore

$$z_n^2+z_{n+1}z_{n-1} = x_{n-1}^2+x_nx_{n-2}+x_nx_0^{n-1}+x_{n-2}x_0^{n+1}$$

and

$$\Gamma^{n+1}z_{n-1}+\Gamma^{n-1}z_{n+1} = x_nx_0^{n-1}+x_{n-2}x_0^{n+1}+[z_{n-1}]_2x_0^{n+1}+[\Gamma z_{n-1}]_2x_0^n+$$
$$+[\Gamma^2z_{n-1}+z_{n+1}]_2x_0^{n-1}+\ldots+[\Gamma^nz_{n-1}+\Gamma^{n-2}z_{n+1}]_2x_0 .$$

Given $\alpha\in\mathfrak{N}_n^{Z_2}$, Stong [8] has proved that $\alpha\in F(j, n-j)$ and $\notin F(j-1, n-j+1)$ if and only if j is the smallest integer such that the image of α under the map

$$\mathfrak{N}_n^{Z_2}\xrightarrow{i}\bigoplus_{k=0}^{n}\mathfrak{N}_{n-k}(\mathrm{BO}(k))\xrightarrow{\pi_k}\mathfrak{N}_{n-k}(\mathrm{BO}(k))\to\mathfrak{N}_{n-k}(\mathrm{BO})\xrightarrow{(-1)_*}\mathfrak{N}_{n-k}(\mathrm{BO})$$

lies in the image of the map $\mathfrak{N}_{n-k}(\mathrm{BO}(j-k))\to\mathfrak{N}_{n-k}(\mathrm{BO})$ for all k with $0\leqslant k\leqslant n$, where $(-1)_*$, which we shall denote by a bar, is the conjugation map. So $z_n^2+z_{n+1}z_{n-1}\in F(2n, 0)$ and $\notin F(2n-1, 1)$, since $j-k = n$ for $k = n$, $j-k = n-2$ for $k = n+2$, and $j-k<2n-2$ for $k = 2$. And $\Gamma^{n+1}z_{n-1}+\Gamma^{n-1}z_{n+1}\in F(2n, 0)$ and $\notin F(2n-1, 1)$, because $j-k = n-2$ for $k = n+2$, $j-k = n$ for $k = n$, and $j-k = 0$ for $1\leqslant k\leqslant n-1$ and $k = n+1$, whenever $[\Gamma^{n+1-k}z_{n-1}+\Gamma^{n-1-k}z_{n+1}]_2 \neq 0$.

It only remains to examine

$$z_n^2+z_{n+1}z_{n-1}+\Gamma^{n+1}z_{n-1}+\Gamma^{n-1}z_{n+1}$$
$$= x_{n-1}^2+x_nx_{n-2}+[z_{n-1}]_2x_0^{n+1}+[\Gamma z_{n-1}]_2x_0^n+[\Gamma^2z_{n-1}+z_{n+1}]_2x_0^{n-1}+\ldots$$
$$\ldots+[\Gamma^nz_{n-1}+\Gamma^{n-2}z_{n+1}]_2x_0 .$$

Since $j-k = 0$ for $k = 1$ and $3\leqslant k\leqslant n+1$, whenever $[\Gamma^{n+1-k}z_{n-1}+\Gamma^{n-1-k}z_{n+1}]_2\neq 0$, it suffices to show that $j-k = n+1$ when $k = 2$. That is, we must show that $\bar{x}_{n-1}^2+\bar{x}_n\bar{x}_{n-2}$ has algebraic degree $n-1$. The algebraic degree can be calculated by means of the algebra homomorphism $\Delta\colon H_*(\mathrm{MO}; Z_2)\to H_*(\mathrm{MO}; Z_2)[[s]]$ defined by $\Delta(a) = \sum_{i=0}^{\infty}\Delta_i(a)s^i$, where $\Delta_i\colon H_*(\mathrm{MO}; Z_2)\to H_{*-i}(\mathrm{MO}; Z_2)$ is the map which is dual to the map defined by the cup product with w_i (Liulevicius [6]). The Thom isomorphism is used to identify $H_*(\mathrm{MO}; Z_2)$ with $H_*(\mathrm{BO}; Z_2)$. Now $\Delta x_n = x_n+x_{n-1}s$. Given an element $a\in H_*(\mathrm{MO}; Z_2)$, its algebraic degree is equal to the highest power of s in $\Delta(a)$.

LEMMA. $\Delta\bar{x}_n = \bar{x}_n+\bar{x}_{n-1}s+\ldots+\bar{x}_1s^{n-1}+s^n$.

Proof. Let

$$X = \sum_{i=0}^{\infty}x_it^i \quad\text{and}\quad X^{-1} = \sum_{i=0}^{\infty}\bar{x}_it^i .$$

Then $\Delta X = X+stX = (1+st)X$. Since $(\Delta(X^{-1}))(\Delta X) = 1$, $\Delta(X^{-1}) = (\Delta X)^{-1} = (1+st)^{-1}X^{-1} = (1+st+s^2t^2+\ldots)X^{-1}$. So $\Delta\bar{x}_n$ is equal to the coefficient of t^n in $(1+st+s^2t^2+\ldots)X^{-1}$, which is $\bar{x}_n+\bar{x}_{n-1}s+\ldots+\bar{x}_1s^{n-1}+s^n$.

Now to complete the proof of the theorem it suffices to show that the highest power of s in $\Delta(\bar{x}_{n-1}^2+\bar{x}_n\bar{x}_{n-2})$ is $n-1$. But

$$\Delta(\bar{x}_{n-1}^2+\bar{x}_n\bar{x}_{n-2}) = \Big(\sum_{i=0}^{n-1}\bar{x}_is^{n-1-i}\Big)^2+\Big(\sum_{j=0}^{n}\bar{x}_js^{n-j}\Big)\Big(\sum_{k=0}^{n-2}\bar{x}_ks^{n-2-k}\Big)$$
$$= \Big(\bar{x}_{n-1}^2+\sum_{i=0}^{n-2}\bar{x}_i^2s^{2n-2-2i}\Big)+\Big(\bar{x}_n+\bar{x}_{n-1}s+\sum_{j=0}^{n-2}\bar{x}_js^{n-j}\Big)\Big(\sum_{k=0}^{n-2}\bar{x}_ks^{n-2-k}\Big)$$
$$= \bar{x}_{n-1}^2+s^2\Big(\sum_{i=0}^{n-2}\bar{x}_is^{n-2-i}\Big)^2+(\bar{x}_n+\bar{x}_{n-1}s)\Big(\sum_{k=0}^{n-2}\bar{x}_ks^{n-2-k}\Big)+$$
$$+s^2\Big(\sum_{j=0}^{n-2}\bar{x}_js^{n-2-j}\Big)^2 = \bar{x}_{n-1}^2+(\bar{x}_n+\bar{x}_{n-1}s)\Big(\sum_{k=0}^{n-2}\bar{x}_ks^{n-2-k}\Big)$$
$$= \bar{x}_{n-1}s^{n-1} + (\text{terms involving powers of } s\leqslant n-2).$$

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Added in proof. R. Stong has pointed out that, to complete the proof of the above theorem, one also has to check the Stiefel-Whitney numbers of the form $w_{\omega'}(\tau)w_{\omega''}(\eta)[M]$.

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