Relatively constructible transitive models

by

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Abstract. In this paper we consider standard transitive models of set theory and its fragments and investigate whether a given model is a minimal one containing a fixed set of natural numbers. The answer is particularly elegant in the case of β-models of second-order number theory. We give a necessary and sufficient condition for a β-model of second-order arithmetic plus the axiom of constructibility to consist exactly of relatively ramified analytical sets. We apply this result to an investigation of arithmetically regular ordinals in the sense of generalized recursion theory, and prove that a countable arithmetically regular ordinal α has the form $\alpha = \beta_0^X$ for some $X \in \wp(\omega) \cap L_\alpha$ iff it is not arithmetically inaccessible and each $\beta < \alpha$ is α-projectable into ω. We consider also similar questions in the uncountable case.

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1. Preliminaries. Throughout the paper we work in the Zermelo–Fraenkel set theory, informally using classes, e.g. *On* and *L*. We use the standard set theoretic notation. We use $\in$ for the real world membership relation. A set a is *transitive* if $x \in y \in a$ implies $x \in a$ for all x, y. For any x there is a smallest transitive set a such that $x \subseteq a$, called the *transitive closure* of x, $\mathrm{TC}(x)$. The hereditary cardinality of a set x is the cardinality of $\mathrm{TC}(x)$. We use $H_\varkappa$ to denote the collection of sets of hereditary cardinality less than $\varkappa$, $\varkappa$ being a cardinal. Thus, $H_{\aleph_1}$ is just the collection of the hereditarily countable sets, HC. By ZF^- we denote the ZF set theory minus the power set axiom. We use $\mathrm{V} = \mathrm{HC}$ to denote the set-theoretical statement "every set is countable".

We begin by recalling the notion of relative constructibility. This notion was introduced by A. Lévy in [13] as a generalization of Gödel's definition of constructibility from [9]. However, we consider a generalization of Gödel's-original definition from [8]. See J. Silver [17]. Given a set x, define

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$$L_0[x] = \mathrm{TC}(\{x\}),$$
$$L_{\alpha+1}[x] = \mathrm{Def}(\langle L_\alpha[x], \in\rangle),$$
$$L_\lambda[x] = \bigcup\{L_\alpha[x]\colon \alpha\in\lambda\} \quad \text{for limit } \lambda,$$
$$L[x] = \bigcup\{L_\alpha[x]\colon \alpha\in On\},$$

where $\mathrm{Def}(M)$ stands for the collection of subsets first-order definable in the model M with parameters allowed. All basic facts concerning the notion of constructibility can be reproved in the relativized version. A reader unfamiliar with those proofs might want to read Section 3 of Silver [17].

Let us also recall some basic facts concerning second-order arithmetic A_2. A_2 is formalized in a two-sorted language: small letters $x, y, \ldots$ are intended to vary over natural numbers and capital letters $X, Y, Z, \ldots$ — over sets of natural numbers. The axioms of A_2 consist of the following statements: the Peano axioms for natural numbers; the axiom of extensionality; the comprehension schema and the following schema of choice (AC_{01} in Kreisel's [11] notation):

$$(x)(\mathrm{E}Y)\Phi(x, Y)\to(\mathrm{E}Y)(x)\Phi(x, Y^{(x)})$$

for every Φ in the language of A_2, where

$$Y^{(x)} = \{z\colon J(z, x)\in Y\}$$

and J is a standard pairing function on natural numbers. A β-model is a model for which the notion of a well-ordering is absolute. β-models of second-order number theory correspond to transitive models of set theory. It is well-known that A_2 is mutually interpretable with $\mathrm{ZF}^-+\mathrm{V}=\mathrm{HC}$. This is usually proved by the method of hereditarily countable well-founded trees on ω. A reader unfamiliar with that subject might consult Kreisel [11] or Zbierski [19]. This result remains true when one adds the axiom of constructibility to both theories. In the arithmetical case this axiom was formulated by Addison [1]; we denote this sentence by CONSTR. We need the following model counterpart of these important results.

1.1. LEMMA (Zbierski). *$\mathfrak{M}$ is a β-model of A_2+CONSTR iff $\mathfrak{M} = L_\alpha\cap\wp(\omega)$ for a unique ordinal α such that $L_\alpha\models \mathrm{ZF}^-+\mathrm{V}=\mathrm{HC}$.*

The ordinal α in this lemma is just equal to $h(\mathfrak{M})$ (the height of $\mathfrak{M}$), i.e. the least ordinal not representable in $\mathfrak{M}$. We use L_α to denote the set of sets constructible before α.

2. Second-order arithmetic case. We begin by recalling some fundamental results on the ramified analytical hierarchy. This hierarchy was introduced by S. C. Kleene. It is instructive to regard it as the analytic counterpart of the constructible hierarchy in set theory. Let $\mathfrak{M}$ be a collection of reals. Then $D(\mathfrak{M})$ denotes the set of subsets of ω definable in $\langle\mathfrak{M}, \omega, 0, +, \cdot, \in\rangle$, possibly with parameters. Given a real X, define

$$\mathrm{RA}_0[X] = \{X\},$$
$$\mathrm{RA}_{\alpha+1}[X] = \mathrm{D}(\mathrm{RA}_\alpha[X]),$$
$$\mathrm{RA}_\lambda[X] = \bigcup\{\mathrm{RA}_\alpha[X]\colon \alpha\in\lambda\},$$
$$\mathrm{RA}[X] = \bigcup\{\mathrm{RA}_\alpha[X]\colon \alpha\in On\}.$$

By an easy cardinality argument one sees that there is an ordinal α such that $\mathrm{RA}[X] = \mathrm{RA}_\alpha[X] = \mathrm{RA}_{\alpha+1}[X]$. Define β_0^X to be the least such ordinal. For $X=\varnothing$, we omit writing X in all the above definitions and for $Y\in\mathrm{RA}$ we say that Y is *ramified analytical*. For $Y\in\mathrm{RA}[X]$ we say that Y is *ramified analytical from X*. The fundamental result, due to R. O. Gandy and H. Putnam, states that the collection $\mathrm{RA}[X]$ forms the minimal β-model of A_2 containing X as an element: see Boyd, Hensel, Putnam [3]. Moreover, for every $X\subseteq\omega$, β_0^X is countable and $h(\mathrm{RA}[X]) = \beta_0^X$. The level by level comparison of the ramified analytical hierarchy and the constructible hierarchy was made by G. Boolos in [2] and R. B. Jensen in his Habilitationschrift: If $\alpha\leqslant\beta_0^X$, then $\mathrm{RA}_\alpha[X] = L_\alpha[X]\cap\wp(\omega)$. Let us also recall that $\mathrm{RA}\models\mathrm{CONSTR}$, i.e. the axiom of constructibility holds in RA.

The following is the main question of the present section: Let $\mathfrak{M}$ be a β-model of A_2. Does $\mathfrak{M}$ have the form $\mathfrak{M} = \mathrm{RA}[X]$ for some $X\subseteq\omega$? In other words, we ask whether there is an $X\subseteq\omega$ such that $\mathfrak{M}$ is the minimal β-model of A_2 containing X as an element.

In the language of A_2 we can formalize the statement that a collection of reals is countable by saying that it can be encoded by a single real. This means that there is a real Y such that our collection is just equal to

$$\{Y^{(0)}, Y^{(1)}, Y^{(2)}, \ldots\}.$$

2.1. THEOREM. *Suppose that $\mathfrak{M}$ is a β-model of A_2+CONSTR. $\mathfrak{M} = \mathrm{RA}[X]$ for some $X\subseteq\omega$ iff $\mathfrak{M}\models$ "there exist at most countably many β-models of A_2+CONSTR".*

Proof. First assume that $\mathfrak{M}$ is a β-model of A_2+CONSTR such that $\mathfrak{M}\models$ "there are at most countably many β-models of A_2+CONSTR". By Zbierski's lemma 1.1, $\mathfrak{M} = L_\alpha\cap\wp(\omega)$. Consider

$$\gamma = \sup\{\beta<\alpha\colon L_\beta\cap\wp(\omega) \text{ is a } \beta\text{-model of } A_2\}.$$

Then γ is the supremum of the heights of all β-models of A_2+CONSTR which are encoded within $\mathfrak{M}$. By our assumptions this subset is countable in $\mathfrak{M}$. Therefore $\gamma<\alpha$, since the operation of supremum is well-defined in A_2. Let X be a well-ordering of ω of type γ belonging to L_α. Such a well-ordering does exist because $L_\alpha\models$ "every set is countable".

CLAIM 1. $L_\alpha[X]\models\mathrm{ZF}^-$.

Proof of Claim 1. In fact, $L_\alpha[X] = L_\alpha$ and $L_\alpha\models\mathrm{ZF}^-$. To prove this equality, note first that $L_\alpha\subseteq L_\alpha[X]$, by the definition of relative constructibility, and secondly that $L_\alpha[X]\subseteq L_\alpha$, because of the following, slightly more general well-known fact:

(*) If M is a transitive model of a reasonable fragment of set theory and $x\in M$, then $L_{h(M)}[x]\subseteq M$.

The proof of (*) follows by a simple application of the principle of transfinite induction. Also, the absoluteness of the formula $x\in L_\mu$ plays an important role. By a "reasonable fragment" we mean a fragment which allows us to prove (*). Clearly, the Kripke-Platek set theory is a reasonable fragment and ZF^- does contain KP.

CLAIM 2. *If* $\beta<\alpha$ *then* $L_\beta[X]$ *not* $\models ZF^-$.

Proof of Claim 2. Assume that the above does not hold and work for a contradiction. Assume that there is a $\beta<\alpha$ such that $L_\beta[X]\models ZF^-$. Then $\beta>\gamma$, since X is a real world well-ordering of type γ and one can prove that every well-ordering is isomorphic to an ordinal in ZF^-. Moreover, $L_\beta\models ZF^-$ because it is just the constructible part of model $L_\beta[X]$. Hence $\beta\geqslant\alpha$, by the choice of γ. A contradiction. Thus $L_\alpha[X]$ is the smallest transitive model of ZF^- containing X as an element. Therefore its analytic part $L_\alpha[X]\cap\wp(\omega)$ is the smallest β-model of A_2 containing X as an element. But $L_\alpha=L_\alpha[X]$ and $\mathfrak{M}=L_\alpha\cap\wp(\omega)$. Applying the Gandy–Putnam result on relative ramified analysis, we conclude that $\mathfrak{M}=RA[X]$. This completes the proof of one direction of the theorem.

Suppose now that $\mathfrak{M}$ is a β-model of A_2 and $\mathfrak{M}=RA[X]$ for some $X\subset\omega$. Assume that $\mathfrak{M}\models\neg$ "there exist at most countably many β-models of $A_2+CONSTR$". Again, $\mathfrak{M}=L_\alpha\cap\wp(\omega)$ by Zbierski's lemma 1.1. Clearly

$$\alpha=\sup\{\beta<\alpha:\ L_\beta\cap\wp(\omega)\text{ is a }\beta\text{-model of }A_2\}.$$

This means

$$\mathfrak{M}=\bigcup\{L_\beta\cap\wp(\omega):\ L_\beta\cap\wp(\omega)\text{ is a }\beta\text{-model of }A_2\}.$$

Consider an arbitrary $X\in\mathfrak{M}$. There is a $\beta<\alpha$ such that $X\in L_\beta\cap\wp(\omega)$ and $L_\beta\cap\wp(\omega)$ is a β-model of A_2 plus the axiom of constructibility. The latter β-model is essentially smaller than $\mathfrak{M}$. Therefore, we have $RA[X]\not\subseteq\mathfrak{M}$ by applying the Gandy–Putnam result once again. This completes the proof.

2.2. COROLLARY. *There is a countable β-model $\mathfrak{M}$ of $A_2+CONSTR$ which does not have the form $\mathfrak{M}=RA[X]$ for any $X\subseteq\omega$.*

Proof. It is sufficient to find an ordinal α such that $L_\alpha\models ZF^-+V=HC$ and

$$\alpha=\sup\{\beta<\alpha:\ L_\beta\models ZF^-+V=HC\}.$$

Then let $\mathfrak{M}=L_\alpha\cap\wp(\omega)$. It is fairly easy to prove that such ordinals do exist (we refer the reader to Marek and Srebrny [15], where such ordinals are called gap inaccessible). For example, take an $\alpha<\omega_1^L$ such that $L_\alpha\prec L_{\omega_1^L}$.

2.3. COROLLARY. *There are $\aleph_1^L$ constructibly countable β-models $\mathfrak{M}$ of $A_2+CONSTR$ which do not have the form $\mathfrak{M}=RA[X]$ for any $X\subset\omega$.*

Let us consider also β-models of A_2 in which the axiom of constructibility is false. The following result is an introductory one.

2.4. THEOREM. *There is a constructible real X such that* $RA[X]$ *not* $\models$ CONSTR.

Proof. Apply a Cohen forcing argument within the universe L. A Cohen generic real is then constructible, because we have made it inside L. (This real is marked "MADE IN L".) For instance, consider such an extension of L_{β_0}, $L_{\beta_0}[G]$. By the well-known forcing lemmas, $L_{\beta_0}[G]$ is the smallest transitive model of $ZF^-+V=HC$ containing L_{β_0} and containing G as an element. Therefore, $L_{\beta_0}[G]$ is the smallest transitive model of ZF^- containing G as an element. Hence, $L_{\beta_0}[G]\cap\wp(\omega)=RA[G]$, by the Gandy–Putnam result. Moreover, $RA[G]$ not $\models$ CONSTR, by Zbierski's lemma 1.1. Q.E.D.

A similar result was proved by Enderton in [5]. Namely, he proved that there is a β-model $\mathfrak{M}$ of A_2 such that $\mathfrak{M}\subset L$ but $\mathfrak{M}$ not $\models$ CONSTR, essentially by the same method.

It is much more difficult to answer our main question in the case of β-models not satisfying the axiom of constructibility. By the same reasoning as above one can prove the lemma below.

2.5. LEMMA. *Let $\mathfrak{M}$ be a β-model of A_2. $\mathfrak{M}=RA[X]$, for some $X\subset\omega$ iff $\mathfrak{M}$ is not a union of β-models properly included in $\mathfrak{M}$.*

However, there are β-models $\mathfrak{M}$ which are not of the form $RA[X]$ and are not unions of any family of β-models each encoded in $\mathfrak{M}$. For instance, one can extend the minimal β-model to a β-model which is the union of a family of β-models of the same height.

3. Arithmetically regular ordinals. In the previous section we studied β-models of A_2 and asked whether they have the form $RA[X]$. In the present section we consider ordinals which are the heights of β-models of A_2 and ask whether they have the form β_0^X. First, we reformulate some notions concerning β-models into the language of generalized recursion theory, in the sense of Kripke and Platek. The reader is referred to R. Platek [16] for recursion-theoretic intuitions and the development of this theory.

The main notion of generalized recursion is that of a recursively regular ordinal, called also admissible. An ordinal is recursively regular iff L_α is an admissible set, i.e., L_α models Kripke–Platek set theory, which consists of the following axioms: extensionality, regularity, pairing, union and the schemas of Δ_0-separation and Δ_0-collection. The name "recursively regular" is justified by the condition that no α-recursive function maps an α-bounded subset of α cofinally into α, where a function mapping α into α is said to be *α-recursive* iff its graph is Δ_1-definable in $\langle L_\alpha,\in\rangle$ a subset of α is said to be *α-bounded* iff there is a $\beta<\alpha$ greater than each member of that subset. Roughly, this means that α cannot be reached from below by any α-recursive function with an α-bounded domain. Following Platek [16], define a function mapping α into α to be α-arithmetical iff its graph is Σ_n-definable in $\langle L_\alpha,\in\rangle$ for some $n\in\omega$. This is natural, because arithmeticalness corresponds to first-order quantification followed by a primitive recursive predicate and Δ_0-ness is an analogue of primitive recursiveness. As is well known, functions Σ_1-definable in $\langle L_\alpha,\in\rangle$ are just α-recursively enumerable. Define α to be arithmetically regular iff it cannot be reached from below by any α-arithmetical function. It is easy to see that α is arithmetically regular just in case L_α models ZF^-. Define α to be locally countable iff $L_\alpha\models V=HC$. This property can also be expressed in the α-recursion-theoretic way. In fact, α is locally countable iff each $\beta<\alpha$ is α-projectable into ω, i.e., projectable by an α-finite function.

We would like to point out that these notions provide a link between α-recursion and β-models of analysis. This link is formed by Lemma 1.1, which — in this terminology — says that α is *arithmetically regular* and *locally countable* iff it is the height of a β-model of analysis plus the axiom of constructibility. Such ordinals were investigated in W. Marek and M. Srebrny [15]. They showed that these ordinals are just the ordinals starting the so called *gaps in the constructible universe*. This is justified by the following result of [15]: α is arithmetically regular and locally countable iff there is no new real in $L_{\alpha+1}$ and for each $\beta<\alpha$ there are reals in $L_\alpha - L_\beta$. A reader not familiar with [15] might adopt one of these equivalent conditions as the definition of the notion of ordinal starting a gap. Also, let us note that by some considerations of A. Zarach [18], an ordinal $\alpha\leqslant\omega_1$ is arithmetically regular iff it is the height of a β-model of A_2. The idea is to collapse α to ω by an argument of forcing with proper classes. If α is arithmetically regular, then such a generic extension of L_α forms a standard transitive model of $ZF^-+V = HC$. Its analytic part is a β-model of A_2 of height α.

The above-mentioned link between arithmetically regular ordinals and β-models of A_2 is much the same in nature as the link between admissible ordinals and relative hyperarithmeticalness. The most important feature of the latter is the well-known theorem of G. E. Sacks concerning the form of countable admissible ordinals. We give here some results tending in the same direction for the case of arithmetically regular ordinals.

3.1. THEOREM. *β_0 is the least arithmetically regular ordinal.*

Proof. L_{β_0} is the smallest transitive model of ZF^-.

The ordinals of the form β_0^X are natural analogues of the ordinals of the form ω_1^X, for $X\subset\omega$, which are just the least α recursively regular in X, i.e., $L_\alpha[X]$ models KP but $L_\beta[X]$ does not model KP for any $\beta<\alpha$. Compare the following with analogous results on ω_1^X, due to Kripke [12].

3.2. THEOREM. *For every $X\subset\omega$, β_0^X is arithmetically regular.*

Proof. Consider $RA[X]$. By the considerations of Section 2 of this paper it is clear that $RA[X] = L_{\beta_0^X}[X]\cap\wp(\omega)$ and $L_{\beta_0^X}[X]$ models ZF^-. Thus $L_{\beta_0^X} = ZF^-$, for it is the constructible part of $L_{\beta_0^X}[X]$. Q.E.D.

One can relativize the notion of an arithmetically regular ordinal. A function mapping α into α is said to be *α-arithmetical* in X iff its graph is Σ_n-definable in $L_\alpha[X]$ for some $n\in\omega$. The ordinals not reached from below by any function α-arithmetical in X are called *arithmetically regular* in X. Again, it is easy to see that α is arithmetically regular in X iff $L_\alpha[X]$ models ZF^-. It is worth mentioning that in the same way one can generalize the notion of an α-arithmetical function to an arbitrary transitive domain. In this way one comes to the transitive models of ZF^- as convenient structures to develop the theory of abstract arithmetical functions. They play the same role here as admissible sets in the theory of generalized recursive functions.

3.3. THEOREM. *Let $X\subset\omega$. β_0^X is the least ordinal arithmetically regular in X.*

Proof. We have to prove that $L_{\beta_0^X}[X]$ models ZF^- and $L_\gamma[X]$ does not model ZF^- for any $\gamma<\beta_0^X$. $L_{\beta_0^X}[X]\cap\wp(\omega) = RA[X]$ and the theorem follows immediately by the Gandy–Putnam result.

The converse theorem to 3.3 is the main question of this section. Given a countable arithmetically regular ordinal α, does α have the form $\alpha = \beta_0^X$, for some $X\subset\omega$? Once again we are faced with an analogy to the case of recursively regular ordinals. The theorem of Sacks says that every countable recursively regular α has the form $\alpha = \omega_1^X$. The reader might consult H. Friedman and R. Jensen [7]. Unfortunately the same proof does not work. The proof, due to Grilliot and Simpson, by the Omitting Types Theorem does not work in this case either. Both these proofs use the lemma that ω-models of ZF^- are closed under relative recursiveness. While trying to apply these proofs to our case, one is faced with the question of closure under relative ramified analyticalness, which is easy to be answered negatively [1].

Let us introduce the notion of arithmetical inaccessibility and gap inaccessibility.

3.4. DEFINITION. Let β_ξ be the monotonic enumeration of the ordinals which start gaps and let γ_ξ be the monotonic enumeration of the arithmetically regular ordinals. β_ξ is *gap inaccessible* iff $\xi = \beta_\xi$. γ_ξ is *arithmetically inaccessible* if $\xi = \gamma_\xi$. Thus α is *gap inaccessible* iff it starts a gap and is a limit of ordinals starting gaps. Similarly, an arithmetically regular α is *arithmetically inaccessible* iff it is a limit of arithmetically regular ordinals.

3.5. THEOREM. *If α is countable and arithmetically regular but not arithmetically inaccessible, then α has the form $\alpha = \beta_0^X$ for some $X\subset\omega$.*

Proof. Consider

$$\gamma = \sup\{\beta<\alpha:\ \beta \text{ is arithmetically regular}\}.$$

$\gamma<\alpha$, by our assumption. Let X be a Cohen generic real collapsing γ into ω. Then $L_\alpha[X]\models ZF^-$. We have to show that there is no $\beta<\alpha$ such that $L_\beta[X]\models ZF^-$. Then by the results of previous sections $L_\alpha[X]\cap\wp(\omega) = RA[X]$ and $L_\alpha[X]\models V = HC$. Therefore, $\alpha = h(RA[X]) = \beta_0^X$. So, assume that $\beta<\alpha$ and $L_\beta[X]\models ZF^-$ and work for a contradiction. By this assumption, L_β models ZF^-, for it is the constructible part of $L_\beta[X]$. But X is a characteristic function of a well-ordering of ω of type γ. Hence, $\beta>\gamma$, because every well-ordering is isomorphic to an ordinal in ZF^-. Thus $\beta\geqslant\alpha$, by the choice of γ. A contradiction. Q.E.D.

One can strenghten the conclusion of 3.5 by choosing X constructible. In fact, we did the above proof in the Zermelo–Fraenkel set theory, and so one can do the same within the constructible universe. All the notions used in the above proof are absolute for L. The following result is a complementary one to 3.5.

[1] After completing the paper the author was informed by Professor G. E. Sacks that, actually, every countable arithmetically regular ordinal has the form β_0^X for some $X\subseteq\omega$. This result is proved in G. E. Sacks "F-recursiveness", in the Logic Colloquium 69 (1971).

3.6. THEOREM. *Let α be arithmetically regular. α has the form $\alpha = \beta_0^X$ for some $X \in \wp(\omega) \cap L_\alpha$, iff α is locally countable and not arithmetically inaccessible.*

Proof. This is an immediate consequence of 2.1.

3.7. COROLLARY. *Let α be a gap ordinal. Then α has the form $\alpha = \beta_0^X$ for some $X \in \wp(\omega) \cap L_\alpha$ iff α is not gap inaccessible.*

Let us also mention two facts following from 2.1.

3.8. COROLLARY. *Let $\mathfrak{M}$ be a β-model of A_2+CONSTR. Then $\mathfrak{M} = \mathrm{RA}[X]$ for some $X \subset \omega$, iff $h(\mathfrak{M})$ is not gap inaccessible.*

3.9. COROLLARY. *Let $X \subset \omega$. If $\mathrm{RA}[X] \models$ CONSTR, then β_0^X is not gap inaccessible.*

Remark. The converse implication does not hold. For instance, β_0 is not gap inaccessible, while the proof of Theorem 2.4 gives us a real G such that $\beta_0^G = \beta_0$ and $\mathrm{RA}[G]$ not $\models$ CONSTR.

We close this section with some considerations on relatively constructible transitive models of ZF. These considerations are actually similar in nature to the previous ones. We consider the following question of G. E. Sacks: Let L_α be a countable model of ZF. Can one always find a real X such that $L_\alpha[X] \models$ ZF but for no $\beta<\alpha$ $L_\beta[X] \models$ ZF? The word "always" is important here, because there are some particular cases which were, most probably, known to Sacks. This paper has arrisen from the author's unsuccessful attempts to answer this question. This section has so far been devoted to considerations of a slightly easier version of this question. Namely, for L_α a model of ZF^-. However the ZF case is very different.

Our result in the ZF case is the following version of Theorem 2.1.

3.10. LEMMA. *Let L_α be a countable model of ZF. Then there is a real $X \in L_\alpha$ required by Sacks iff $L_\alpha \models$ "there exist at most countably many transitive models of ZF+V = L".*

The following three results are due to A. R. D. Mathias (private communication).

3.11. THEOREM. *Suppose that $\alpha<\omega_1$ and $L_\alpha \models$ ZF. If $L_\alpha \models$ "the collection of all transitive models of ZF+V = L is bounded", then there is a real required by Sacks.*

Proof. Take a Cohen generic real collapsing the ordinal

$$\gamma = \sup\{\beta<\alpha\colon L_\beta \models \mathrm{ZF}\}<\alpha. \quad \text{Q.E.D.}$$

3.12. THEOREM. *If $L_\alpha \models$ ZF and $\alpha = \sup\{\beta<\alpha\colon L_\beta \models \mathrm{ZF}\}$, then no such X can be added by a forcing extension of L_α.*

Proof. Suppose that there is such an X added by a forcing extension of L_α. Then there is a $\boldsymbol{B} \in L_\alpha$ such that $\boldsymbol{B}$ is a complete Boolean algebra in L_α, and $L_\alpha[X] = L_\alpha^{\boldsymbol{B}}/F$, where F is some L_α-complete filter on $\boldsymbol{B}$. Now let $\beta<\alpha$ be such that $L_\beta \models$ ZF, $\boldsymbol{B} \in L_\beta$ and $P(\boldsymbol{B})^{L_\alpha} \in L_\beta$. Then $L_\beta^{\boldsymbol{B}}/F = L_\beta[X]$, so $L_\beta[X] \models$ ZF. Q.E.D.

3.13. Remark. On the other hand, let $\alpha<\omega_1$ be the least ordinal such that $L_\alpha[O^\#] \models$ ZF. Then

$$\sup\{\beta<\alpha\colon L_\beta \models \mathrm{ZF}\} = \alpha .$$

Finally let us point out that the proof of Theorem 3.12 makes essential use of the power set axiom.

4. Uncountable case. In this section we consider the forty sixth problem of H. Friedman [6]. He conjectures that. If $\omega_1<\alpha<\omega_2$ is admissible, then α is the first ordinal admissible in some $x \subset \omega_1$. We do not prove this conjecture in full generality. We only have a partial answer to this question in some simple cases. We use once again the same reasoning as in the previous sections of this paper.

We say that α is *admissible in* x iff $L_\alpha[x]$ models the Kripke–Platek set theory. By $A \prec_1 B$ we mean that $\langle A, \in\rangle$ is a Σ_1-elementary submodel of $\langle B, \in\rangle$, i.e., $A \subseteq B$ and for every Σ_1-formula Φ and $a_1, \ldots, a_k \in A$

$$A \models \Phi[a_1, \ldots, a_k] \Leftrightarrow B \models \Phi[a_1, \ldots, a_k] .$$

Define α to be locally of power $\aleph_\mu$ iff $L_\alpha \models$ "every set is of cardinality $\leqslant \aleph_\mu$", where $\aleph_\mu$ is defined, as usual, to be the μth infinite cardinal. α is said to be *projectable into* β iff there is a total α-recursive one-one function into β.

4.1. LEMMA. *Suppose that α is admissible but not recursively inaccessible. Let $\aleph_\mu^{L_\alpha}$ be the $\aleph_\mu$ in the sense of $\langle L_\alpha, \in\rangle$. Then α is locally of power $\aleph_\mu$ iff α is projectable into $\aleph_\mu^{L_\alpha}$.*

4.2. THEOREM. *Assume* V = L. *Suppose $\omega_1<\alpha$. Then α is the first ordinal admissible in some $x \in \wp(\omega_1) \cap L_\alpha$ iff*

(a) *α is not recursively inaccessible,*

(b) *α is locally of power $\aleph_1$.*

Proof. First, we prove the implication from the left to the right. Suppose that α is the first ordinal admissible in $x \in \wp(\omega_1) \cap L_\alpha$. To prove (a) assume that α is recursively inaccessible. Then there is an admissible ordinal $\gamma<\alpha$ such that $x \in L_\gamma$. But $L_\gamma = L_\gamma[x]$, by (*) in Section 2. Then α cannot be the first ordinal admissible in x, contrary to our assumption.

Now we prove that α is locally of power $\aleph_1$. Suppose that it is not, i.e. that $L_\alpha \models \mathrm{V} \neq H_{\aleph_2}$. Consider $(H_{\aleph_2})^{L_\alpha}$, the collection of sets of hereditary cardinality $\leqslant \aleph_1$ within L_α, i.e.

$$\{x \in L_\alpha\colon L_\alpha \models (\mathrm{E}f)[\mathrm{Func}(f) \,\&\, f\colon \aleph_1^{L_\alpha} \xrightarrow{\text{onto}} TC(x)]\} .$$

CLAIM 1. $\langle (H_{\aleph_2})^{L_\alpha}, \in\rangle \prec_1 \langle L_\alpha, \in\rangle$.

This is just the well-known lemma of Lévy [13], but proved in KP instead of ZF. A proof is essentially the same as Lévy's original proof with the exception that in our case $(H_{\aleph_2})^L$ is not proved to be a set of L_α yet. However, we are in a slightly better situation here, because we can additionally use the axiom of constructibility. The proof below follows some ideas of D. Guaspari.

Proof of Claim 1. Let $\Phi(x_1, \ldots, x_n)$ be a Σ_1-formula and $y_1, \ldots, y_n \in (H_{\aleph_2})^{L_\alpha}$. W.l.o.g. we can assume that Φ has the form $(\mathrm{E}x)\, \Psi(x, x_1, \ldots, x_n)$ for a Δ_0-formula Ψ. Suppose $\langle L_\alpha, \in\rangle \models (\mathrm{E}x)\, \Psi(x, \bar{y})$ and let y be a witness for Ψ. There is a $\delta<\alpha$ such that $y \in L_\delta$ and $TC(\{y_1, \ldots, y_n\}) \in L_\delta$. Then, by the absoluteness of Δ_0-formulas for transitive models, we get $\langle L_\delta, \in\rangle \models (\mathrm{E}x)\, \Psi(x, \underline{y}_1, \ldots, \underline{y}_n)$. Now consider the hull of $TC(\{y_1, \ldots, y_n\})$ for the skolem functions of formula Φ and extensionality in $\langle L_\delta, \in\rangle$. Since the number of these functions is finite and all of them are definable, the hull belongs to L_α. Collapse the hull. It is easy to see that the collapse is of cardinality $\leqslant \aleph_1$ in L_α. Again by the absoluteness we can pick up another witness y' for Ψ belonging to the collapse. Thus we have a witness of cardinality $\leqslant \aleph_1$ in L_α and the proof is complete.

Claim 2. *$(H_{\aleph_2})^{L_\alpha}$ is transitive.*

Proof of Claim 2. Take an arbitrary $a \in (H_{\aleph_2})^{L_\alpha}$. We have to show that $a \subset (H_{\aleph_2})^{L_\alpha}$. But there is a function $f\colon \aleph_1^{L_\alpha} \xrightarrow{\text{onto}} a$ in L_α. This is a Σ_1-statement with parameters $\aleph_1^{L_\alpha}$ and a, both belonging to $(H_{\aleph_2})^{L_\alpha}$. Then it holds in $(H_{\aleph_2})^{L_\alpha}$ by Claim 1. Let $f \in (H_{\aleph_2})^{L_\alpha}$ be such a function. The same f works in L_α as well. If $\not\subset (H_{\aleph_2})^{L_\alpha}$, then f would not be "onto" in L_α. This completes the proof of the claim.

Now, $(H_{\aleph_2})^{L_\alpha} = L_\gamma$ for some $\gamma<\alpha$, by the condensation lemma. Simply γ plays the role of $\aleph_2$ within $\langle L_\alpha, \in\rangle$. Moreover, $x \in L_\gamma$, since x is of L_α-cardinality $\aleph_1$. Also, L_γ is admissible, as a transitive Σ_1-elementary submodel of an admissible set. Then $L_\gamma[x]$ is admissible, because it is equal to L_γ, by (*) in Section 2. This contradicts the assumption that α is the first ordinal admissible in x. This completes the proof of this direction.

We now proceed from the right to the left. Assume (a) and (b). Consider $\gamma = \sup\{\beta<\alpha\colon L_\beta \models \mathrm{KP}\}$. $\gamma<\alpha$, since α is not recursively inaccessible. Then there is a well-ordering $x \subset \omega_1$ of type γ in L_α. We identify x with the relation $\{\langle\mu, \nu\rangle\colon J(\mu, \nu) \in x\}$, where J is a standard pairing function on countable ordinals. The existence of such a well-ordering follows easily by (b). This x does work. In fact, $L_\alpha[x] \models \mathrm{KP}$, because $L_\alpha[x] = L_\alpha$. It remains to show that if $\beta<\alpha$ then $L_\beta[x]$ not $\models \mathrm{KP}$. Assume that $\beta<\alpha$ and $L_\beta[x] \models \mathrm{KP}$. Then $L_\beta \models \mathrm{KP}$, because it is the constructible part of $L_\beta[x]$. Therefore $\gamma>\beta$, because γ is an upper bound of ordinals with these properties in L_α. But x is a real well-ordering and therefore it is isomorphic to an ordinal in L_β. Hence $\beta>\gamma$. A contradiction. Q.E.D.

4.3. Corollary (Perhaps $V \neq L$). *Suppose that $\alpha>\omega_1^L$ is admissible. Then α is the first ordinal admissible in some $x \in \wp(\omega_1^L) \cap L_\alpha$ iff*

(a) *α is not recursively inaccessible,*

(b) *α is locally of power $\aleph_1$.*

The rest of this section is devoted to the listing of several results of the same kind as 4.2 and 4.3. The reader can modify the proof of 4.2 to obtain these results. By $\varkappa^+$ we mean the least cardinal, i.e. the least initial ordinal, greater than $\varkappa$, and by $\varkappa^{+L}$ we mean the least L-cardinal greater than $\varkappa$.

4.4. Theorem. *Assume $V = L$. Suppose that $\varkappa$ is a cardinal and $\alpha>\varkappa$ is admissible. Then α is the first ordinal admissible in some $x \in \wp(\varkappa) \cap L_\alpha$ iff*

(a) *α is not recursively inaccessible,*

(b) *α is locally of power $\varkappa$.*

4.5. Corollary (Perhaps $V \neq L$). *Suppose that $\varkappa$ is an L-cardinal and $\alpha>\varkappa$ is admissible. Then α is the first ordinal admissible in some $x \in \wp(\varkappa) \cap L_\alpha$ iff*

(a) *α is not recursively inaccessible,*

(b) *α is locally power $\varkappa$.*

The remaining cases of the forty sixth question of H. Friedman are left open. They seem to be really difficult at the present state of knowledge about uncountable models. However, it seems possible to use Devlin [4] generalization of Martin's axiom to solve these problems. In fact, that form of Martin's axiom enables us to carry over the Cohen generic extension construction to an uncountable case. We feel that this can be the way leading to the uncountable analogue of Sacks' theorem. This would be complete positive answer to Friedman's question but with the help of an additional axiom. We put this off to future research.

One can re-prove the above results for the case of uncountable arithmetically regular ordinals, and also for Σ_n-admissible ordinals. Note also the following results on uncountable models of ZF and on uncountable stable ordinals.

4.6. Theorem. *Suppose that $\varkappa$ is an L-cardinal and $\alpha>\varkappa$. If $L_\alpha \models \mathrm{ZF}+$ "there exist at most $\varkappa$ transitive models of $\mathrm{ZF}+V = L$", then there is an $x \subset \varkappa$ such that $L_\alpha[x]$ is the smallest transitive model of ZF containing x as an element.*

4.7. Theorem. *Suppose $\varkappa \leqslant \sigma < \varkappa^{+L}$ and let σ' denote the first stable ordinal greater than σ. Then σ' is the first ordinal stable in some $x \subset \varkappa$. In particular, take $\varkappa = \omega_1$.*

4.8. Lemma. *If σ is the first ordinal stable in some $x \in L_\sigma$, then σ is not limit in the monotonic enumeration of stable ordinals.*

4.9. Lemma. *If σ is stable in x, then σ is stable.*

4.10. Corollary. *If $\varkappa$ is an L-cardinal and every stable σ of L-cardinality $\varkappa$ is the first stable ordinal in some $x \subset \varkappa$, then $\wp(x) \not\subset L$.*

Proof. Consider a stable ordinal $\sigma = \bigcup\{\sigma_\xi\colon \xi<\lambda\}$ where σ_ξ are also stable. Suppose that there is an $x \subset \varkappa$ such that σ is the first ordinal stable in x. If x is constructible, then $L[x] = L$ and so $L_\sigma[x]$ satisfies the axiom of constructibility. Since it is a transitive model of height σ, we have $L_\sigma[x] = L_\sigma$. Thus $x \in L_{\sigma_\xi}$ for some $\xi<\lambda$. Moreover, $L_{\sigma_\xi}[x] \prec_1 L[x]$, which contradicts the minimality of σ. Q.E.D.

The converse to 4.10 remains an open question.

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A geometric filtration of $\mathfrak{N}_*^{Z_2}$

by

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Abstract. If a is the cobordism class of a manifold with involution, $e(a)$ is defined to be the smallest integer n such that a representative of a can be Z_2-equivariantly embedded in R^{n+s}, for some s, where Z_2 acts on R^{n+s} by multiplying the first n coordinates by -1. Z_2-cobordism classes α_k, β_k are exhibited such that $e(\alpha_k) = e(\beta_k)$ but $e(\alpha_k+\beta_k) = e(\alpha_k)-k$, for arbitrarily large integers k.

Let $\mathfrak{N}_*^{Z_2}$ be the cobordism ring of manifolds with involutions. Let z_n denote RP^n with the Z_2-action given in homogeneous coordinates by $[x_0, x_1, \ldots, x_n] \to [-x_0, x_1, \ldots, x_n]$. Define $\Gamma\colon \mathfrak{N}_*^{Z_2} \to \mathfrak{N}_{*+1}^{Z_2}$ by

$$\Gamma[M, T] = \left[\frac{M\times S^1}{(m, z)\sim(Tm, -z)},\ [m, z]\to[m, \bar{z}]\right].$$

Then $\{1\} \cup \{(\Gamma^i z_{n_1}) z_{n_2} \ldots z_{n_k} |\ i \geqslant 0, k \geqslant 1, n_1 \geqslant n_2 \geqslant \ldots \geqslant n_k > 1\}$ is a set of generators for $\mathfrak{N}_*^{Z_2}$ as an $\mathfrak{N}_*$-module (Alexander [1], Stong [7]), where Γ^0 denotes the identity map.

To define a geometric filtration of $\mathfrak{N}_{i+j}^{Z_2}$, let $F(i, j)$ be the set of Z_2-cobordism classes having representatives which, for some s, can be Z_2-equivariantly embedded in R^{i+s}, furnished with the Z_2-action $(x_1, \ldots, x_{i+s}) \to (-x_1, \ldots, -x_i, x_{i+1}, \ldots, x_{i+s})$. It is known that $(\Gamma^i z_{n_1}) z_{n_2} \ldots z_{n_k} \in F(i+n_1+n_2+\ldots+n_k, 0)$ and $\notin F(i+n_1+n_2+\ldots+n_k-1, 1)$ (Bix [2]). But even if $a, b \in F(i, j)$ and $\notin F(i-1, j+1)$, it is possible that $a+b \in F(i-k, j+k)$ for some $k>0$. The main result of this paper is that such drops in dimension occur with k arbitrarily large.

THEOREM. *$z_n^2 + z_{n+1} z_{n-1} + \Gamma^{n+1} z_{n-1} + \Gamma^{n-1} z_{n+1} \in F(n+1, n-1)$ and $\notin F(n, n)$, while $z_n^2 + z_{n+1} z_{n-1}$ and $\Gamma^{n+1} z_{n-1} + \Gamma^{n-1} z_{n+1} \in F(2n, 0)$ and $\notin F(2n-1, 1)$, for all $n \geqslant 3$.*

Proof. The classifying map of the normal bundle to the fixed-point set of a manifold with involution defines a monomorphism $i\colon \mathfrak{N}_*^{Z_2} \to \bigoplus_{k=0}^{*} \mathfrak{N}_{*-k}(BO(k)) \cong \mathfrak{N}_*[x_0, x_1, \ldots]$, where x_n is the bordism class of the canonical line bundle over RP^n (Boardman [3], [4], Conner and Floyd [5]). We identify a manifold with involution with the image under i of its Z_2-cobordism class. So $z_n = x_{n-1} + x_0^n$. And

$$\Gamma^i z_n = x_{n-1} x_0^i + x_0^{n+i} + [z_n]_2 x_0^i + [\Gamma z_n]_2 x_0^{i-1} + [\Gamma^2 z_n]_2 x_0^{i-2} + \ldots + [\Gamma^{i-1} z_n]_2 x_0,$$