EXAMPLE 4.6. Let M' denote the reflection of M in Example 4.4 in the line $x = 2/\pi$, and let $H = M \cup M'$. A pattern for H is $(2, 1, 4, 3)$.

EXAMPLE 4.7. $(1, 3, 1)$ is a pattern for a well-known indecomposable continuum with only one endpoint (see [4], p. 332, Figure 8–6).

EXAMPLE 4.8. The union of two copies of Example 4.7 joined at their endpoints is used by Bing as an example of a chainable continuum with no endpoint ([1], p. 662, Example 7). A pattern is $(3, 1, 3, 5, 3)$.

EXAMPLE 4.9. $(2, 3, 1)$ is a pattern for an indecomposable continuum with three endpoints, which is chainable (and hence irreducible) between any two of them (compare with [4], p. 142).

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Extensions, retracts, and absolute neighborhood retracts in proper shape theory

by

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Abstract. The notion of an *extension* of a proper fundamental net is defined and studied. Various results concerning this notion are obtained; these include a homotopy extension theorem and results relating the idea of extension to the concept of proper fundamental retraction. We also define absolute neighborhood proper shape retract (ANPSR), and show that the property of being an ANPSR is a hereditary proper shape invariant.

1. Introduction. In [5] Borsuk introduced the notions of *fundamental retract*, the *extension* of a fundamental sequence, *fundamental absolute retract* (FAR), and *fundamental absolute neighborhood retract* (FANR) for compacta in the Hilbert cube Q. These ideas were later studied by Mardešić [13] for compact Hausdorff spaces using the ANR-system approach to shape theory developed by Mardešić and Segal [14]. In [15], Patkowska proved the important homotopy extension theorem for fundamental sequences on compacta in Q, and this result was then used by Borsuk [6] to show that the property of being an FANR-space is a hereditary shape invariant. Results similar to these have recently been obtained for the shape theory due to Fox [9] by Godlewski ([10], [11], [12]). In a seminar at the University of Georgia during the spring of 1974, Godlewski presented an example to show that similar results do not hold in the theory of shape for metrizable spaces described by Borsuk in [7], [8]. (It was this example and its implications which, to a degree, stimulated the ideas that led to this paper.)

In [1], Ball introduced the notions of *proper fundamental retract* and *absolute proper shape retract* (APSR), which are in some sense the natural analogues in *proper* shape theory ([2], [3]) of Borsuk's fundamental retract and FAR. It is our purpose in the present paper to introduce and study the concepts of *extension* of a proper fundamental net and of *absolute neighborhood proper shape retract* (ANPSR). Perhaps it should be now noted that the notion of extension studied here is not an exact word-for-word carry over into the proper shape theory of the extension of a fundamental sequence; indeed, as noted in Section 2, the precise carry over would not yield the main results here established, notably the (proper) homotopy extension theorem (Theorem 4.1) which yields the fact that the property of being an ANPSR is a hereditary proper shape invariant (Theorem 6.5). Theorems relating the ideas of proper fundamental retraction and the extension of a proper fundamental net appear in Sections 2 and 5. Section 4 is occupied by the proof of the homotopy extension theorem, while Section 3 contains some technical results which are used in Section 4 but may be of value elsewhere. In Section 6 the foundations are layed for the study of the ANPSR's.

Our terminology and notation is that of [2], [3] and, to avoid a gross repetition of what appears there, we assume familiarity with those papers. For convenience, though, we recall the following: If X and Y are closed subsets of the spaces M and N, respectively, then a *proper fundamental net*, $\underline{f} = \{f_\lambda|\ \lambda \in \Lambda\}$: $X \to Y$ in (M, N), is a family $\{f_\lambda\colon M \to N\}_{\lambda \in \Lambda}$ of maps, indexed by the directed set Λ, such that if V is a closed neighborhood of Y in N, then there exist a closed neighborhood U of X in M and an index $\lambda_0 \in \Lambda$ such that if $\lambda \geqslant \lambda_0$, then $f_\lambda | U \underset{p}{\simeq} f_{\lambda_0} | U$ in V. Here, and throughout, $\underset{p}{\simeq}$ stands for "is properly homotopic to" and shall refer either to maps or proper fundamental nets, context supplying the correct interpretation.

Our primary interest, here and throughout the study of proper shape theory, centers on locally compact separable metric spaces. Therefore, while remarking that our results can be extended in an obvious way to the (non-proper) shape theory based on nets which is briefly mentioned in [2], Section 3, and or to nonseparable local compacta as outlined in [2], Section 5, we now call the readers attention to the following standing hypothesis: Henceforth, all spaces considered in this paper shall be *locally compact*, *separable*, and *metrizable*. In particular $X \in$ ANR ($X \in$ AR) means that X is a locally compact, separable absolute neighborhood retract (absolute retract) for metric spaces.

2. Extensions and restrictions of proper fundamental nets. Suppose that M and N are AR's, $X \subset X'$ are closed subsets of M, Y is a closed subset of N, and $\underline{f} = \{f_\lambda|\ \lambda \in \Lambda\}$: $X \to Y$ is a proper fundamental net in (M, N). Then the proper fundamental net $\underline{f}^* = \{f_\delta^*|\colon \delta \in \Delta\}$: $X' \to Y$ in (M, N) is an *extension* of $\underline{f}$ (alternatively, $\underline{f}$ is a restriction of $\underline{f}^*$) if for each $\delta \in \Delta$ there exists $\lambda(\delta) \in \Lambda$ such that

(1) if $\delta \leqslant \delta' \in \Delta$, then $\lambda(\delta) \leqslant \lambda(\delta')$,

(2) $\{\lambda(\delta)|\ \delta \in \Delta\}$ is cofinal in Λ, and

(3) for all $\delta \in \Delta$, $f_\delta^* | X = f_{\lambda(\delta)} | X$.

We also say that $\underline{f}$ *extends* to $\underline{f}^*$ if $\underline{f}^*$ is an extension of $\underline{f}$.

The above definition may not seem particularly natural, in view of the results of [5]; more precisely, our initial solution to the problem of defining "extension" for proper fundamental nets, based on [5], would be (paraphrased) as follows: $\underline{f}^* = \{f_\lambda^*|\ \lambda \in \Lambda\}$: $X' \to Y$ is an "extension" of $\underline{f} = \{f_\lambda|\ \lambda \in \Lambda\}$: $X \to Y$ if for all $\lambda \in \Lambda$, $f_\lambda^* | X = f_\lambda | X$. Note that this is quite restrictive inasmuch as it requires the *same* indexing set for $\underline{f}$ and $\underline{f}^*$, and that this causes certain desirable results to fail ipso facto. For example, if $\underline{r}\colon X' \to X$ is a proper fundamental retraction (as defined below) and $\underline{f}\colon X \to Y$ is a proper fundamental net, then $\underline{f}\underline{r}$ (when this composition is defined) would not generally be an extension of $\underline{f}$, nor would $\underline{r}$ be an extension of $\underline{i}_X$ (cf. Theorems 5.1 and 2.15). In addition, the important homotopy extension theorem (Theorem 4.1) would fail, as the reader may easily verify. Our definition, while weaker, retains enough of the features of the "restrictive" one given above to make good geometric sense, and it seems as though there is little to be lost by adopting it.

There is, however, one potential area of dificulty to be considered. This involves the case in which X' and Y are compact and $\underline{f}$ is a fundamental *sequence* from X to Y; for, as observed in [2], Lemma 3.13, $\underline{f}$ is also a proper fundamental net, while Borsuk [5], Section 1 and [8], Section 17 has defined an "extension" of $\underline{f}$ to be a fundamental sequence from X' to Y essentially satisfying the "restrictive" definition of the preceding paragraph. This has proven to be very fruitful as regards the theory of shape for compacta. However, the two concepts are actually equivalent in this case, as shown by the following result.

(2.1.) THEOREM. *Suppose $X \subset X'$ and Y are compacta such that $X' \subset M \in$ AR and $Y \subset N \in$ AR, and suppose further that $\underline{f} = \{f_i\}_{i=1}^\infty$: $X \to Y$ is a fundamental sequence in (M, N). Then $\underline{f}$ extends to the proper fundamental net $\underline{f}^*$: $X' \to Y$ in (M, N) if and only if there exists a fundamental sequence $\underline{f}' = \{f_i'\}_{i=1}^\infty$: $X' \to Y$ in (M, N) such that $f_k' | X = f_k | X$ for $k = 1, 2, \ldots$*

Proof. The sufficiency of the given condition is apparent, since a fundamental sequence $\underline{f}'$ as in the statement of the theorem is immediately seen to be a proper fundamental net extending $\underline{f}$. Assume then, for the converse, that $\underline{f}^* = \{f_\delta|\ \delta \in \Delta\}$: $X' \to Y$ is a proper fundamental net in (M, N) extending $\underline{f}\colon X \to Y$. Let $V_1 \supset V_2 \supset \ldots$ be open neighborhoods of Y in N such that

(2.2) if V is a neighborhood of Y in N, then $V \supset V_k$ for some k.

Since N is normal and $\underline{f}^*$ is a proper fundamental net, for each positive integer k there exists an index $\delta_k \in \Delta$ and a closed neighborhood U_k of X' in M such that

(2.3) if $\delta \geqslant \delta_k$, then $f_\delta^* | U_k \simeq f_{\delta_k}^* | U_k$ in V_k.

Clearly, we may also assume that

(2.4) $U_1 \supset U_2 \supset \ldots$ and $\delta_1 \leqslant \delta_2 \leqslant \ldots$

Letting Λ denote the set of positive integers, directed by $\leqslant$, and using the notation of the definition of "extension", we may also assume, by conditions (1) and (2) of the definition, that

(2.5) if $i \geqslant \lambda(\delta_k)$, then $f_i | X \simeq f_{\lambda(\delta_k)} | X$ in V_k, and

(2.6) $\lambda(\delta_1) < \lambda(\delta_2) < \ldots$

By (2.6), we may let $f_i' = f_{\delta_k}^*$ for $i = \lambda(\delta_k)$, and in this way we have defined a "skelton" of the sequence $\{f_i'\}_{i=1}^\infty$. We note that if $i = \lambda(\delta_k)$, then

$$f_i' | X = f_{\delta_k}^* | X = f_{\lambda(\delta_k)} | X = f_i | X.$$

The skelton may now be "fleshed out" to a full sequence $\{f_i'\}_{i=1}^\infty$ such that

(2.7) $f_i' | X = f_i | X$ for $i = 1, 2, \ldots$, and

(2.8) if $i \geqslant \lambda(\delta_k)$, then $f_i'|U_k \simeq f_{\delta_k}^*|U_k$ in V_k.

(This fleshing out is accomplished using (2.3), (2.4), (2.5) and the technique of the proof of [15], Theorem; or, what amounts to the same thing, using the technique of the proof of Lemma 3.2 of the current paper.) By (2.2) and (2.8), $\underline{f}' = \{f_i'\}_{i=1}^{\infty}: X' \to Y$ is a fundamental sequence in (M, N) and, by (2.7), the proof is complete.

The following observation is immediate.

(2.9) *If $X \subset X'$ are closed subsets of $M \in$ AR, Y is a closed subset of $N \in$ AR, and $\underline{f}^* = \{f_\delta^*| \ \delta \in \Delta\}: X' \to Y$ is a proper fundamental net in (M, N), then $\underline{f}^*|X = \{f_\delta^*| \ \delta \in \Delta\}: X \to Y$ is a proper fundamental net in (M, N) and $\underline{f}^*|X$ is a restriction of $\underline{f}^*$.*

The relationship between $\underline{f}^*|X$ and an arbitrary restriction of $\underline{f}^*$ to X is given by the following

(2.10) THEOREM. *Suppose $X \subset X'$ are closed subsets of $M \in$ AR, Y is a closed subset of $N \in$ AR, and $\underline{f}: X \to Y$ is a proper fundamental net in (M, N) which extends to the proper fundamental net $\underline{f}^*: X' \to Y$ in (M, N). Then $\underline{f} \underset{p}{\simeq} \underline{f}^*|X$.*

Proof. Suppose $\underline{f} = \{f_\lambda| \ \lambda \in \Lambda\}$ and $\underline{f}^* = \{f_\delta^*| \ \delta \in \Delta\}$ are as given in the hypothesis. Let V be a closed neighborhood of Y in N. Choose W to be a closed neighborhood of Y such that W lies in the interior of V. Then, since $\underline{f}$ is a proper fundamental net, there exist an index $\lambda_1 \in \Lambda$ and a closed neighborhood U_1 of X in M such that

(2.11) if $\lambda \geqslant \lambda_1$, then $f_\lambda|U_1 \underset{p}{\simeq} f_{\lambda_1}|U_1$ in W.

Since $\underline{f}^*|X$ is a proper fundamental net, there exist an index $\delta_1 \in \Delta$ and a closed neighborhood U_1^* of X in M such that

(2.12) if $\delta \geqslant \delta_1$, then $f_\delta^*|U_1^* \underset{p}{\simeq} f_{\delta_1}^*|U_1^*$ in W.

Since $\{\lambda(\delta)| \ \delta \in \Delta\}$ is cofinal in Λ, there exists $\delta_2 \in \Delta$ such that $\lambda(\delta_2) \geqslant \lambda_1$. Let $\delta_3 \in \Delta$ be chosen so that $\delta_3 \geqslant \delta_2$ and $\delta_3 \geqslant \delta_1$. Recall that $\delta_3 \geqslant \delta_2$ implies $\lambda(\delta_3) \geqslant \lambda(\delta_2)$, and thus $\lambda(\delta_3) \geqslant \lambda_1$. Finally, noting that $f_{\delta_3}^*|X = f_{\lambda(\delta_3)}|X$ and that $f_{\delta_3}^*(X) \subset W$, it follows from [2], Lemma 3.4 that there exists a closed neighborhood U_2 of X such that

(2.13) $f_{\delta_3}^*|U_2 \underset{p}{\simeq} f_{\lambda(\delta_3)}|U_2$ in V.

Now, letting $\lambda_0 = \lambda(\delta_3)$, $\delta_0 = \delta_3$, and $U = U_1 \cap U_1^* \cap U_2$, we conclude, using (2.11), (2.12), and (2.13), that

(2.14) if $\lambda \geqslant \lambda_0$ and $\delta \geqslant \delta_0$, then $f_\lambda|U \underset{p}{\simeq} f_\delta^*|U$ in V.

This, of course, implies that $\underline{f} \underset{p}{\simeq} \underline{f}^*|X$, completing the proof.

We now recall some definitions from [1]. Suppose $X \subset X'$ are closed subsets of $M \in$ AR. Then a *proper fundamental retraction* of X' to X in (M, M) is a proper fundamental net $\underline{r} = \{r_\alpha| \ \alpha \in A\}: X' \to X$ in (M, M) such that $r_\alpha(x) = x$ for all $x \in X$ and $\alpha \in A$. A closed subset Y of the space Y' is a *proper shape retract* of Y' provided there exist a space $M \in$ AR, an embedding (understood to be *closed*) $h: Y' \to M$, and a proper fundamental retraction $\underline{r}: h(Y') \to h(Y)$ in (M, M). As shown by [1], Theorem 4.2 the property of a space being a proper shape retract of another is not dependent on a particular choice of h and/or M. The connection between this concept and that of the extendibility of a proper fundamental net is given by the following result. Recall that if X is a closed subset of $M \in$ AR, then $\underline{i}_X = \{\mathrm{id}_M\}: X \to X$ in (M, M).

(2.15) THEOREM. *Suppose $X \subset X'$ are closed subsets of $M \in$ AR. Then X is a proper shape retract of X' if and only if $\underline{i}_X: X \to X$ in (M, M) extends to a proper fundamental net $\underline{r}: X' \to X$ in (M, M).*

Proof. Suppose first that X is a proper shape retract of X'. Then there exists a proper fundamental retraction $\underline{r}: X' \to X$ in (M, M). It is evident that $\underline{r}$ is an extension of $\underline{i}_X$.

Conversely, suppose $\underline{i}_X$ extends to the proper fundamental net

$$\underline{r} = \{r_\alpha| \ \alpha \in A\}: X' \to X \quad \text{in } (M, M).$$

It follows from condition (3) of the definition of "extension" that $r_\alpha|X = \mathrm{id}_M|X$ for all $\alpha \in A$. Thus $\underline{r}$ is a proper fundamental retraction of X' to X in (M, M) and X is a proper shape retract of X'.

3. Preliminary results. In this section we establish some technical results which will be used in the proof of the homotopy extension theorem of Section 4. The first of these is the "proper" version of the Borsuk homotopy extension theorem ([4], Theorem 8.1, p. 94), and the obvious modification of the proof given in [4] suffices once we note that if Z is a closed subset of Z', $f: Z \to Y$ is a proper map having an extension to a map f' defined on a neighborhood V' of Z in Z', then there exists a closed neighborhood V of Z lying in V' such that $f'|V$ is a proper map ([2], Lemma 3.2).

(3.1) THEOREM (Borsuk's homotopy extension theorem for proper maps). *Suppose X is a closed subset of the space X' and $Y \in$ ANR. Then every proper map $H: X \times [0, 1] \cup X' \times \{0\} \to Y$ has an extension to a proper map $H': X' \times [0, 1] \to Y$.*

We now generalize to a "collectionwise" homotopy extension theorem as follows. We note that our proof also shows that the "non-proper" version is true.

(3.2) LEMMA. *Suppose given the following data:*

(i) *a space X_0 and closed subsets $X_1, X_2, \ldots, X_n$;*

(ii) *a space Y_0 and closed subsets $Y_1, Y_2, \ldots, Y_n$ such that if $i_1, i_2, \ldots, i_k \in \{0, 1, \ldots, n\}$, then $\bigcap_{j=1}^{k} Y_{i_j} \in$ ANR;*

(iii) *a proper map $F: X_0 \to Y_0$ such that if $0 \leqslant i \leqslant n$, then $F(X_i) \subset Y_i$.*

Then if X is a closed subset of $\bigcap_{i=1}^{n} X_i$ and $g: X \to Y_0$ is a proper map such that $F|X \underset{p}{\simeq} g$ in $\bigcap_{i=1}^{n} Y_i$, g extends to a proper map $G: X_0 \to Y_0$ such that if $i_1, i_2, \ldots, i_k \in \{0, 1, \ldots, n\}$, then

$$F|\bigcap_{j=1}^{k} X_{i_j} \underset{p}{\simeq} G|\bigcap_{j=1}^{k} X_{i_j} \quad \text{in } \bigcap_{j=1}^{k} Y_{i_j}.$$

Proof. If $1 \leqslant k \leqslant n+1$, let X^k (resp. Y^k) denote the set of points in X_0 (resp. Y_0) lying in at least k of the sets $X_0, X_1, \ldots, X_n$ (resp. $Y_0, Y_1, \ldots, Y_n$). Let

$$\hat{G}: X \times [0,1] \cup X_0 \times \{0\} \to Y_0$$

be a proper map such that $\hat{G}(x, 0) = F(x)$ for all $x \in X_0$, $\hat{G}(x, 1) = g(x)$ for all $x \in X$, and $\hat{G}(X \times [0,1]) \subset \bigcap_{i=1}^{n} Y_i$. By Theorem 3.1, $\hat{G}$ extends to a proper map $G^{n+1}: X^{n+1} \times [0,1] \cup X_0 \times \{0\} \to Y_0$ such that $G^{n+1}(X^{n+1} \times [0,1]) \subset Y^{n+1}$. (Note that $X^{n+1} = \bigcap_{i=1}^{n} X_i$ and $Y^{n+1} = \bigcap_{i=1}^{n} Y_i$.)

Inductively, suppose $1 < k \leqslant n+1$ and that $G^k: X^k \times [0,1] \cup X_0 \times \{0\} \to Y_0$ is a proper map which extends $\hat{G}$ and has the following property:

(3.3) if $k \leqslant j \leqslant n+1$ and $i_1, i_2, \ldots, i_j \in \{0, 1, \ldots, n\}$, then

$$G^k((X_{i_1} \cap X_{i_2} \cap \ldots \cap X_{i_j}) \times [0,1]) \subset Y_{i_1} \cap Y_{i_2} \cap \ldots \cap Y_{i_j}.$$

Let $i_0 = \binom{n+1}{k-1}$ and let $A_1, A_2, \ldots, A_{i_0}$ denote those subsets of $\{0, 1, \ldots, n\}$ containing exactly $k-1$ elements. If $1 \leqslant j \leqslant i_0$, let $X'_j = X^k \cup (\bigcap_{i \in A_j} X_i)$. Applying Theorem 3.1, if $1 \leqslant j \leqslant i_0$, then G^k extends to a proper map $G^k_j: X'_j \times [0,1] \cup X_0 \times \{0\} \to Y_0$ such that

$$G^k_j((\bigcap_{i \in A_j} X_i) \times [0,1]) \subset \bigcap_{i \in A_j} Y_i.$$

Now, noting that $X^{k-1} = X'_1 \cup X'_2 \cup \ldots \cup X'_{i_0}$, we may define $G^{k-1}: X^{k-1} \times [0,1] \cup X_0 \times \{0\} \to Y_0$ by $G^{k-1}(x, t) = G^k_j(x, t)$ if $(x, t) \in X'_j \times [0,1] \cup X_0 \times \{0\}$. It is evident that G^{k-1} is a well-defined proper map and that

(3.4) if $k-1 \leqslant j \leqslant n+1$ and $i_1, i_2, \ldots, i_j \in \{0, 1, \ldots, n\}$, then

$$G^{k-1}((X_{i_1} \cap X_{i_2} \cap \ldots \cap X_{i_j}) \times [0,1]) \subset Y_{i_1} \cap Y_{i_2} \cap \ldots \cap Y_{i_j}.$$

Noting that $X^1 = X_0$, this induction yields a proper map $G^1: X_0 \times [0,1] \to Y_0$ which extends $\hat{G}$ and has the property that

(3.5) if $1 \leqslant j \leqslant n+1$ and $i_1, i_2, \ldots, i_j \in \{0, 1, \ldots, n\}$, then

$$G^1((X_{i_1} \cap X_{i_2} \cap \ldots \cap X_{i_j}) \times [0,1]) \subset Y_{i_1} \cap Y_{i_2} \cap \ldots \cap Y_{i_j}.$$

The proof is now complete, letting $G(x) = G^1(x, 1)$ for all $x \in X_0$.

Now, let ω be a point of the Hilbert cube Q and let $K = Q - \{\omega\}$. By the homogeneity of Q, K is not (topologically) dependent on the choice of ω. Since Q is homeomorphic with the cone on Q, it follows that $K \cong Q \times [0, 1)$. Introducing one final bit of notation, if S is a set, let $\sum(S)$ denote the set of all finite subsets of S regarded as a directed set, where $\sigma_1 \leqslant \sigma_2$ if and only if $\sigma_1 \subset \sigma_2$. We note that $\sum(S)$ is closure-finite; that is, each member of $\sum(S)$ has but a finite number of predecessors.

The following proposition was called to the author's attention by B. J. Ball.

(3.6) Lemma. *If S is a set such that $|S| \geqslant \mathfrak{C}$ and X is a closed subset of K, then there exists a cofinal system $\mathcal{U}$ of closed neighborhoods of X in K and a function $\Phi: \sum(S) \to \mathcal{U}$ such that*

(i) *if $U \in \mathcal{U}$, then $U \in$ ANR,*

(ii) *$\mathcal{U}$ is closed under finite intersection, and*

(iii) *if $\sigma_1, \sigma_2 \in \sum(S)$ and $\sigma_1 \leqslant \sigma_2$, then $\Phi(\sigma_2) \subset \Phi(\sigma_1)$.*

Proof. Let $\mathcal{W}$ be a cofinal system of closed neighborhoods of X in K such that the intersection of any finite number of elements of $\mathcal{W}$ is an ANR. Since $|\mathcal{W}| \leqslant \mathfrak{C} \leqslant |S|$, there is a function $\Psi: S \to\!\!\!\!\to \mathcal{W}$. If $\sigma = \{s_1, s_2, \ldots, s_n\} \in \sum(S)$, let $\Phi(\sigma) = \Psi(s_1) \cap \Psi(s_2) \cap \ldots \cap \Psi(s_n)$ and let $\mathcal{U} = \{\Phi(\sigma) |\ \sigma \in \sum(S)\}$. Then (i)-(iii) are easily verified.

4. The homotopy extension theorem for proper fundamental nets. We are now prepared to prove the following result, which will be quite important to the remainder of the paper.

(4.1) Theorem. *Suppose $X \subset X'$ are closed subsets of $M \in$ AR, Y is a closed subset of $N \in$ AR, $\underline{f}: X \to Y$ is a proper fundamental net in (M, N) which extends to the proper fundamental net $\underline{f}^*: X' \to Y$ in (M, N), and $\underline{g}: X \to Y$ is a proper fundamental net in (M, N) such that $\underline{f} \underset{p}{\simeq} \underline{g}$. Then $\underline{g}$ extends to a proper fundamental net $\underline{g}^*: X' \to Y$ in (M, N) such that $\underline{f}^* \underset{p}{\simeq} \underline{g}^*$.*

Proof. By Theorem 2.10, there is no loss of generality in supposing that $\underline{f} = \underline{f}^*|X$, and so we henceforth make this assumption. We suppose $\underline{f}^* = \{f^*_\alpha |\ \alpha \in A\}$ and $\underline{g} = \{g_\lambda |\ \lambda \in \Lambda\}$.

To begin with, consider the special case $M = N = K$. By Lemma 3.6 there exist a closure-finite directed set Δ and a cofinal system $\{V_\delta |\ \delta \in \Delta\}$ of closed neighborhoods of Y in K such that

(4.2) if $\delta' \leqslant \delta$, then $V_\delta \subset V_{\delta'}$,

(4.3) if $\delta_1, \delta_2, \ldots, \delta_n \in \Delta$, then $\bigcap_{i=1}^{n} V_{\delta_i} \in$ ANR,

and

(4.4) $|\Delta| \geqslant |\Lambda|$.

By (4.4), there exists a function $\theta: \Delta \to\!\!\!\!\to \Lambda$.

Now, let k be a positive integer and suppose that for each $\delta \in \Delta$ having fewer than k (strict) predecessors there has been defined a closed neighborhood U_δ of X'

in K, an index $\alpha_{\hat{\delta}} \in \Lambda$, an index $\lambda(\hat{\delta}) \in \Lambda$, and a proper map $g_{\hat{\delta}}^*\colon K \to K$ such that

(4.5) if $\delta' \leqslant \hat{\delta}$, then $\lambda(\delta') \leqslant \lambda(\hat{\delta})$,

(4.6) $\lambda(\hat{\delta}) \geqslant \theta(\hat{\delta})$,

(4.7) if $\alpha \geqslant \alpha_{\hat{\delta}}$, then $f_\alpha^* | U_{\hat{\delta}} \underset{p}{\simeq} f_{\alpha_{\hat{\delta}}}^* | U_{\hat{\delta}}$ in $V_{\hat{\delta}}$,

(4.8) $g_{\hat{\delta}}^* | U_{\hat{\delta}} \underset{p}{\simeq} f_{\alpha_{\hat{\delta}}}^* | U_{\hat{\delta}}$ in $V_{\hat{\delta}}$,

(4.9) if $\delta' \leqslant \hat{\delta}$, then $g_{\hat{\delta}}^* | U_{\delta'} \underset{p}{\simeq} g_{\delta'}^* | U_{\delta'}$ in $V_{\delta'}$,

and

(4.10) $g_{\hat{\delta}}^* | X = g_{\lambda(\hat{\delta})} | X$.

Now suppose $\delta \in \Delta$ has exactly k predecessors, say $\delta_1, \delta_2, \ldots, \delta_k$. Then there exist a closed neighborhood U_δ of X' in K, an index $\alpha_\delta \in \Lambda$, and an index $\lambda(\delta) \in \Lambda$, such that

(4.11) $\alpha_\delta \geqslant \alpha_{\delta_1}, \alpha_{\delta_2}, \ldots, \alpha_{\delta_k}$,

(4.12) if $\delta' \leqslant \delta$, then $\lambda(\delta') \leqslant \lambda(\delta)$,

(4.13) $\lambda(\delta) \geqslant \theta(\delta)$,

(4.14) if $\alpha \geqslant \alpha_\delta$, then $f_\alpha^* | U_\delta \underset{p}{\simeq} f_{\alpha_\delta}^* | U_\delta$ in V_δ,

and

(4.15) $f_{\alpha_\delta}^* | X \underset{p}{\simeq} g_{\lambda(\delta)} | X$ in V_δ.

By (4.2), $V_\delta \subset \bigcap_{i=1}^{k} V_{\delta_i}$. Hence, using (4.3), (4.11), (4.7), (4.14), and (4.15), we may apply Lemma 3.2 to obtain a proper map $g_\delta^*\colon K \to K$ such that

(4.16) $g_\delta^* | X = g_{\lambda(\delta)} | X$,

(4.17) for $i = 1, 2, \ldots,$ or k, $g_\delta^* | U_{\delta_i} \underset{p}{\simeq} f_{\alpha_\delta}^* | U_{\delta_i}$ in V_{δ_i},

and

(4.18) $g_\delta^* | U_\delta \underset{p}{\simeq} f_{\alpha_\delta}^* | U_\delta$ in V_δ.

By (4.11) and (4.7), $f_{\alpha_\delta}^* | U_{\delta_i} \underset{p}{\simeq} f_{\alpha_{\delta_i}}^* | U_{\delta_i}$ in V_{δ_i}. By (4.8), $f_{\alpha_{\delta_i}}^* | U_{\delta_i} \underset{p}{\simeq} g_{\delta_i}^* | U_{\delta_i}$ in V_{δ_i}. Using these facts along with (4.17), we have

(4.19) if $\delta' \leqslant \delta$, then $g_\delta^* | U_{\delta'} \underset{p}{\simeq} g_{\delta'}^* | U_{\delta_i}$ in V_{δ_i}.

Note that (4.12), (4.13), (4.14), (4.18), (4.19), and (4.16) correspond to the inductive assumptions (4.5)-(4.10) with $\hat{\delta}$ replaced by δ. We may thus suppose that U_δ, α_δ, $\lambda(\delta)$, and g_δ^* have been defined for all $\delta \in \Delta$ in such a way that (4.12), (4.13), (4.14), (4.16), (4.18), and (4.19) hold. By (4.19) and the fact that $\{V_\delta | \ \delta \in \Delta\}$ is cofinal, $\underline{g}^* = \{g_\delta^* | \ \delta \in \Delta\}\colon X' \to Y$ is a proper fundamental net in (K, K), while by (4.18), (4.19), and (4.14), $\underline{f}^* \underset{p}{\simeq} \underline{g}^*$. The fact that θ is an onto function along with (4.12), (4.13) and (4.16), implies that $\underline{g}^*$ is an extension of $\underline{g}$.

Now suppose that M and N are arbitrary. Let h_0 be a homeomorphism from X' onto a closed set $X_0' \subset K$, let $h\colon M \to K$ be a map such that $h(x) = h_0(x)$ for all $x \in X'$

and let $\hat{h}\colon K \to M$ be a map such that $\hat{h}(x) = h_0^{-1}(x)$ for all $x \in X_0'$. Let $X_0 = h_0(x)$. Similarly, let k_0 be a homeomorphism from Y onto a closed set $Y_0 \subset K$, let $k\colon N \to K$ be a map such that $k(y) = k_0(y)$ for all $y \in Y$ and let $\hat{k}\colon K \to N$ be a map such that $\hat{k}(y) = k_0^{-1}(y)$ for all $y \in Y_0$. Then $\underline{\hat{f}}^* = \{k f_\alpha^* \hat{h} | \ \alpha \in \Lambda\}\colon X_0' \to Y_0$ is a proper fundamental net in (K, K), as is $\underline{\hat{g}} = \{k g_\lambda \hat{h} | \ \lambda \in \Lambda\}\colon X_0 \to Y_0$, and it is easy to verify that $\underline{\hat{f}}^* | X_0 \underset{p}{\simeq} \underline{\hat{g}}$. By the special case established above, $\underline{\hat{g}}$ extends to a proper fundamental net $\underline{\hat{g}}^* = \{\hat{g}_\delta^* | \ \delta \in \Delta\}\colon X_0' \to Y_0$ in (K, K) such that $\underline{\hat{g}}^* \underset{p}{\simeq} \underline{\hat{f}}^*$. If $\delta \in \Delta$, let $g_\delta^* = \hat{k} \hat{g}_\delta^* h\colon M \to N$. Then it is easy to verify that $\underline{g}^* = \{g_\delta^* | \ \delta \in \Delta\}\colon X' \to Y$ is a proper fundamental net in (M, N) extending $\underline{g}$ and satisfying the condition $\underline{g}^* \underset{p}{\simeq} \underline{f}^*$. Thus the proof is complete.

The indexing set used for the extension $\underline{g}^*$ of Theorem 4.1 is of the form $\Sigma(S)$ for some sufficiently large set S, since it was obtained from Lemma 3.6. The following result, while not made use of in the current paper, shows that this is not a completely unexpected nor undesirable state of affairs.

(4.20) THEOREM. *Suppose X and Y are closed subsets of the* AR*'s M and N, respectively, and let S be a set such that $|S| \geqslant \mathfrak{C}$. Then if $\underline{f} = \{f_\lambda | \ \lambda \in \Lambda\}\colon X \to Y$ is a proper fundamental net in (M, N), there exists a proper fundamental net*

$$\underline{f}' = \{f_\sigma' | \ \sigma \in \Sigma(S)\}\colon X \to Y \text{ in } (M, N) \text{ such that } \underline{f} \underset{p}{\simeq} \underline{f}'.$$

Proof. As in the proof of Theorem 4.1, we may assume $M = N = K$. By Lemma 3.6, there exists a cofinal system $\{V_\sigma | \ \sigma \in \Sigma(S)\}$ of closed neighborhoods of Y in K such that $\sigma' \leqslant \sigma \in \Sigma(S)$ implies $V_\sigma \subset V_{\sigma'}$. If $\sigma \in \Sigma(S)$, there exists a closed neighborhood U_σ of X in K and an index $\lambda_\sigma \in \Lambda$ such that

(4.21) if $\lambda \geqslant \lambda_\sigma$, then $f_\lambda | U_\sigma \underset{p}{\simeq} f_{\lambda_\sigma} | U_\sigma$ in V_σ.

We may also assume, since $\Sigma(S)$ is closure-finite, that

(4.22) if $\sigma' \leqslant \sigma$, then $\lambda_{\sigma'} \leqslant \lambda_\sigma$.

Now, for each $\sigma \in \Sigma(S)$, let $f_\sigma' = f_{\lambda_\sigma}$. Suppose V is a closed neighborhood of Y in K; then let $\sigma_0 \in \Sigma(S)$ be chosen so that $V_{\sigma_0} \subset V$, and let $U = U_{\sigma_0}$. Then, by (4.21) and (4.22), if $\sigma \geqslant \sigma_0$, $f_\sigma' | U = f_{\lambda_\sigma} | U \underset{p}{\simeq} f_{\lambda_{\sigma_0}} | U = f_{\sigma_0}' | U$ in $V_{\sigma_0} \subset V$. We conclude from this that $\underline{f}'\colon X \to Y$ is a proper fundamental net in (K, K). Furthermore, if $\lambda \geqslant \lambda_{\sigma_0}$ and $\sigma \geqslant \sigma_0$, then

$$f_\lambda | U \underset{p}{\simeq} f_{\lambda_{\sigma_0}} | U \underset{p}{\simeq} f_{\lambda_\sigma} | U = f_\sigma' | U \quad \text{in } V_\sigma \subset V,$$

and we conclude from this that $\underline{f} \underset{p}{\simeq} \underline{f}'$.

5. Proper shape retracts and the extension of proper fundamental nets. In this section we establish the analogues of the extension theorems of [5], Section 5.

(5.1) THEOREM. *Suppose $X \subset X'$ are closed subsets of $M \in$ AR, Y is a closed subset of $N \in$ AR, and X is a proper shape retract of X'. Then every proper fundamental net $\underline{f}\colon X \to Y$ in (M, N) extends to a proper fundamental net $\underline{f}^*\colon X' \to Y$ in (M, N).*

Proof. Since X is a proper shape retract of X', there exists a proper fundamental retraction $\underline{r} = \{r_\alpha | \alpha \in A\}$: $X' \to X$ in (M, M). If $\underline{f} = \{f_\lambda | \lambda \in A\}$: $X \to Y$ in (M, N), then $\underline{f}^* = \underline{f}\underline{r}$: $X' \to Y$ in (M, N) is an extension of $\underline{f}$; for $\underline{f}^* = \{f_\lambda r_\alpha | (\lambda, \alpha) \in A \times A\}$ and we may let, using the notation of the definition of "extension", $\lambda(\sigma) = \lambda_0$ where $\sigma = (\lambda_0, \alpha_0) \in A \times A = \Sigma$.

(5.2) THEOREM. *Suppose $X \subset X'$ are closed subsets of $M \in$ AR, $Y \subset Y'$ are closed subsets of $N \in$ AR, and that Y is a proper shape retract of Y'. Let $\underline{i} = \{\mathrm{id}_N\}$: $Y \to Y'$ in (N, N) and suppose that $\underline{f}$: $X \to Y$ is a proper fundamental net in (M, N) such that $\underline{i}\underline{f}$: $X \to Y'$ extends to $\underline{f}'$: $X' \to Y'$ in (M, N). Then $\underline{f}$ has an extension $\underline{f}^*$: $X' \to Y$ in (M, N).*

Proof. Since Y is a proper shape retract of Y', there exists a proper fundamental retraction $\underline{r}$: $Y' \to Y$ in (N, N). It is easy to see that $\underline{r}\underline{i} \underset{p}{\simeq} \underline{i}_Y$. Since $\underline{i}\,\underline{f}$ is a restriction of $\underline{f}'$ it follows from Theorem 2.10 that $\underline{f}'|X \underset{p}{\simeq} \underline{i}\,\underline{f}$. Thus

$$\underline{f} \underset{p}{\simeq} \underline{i}_Y \underline{f} \underset{p}{\simeq} (\underline{r}\underline{i})\,\underline{f} = \underline{r}(\underline{i}\,\underline{f}) \underset{p}{\simeq} \underline{r}(\underline{f}'|X)\,.$$

But $\underline{r}(\underline{f}'|X)$ is clearly a restriction of $\underline{r}\underline{f}'$. Hence, by Theorem 4.1, $\underline{f}$ extends to $\underline{f}^*$: $X' \to Y$ in (M, N).

6. APSR's and ANPSR's. Generalizing a useful idea in shape theory for compacta due to Borsuk [5], Ball [1] has defined an *absolute proper shape retract* (APSR) to be a space X which is a proper shape retract of every space X' in which X is properly embedded. (See [1] for the definition of "properly embedded".) Extending this concept in the obvious way, let us say that X is an *absolute neighborhood proper shape retract* (ANPSR) if for each space X' containing X as a closed subset, there exists a closed neighborhood X'' of X in X' such that X is a proper shape retract of X''. As is customary, we write $X \in$ APSR and $X \in$ ANPSR for the statements "X is an APSR" and "X is an ANPSR".

Now, suppose $X \in$ ANR. Then if X is a closed subset of the space X', by [16], Theorem 2.1 there exists a closed neighborhood X'' of X in X' and a proper map r: $X'' \to X$ such that $r(x) = x$ for all $x \in X$ (that is, r is a proper retraction of X'' onto X). It is easily shown that this implies that X is a proper shape retract of X'', and hence we have the following.

(6.1) THEOREM. *If $X \in$ ANR, then $X \in$ ANPSR.*

The following two results are proved by obvious modifications of the proofs of [5], Theorem 6.7 and [5], Corollary 6.8, and the reader is spared a repetition of the details.

(6.2) THEOREM. *If X is a proper shape retract of the space $X' \in$ ANPSR, then $X \in$ ANPSR.*

(6.3) COROLLARY. *The space $X \in$ ANPSR if and only if X is a proper shape retract of some* ANR.

From Corollary 6.3 and [1], Theorem 4.3 we obtain the following.

(6.4) COROLLARY. *If $X \in$ APSR, then $X \in$ ANPSR.*

Finally, we note that the property of being an ANPSR is a hereditary proper shape invariant. The proof is completely analogous to that of [8], Theorem 26.1, only using Theorem 4.1 rather than [15], Theorem, and we again omit detailing the modifications.

(6.5) THEOREM. *If $X' \in$ ANPSR and $\mathrm{Sh}_p X \leqslant \mathrm{Sh}_p X'$, then $X \in$ ANPSR.*

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