Obtaining inverse sequences for certain continua

by

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Abstract. A well-known theorem states that every continuum (compact connected metric space) is homeomorphic to the limit of an inverse sequence whose coordinate spaces are finite dimensional connected polyhedra and whose bonding maps are piecewise-linear. The usual proofs of this theorem, however, give bonding maps which are rather complicated. Simpler bonding maps are referred to in the literature for some special cases: if each coordinate space is the interval $[0, 1]$ and each bonding map is the identity, the limit is an arc; also a certain indecomposable continuum may be obtained by using for each bonding map a function f: $[0, 1]\to[0, 1]$ whose graph resembles an inverted "V".

The purpose of this paper is to provide, for a certain class of one-dimensional continua, general methods for constructing relatively simple inverse sequences.

1. Obtaining inverse sequences. In this section we show that an inverse sequence which follows the pattern (in a sense to be defined) of a sequence of finite open covers of a one-dimensional continuum M has a limit homeomorphic to M. The bonding maps obtained in this proof, however, are fully as complicated as those discussed above, so in Sections 2 and 3 we turn to the problem of "smoothing" the bonding maps. An example at the end of Section 3 provides an application of the method. These results are then applied in Section 4 to certain chainable continua, where they give particularly simple inverse limit sequences, as is shown by a number of examples.

A *map* is a continuous function. We denote by (X,f) the inverse sequence whose coordinate spaces form the sequence $X = X_1, X_2, \ldots$ and whose bonding maps form the sequence $f = f_1, f_2, \ldots$, with f_i: $X_{i+1}\to X_i$ for each i, and call (X,f) *proper* if $f_i(X_{i+1}) = X_i$ for each i. If $L = \lim(X,f)$, the limit of the inverse sequence (X,f), then we denote by π_i the projection map from L into X_i and by f_{ij} the composite $f_i\circ\ldots\circ f_{j-1}$: $X_j\to X_i$, if $i<j$. f_{ii} denotes the identity map on X_i. The metrics $\{d_i\}_{i=1}^{\infty}$ for the coordinate spaces of the inverse sequence (X,f) are always assumed to be such that the diameter of the coordinate spaces are all $\leqslant 1$, and the metric for lim (X,f) is defined by

$$d(x, y) = \sum_{i=1}^{\infty} d_i\big(\pi_i(x), \pi_i(y)\big)\cdot 2^{-i}.$$

DEFINITION 1.1. If C and D are finite collections of open sets then C is *properly embedded* in D if and only if

(1) C is a refinement of D,

(2) if $c \in C$ intersects $c' \in C$, then $c \cup c'$ does not intersect three elements of D, and

(3) if d and d' are two intersecting elements of D, there exist intersecting elements c and c' of C such that $c \subseteq d$, c intersects no element of $D-\{d\}$, and c' intersects d'.

DEFINITION 1.2. A *defining sequence* H for a one-dimensional continuum M is a sequence $H_1, H_2, \ldots$ such that for some sequence $\varepsilon_1, \varepsilon_2, \ldots$ of positive numbers converging to 0 and for each i,

(1) H_i is a finite collection of open sets relative to M which covers M,

(2) H_{i+1} is properly embedded in H_i, and

(3) the mesh of H_i is $<\varepsilon_i$) (i.e. each element of H_i has diameter $<\varepsilon_i$).

DEFINITION 1.3. Unless otherwise noted, we adopt the conventions of [3]. All complexes used here are finite. If H is a finite collection of point sets we denote the nerve of H by $N(H)$, and we identify the elements of H with the corresponding vertices of $N(H)$ and $|N(H)|$. Note that if H is a defining sequence for a one-dimensional continuum, then Conditions 1.2 (2) and 1.1 (2) imply that no point belongs to three elements of H_i for any i; hence $N(H_i)$ is one-dimensional for each i. The barycentric subdivision of the simplicial complex K is denoted by $\mathrm{Sd}(K)$.

DEFINITION 1.4. The inverse sequence *associated with* the defining sequence H for the one-dimensional continuum M is the proper inverse sequence (X, f) such that

(1) for each i, $X_i = |N(H_i)|$,

(2) for each i, $f_i\colon X_{i+1} \to X_i$ is the simplicial map relative to $N(H_{i+1})$ and $\mathrm{Sd}[N(H_i)]$ which assigns to the vertex $h \in H_{i+1}$ of $|N(H_{i+1})|$ the point

(a) h' of $|\mathrm{Sd}[N(H_i)]|$, if h' is the only element of H_i that h intersects, and

(b) $\frac{1}{2}h' + \frac{1}{2}h''$ of $|\mathrm{Sd}[N(H_i)]|$, if h' and h'' are the only two elements of H_i that h intersects.

Remarks 1.5. Several remarks are necessary to justify this definition. First note that conditions (2a) and (2b) above define a vertex map φ_i relative to $N(H_{i+1})$ and $\mathrm{Sd}[N(H_i)]$. Because of Definition 1.1 (2), φ_i maps the vertices of each 1-simplex of $N(H_{i+1})$ either to a point, or to the vertices of a 1-simplex of $\mathrm{Sd}[N(H_i)]$, so φ_i extends uniquely to a simplicial map $f_i\colon N(H_{i+1}) \to \mathrm{Sd}[N(H_i)]$. Condition 1.1 (3) guarantees that each simplex in $\mathrm{Sd}[N(H_i)]$ is the image of a simplex of $N(H_{i+1})$, so that (X, f) is proper.

We now state the main theorem of this section. Its proof, using several lemmas, occupies the remainder of this section.

THEOREM 1.6. *If the inverse sequence (X, f) is associated with the defining sequence H for the one-dimensional continuum M, then* $\lim(X, f)$ *is homeomorphic to M.*

We will denote by K^0 the interior of the point set K and by $\overline{K}$ the closure of K. Since each 1-simplex of $N(H_{i+1})$ is thrown into half of a 1-simplex of $N(H_i)$, we have by induction:

LEMMA 1.7. *If $i<j$ e_i and e_j are 1-simplexes of $N(H_i)$ and $N(H_j)$ respectively, and $f_{ij}(e_j^0)$ intersects e_i, then f_{ij} is linear on e_j, $f_{ij}(e_j) \subseteq e_i$, and*

$$\mathrm{diam}\ (f_{ij}(e_j)) \leqslant 2^{i-j}\,\mathrm{diam}(e_i)\,.$$

For each $h \in H_i$, $\mathrm{st}(h)$ denotes the open star of h in $|N(H_i)|$ and h' denotes $\pi_i^{-1}(\mathrm{st}(h))$. Let G_i denote the collection of all sets h', where h is in H_i. Then G_i is an open cover of $\lim(X, f)$.

LEMMA 1.8. *If $i<j$, $g_j \in H_j$, $h_i \in H_i$ and $g_j \subseteq h_i$, then $f_{ij}(\overline{\mathrm{st}(g_j)}) \subseteq \mathrm{st}(h_i)$; hence $\overline{g}'_j \subseteq h'_i$.*

This follows easily once it is shown by induction that if $i<j$, $g_j \in H_j$, $h_i \in H_i$, and g_j intersects h_i, then $f_{ij}(\mathrm{st}(g_j)) \subseteq \mathrm{st}(h_i)$.

LEMMA 1.9. *If g_i and h_i are elements of H_i for some i, then g_i intersects h_i if and only if g'_i intersects h'_i.*

Proof. Each of the following statements is clearly equivalent to the next: (1) g'_i intersects h'_i, (2) $\mathrm{st}(g_i)$ intersects $\mathrm{st}(h_i)$, (3) g_i and h_i are vertices of a 1-simplex of $N(H_i)$, and (4) g_i and h_i intersect.

LEMMA 1.10. $\lim\limits_{j\to\infty}$*(mesh of G_j)* $= 0$.

Proof. Suppose j is a positive integer, and z is an element of G_j. Then $z = h'_j$ for some element h_j of H_j. So

$$\begin{aligned}\mathrm{diam}(z) &\leqslant \sum_{i=1}^{\infty} \mathrm{diam}\,(\pi_i(z))\cdot 2^{-i}\\ &\leqslant \sum_{i=1}^{j} \mathrm{diam}\,(f_{ij}(\mathrm{st}(h_j)))\cdot 2^{-i} + \sum_{i=j+1}^{\infty} 1\cdot 2^{-i}\,,\end{aligned}$$

since $\mathrm{diam}(X_i) = 1$ for every i. But if e_j is one of the 1-simplexes in $\mathrm{st}(h_j)$, $i \leqslant j$, and e_i is a 1-simplex of $N(H_i)$ that contains $f_{ij}(e_j)$, then by Lemma 1.7,

$$\mathrm{diam}\,(f_{ij}(e_j)) \leqslant 2^{i-j}\mathrm{diam}(e_i) \leqslant 2^{i-j}\,.$$

Hence,

$$\begin{aligned}\mathrm{diam}(z) &\leqslant \sum_{i=1}^{j} (2\cdot 2^{i-j})\cdot 2^{-i} + \sum_{i=j+1}^{\infty} 2^{-i}\\ &= \Big(\sum_{i=1}^{j} 2\cdot 2^{-j}\Big) + 2^{-j} = (2j+1)\cdot 2^{-j}\,,\end{aligned}$$

which has limit 0.

If $p \in M$, we will say that g *determines* p if and only if g is a sequence $g_1, g_2, \ldots$ such that $g_{i+1} \subseteq g_i \in H_i$ for each i, and $p = \bigcap_{i=1}^{\infty} g_i$. If g determines p, let $\lambda(g) = \bigcap_{i=1}^{\infty} g'_i$ (note that, by Lemma 1.8, $\overline{g'_{i+1}} \subseteq g'_i$, and by Lemma 1.10, $\lambda(g)$ is degenerate).

LEMMA 1.11. *If $p \in M$ and g and h are sequences that determine p, then $\lambda(g) = \lambda(h)$.*

Proof. Since $p \in g_i \cap h_i$ for each i, g_i' and h_i' intersect for each i (Lemma 1.9). By Lemma 1.10, $\lim_{i\to\infty} \operatorname{diam}(g_i' \cup h_i') = 0$. Hence

$$\lambda(g) = \bigcap_{i=1}^{\infty} g_i' = \bigcap_{i=1}^{\infty} h_i' = \lambda(h).$$

LEMMA 1.12. *If $p \in M$, there is a sequence that determines p.*

This follows easily from Lemma 1.10 and the fact that (1) for each n, there are at most two elements of H_n that contain p (Definition 1.2 (2)) and (2) each element g of H_n that contains p is the last term of a sequence $g_1 \supseteq g_2 \supseteq \ldots \supseteq g_n = g$ where $g_i \in H_i$ for each $1 \leqslant i \leqslant n$.

Lemmas 1.11 and 1.12 allows us to define a transformation $\theta\colon M \to \lim(X, f)$ such that if $p \in M$, then $\theta(p) = \lambda(g)$ for any sequence g that determines p.

LEMMA 1.13. *If $h_i \in H_i$, then $\theta(h_i) \subseteq h_i'$.*

Proof. If g determines $p \in h_i$, then for some $j > i$, $g_j \subseteq h_i$, by Definition 1.2 (3). Hence $\theta(p) \in g_j' \subseteq h_i'$ by Lemma 1.8.

LEMMA 1.14. *θ is a homeomorphism from M onto* $\lim(X, f)$.

Proof. θ is one-to-one, since if g determines p and h determines $q \neq p$, then there is a positive integer i such that the mesh of $H_i < \frac{1}{2} d(p, q)$. Hence h_i and g_i cannot intersect, and by Lemma 1.9, h_i' and g_i' do not intersect. But $\theta(p) = \lambda(g) \in g_i'$ and $\theta(q) = \lambda(h) \in h_i'$, so $\theta(p) \neq \theta(q)$.

θ is continuous, since if $p \in M$, 0 is an open set in $\lim(X, f)$ containing $\theta(p)$, and g determines p, then for some i, $\theta(g_i) \subseteq g_i' \subseteq 0$, by Lemmas 1.10 and 1.13. But $p \in g_i$, and g_i is open in M.

Finally, θ maps M onto $\lim(X, f)$, since for each i, G_i covers $\lim(X, f)$ and each element of G_i contains a point of $\theta(M)$ by Lemma 1.13, so that $\theta(M)$ is dense in $\lim(X, f)$ by Lemma 1.10. But M is compact, so $\theta(M)$ is closed in $\lim(X, f)$. Hence $\theta(M) = \lim(X, f)$.

2. Modifying bonding maps.

Our object in this section is to show that certain modifications of the bonding maps in certain types of inverse sequences yield sequences with limits homeomorphic to the original. These results will allow us to simplify the bonding maps we obtain in the next section.

DEFINITION 2.1. T is a *triangulation* of the continuum M if and only if T is a pair (t, K), where K is a simplicial complex and t is a homeomorphism from $|K|$ onto M. A *simplex of T* is the image under t of a simplex of K. A *polyhedron* is a continuum with a triangulation, and a *graph* is a one-dimensional polyhedron. A map f from the polyhedron X into the polyhedron Y is *simplicial* (resp. *piecewise monotone*) if and only if there are triangulations $T_1 = (t_1, K_1)$ and $T_2 = (t_2, K_2)$ of X and Y respectively, and a map $s\colon |K_1| \to |K_2|$ such that $f = t_2 s t_1^{-1}$ and s restricted to each simplex of K_1 is a linear (resp. monotone) map *onto* some simplex of K_2; in which case f is called *simplicial* (resp. *piecewise monotone*) relative to (T_1, T_2) (or relative to (K_1, K_2) in case t_1 and t_2 are identity maps). The notion of simplicial map between topological polyhedra used here coincides with that of [3], p. 60. If $f\colon X \to Y$ and $g\colon X \to Y$ are maps and $T = (t, K)$ is a triangulation of X, then f and g are *similar relative to T* if and only if, for each simplex α of T, the set $f(\alpha)$ is the set $g(\alpha)$. The triangulations $T_1 = (t_1, K_1)$ and $T_2 = (t_2, K_2)$ of X are *similar* if and only if $K_1 = K_2$ and t_1 and t_2 are similar relative to K_1, i.e. the subset α of X is a simplex of T_1 if and only if α is a simplex of T_2.

DEFINITION 2.2. (G, T, f) is a *uniformly simplicial inverse sequence on graphs* if and only if (1) G is a sequence $G_1, G_2, \ldots$ of graphs, (2) T is a sequence $T_1, T_2, \ldots$ such that T_i is a triangulation of G_i for each i, and (3) f is a sequence $f_1, f_2, \ldots$ such that for each i, f_i maps G_{i+1} onto G_i, and f_i is simplicial relative to (T_{i+1}, T_i).

THEOREM 2.3. *Suppose that* (1) *M and N are the limits of the uniformly simplicial inverse sequences (G, T, g) and (G, T', f), respectively,* (2) *for each i, $T_i = (t_i, K_i)$ is similar to $T_i' = (t_i', K_i')$, and* (3) *for each i, g and f are similar relative to T_{i+1}. Then M and N are homeomorphic.*

Proof. By hypothesis, $K_i = K_i'$ and for each vertex v of K_i,

$$s_i(v) = t_i^{-1} g_i t_{i+1}(v) = t_i^{-1} f_i t_{i+1}(v) = t_i'^{-1} f_i t_{i+1}'(v) = s_i'(v),$$

where $s_i\colon K_{i+1} \to K_i$ and $s_i'\colon K_{i+1} \to K_i$ are simplicial maps so that $g_i = t_i s_i t_{i+1}^{-1}$ and $f_i = t_i' s_i' t_{i+1}'^{-1}$. But simplicial maps between complexes which agree on the vertices are identical; so $s_i = s_i'$ for each i. Thus

$$t_i' t_i^{-1} g_i = t_i' s_i t_{i+1}^{-1} = t_i' s_i' t_{i+1}' = f_i t_{i+1}' t_{i+1}^{-1}.$$

Hence the homeomorphisms $t_i' t_i^{-1}$ induce a homeomorphism from M onto N.

THEOREM 2.4. *Suppose that (G, g) is an inverse sequence on graphs and there is a sequence $T = T_1, T_2, \ldots$ of triangulations of the terms of $G = G_1, G_2, \ldots$ such that for each i and each simplex α of T_{i+1}, $g_i|\alpha$ is either constant or a homeomorphism onto a simplex of T_i. Then there is a sequence $T' = T_1', T_2', \ldots$ such that* (1) *for each i, T_i' is a triangulation of G_i similar to T_i, and* (2) *(G, T', g) is uniformly simplicial.*

Proof. We define $T_i' = (t_i', K_i')$ $(i = 1, 2, \ldots)$ by first letting $K_i' = K_i$ for all i, and then defining $t_1', t_2', \ldots$ recursively. Let $t_1' = t_1$. Suppose $t_1', \ldots, t_n'$ have been defined; we define t_{n+1}'. Let $s\colon K_{n+1} \to K_n$ denote the simplicial extension of the restriction of $t_n^{-1} g_n t_{n+1}$ to the vertices of K_{n+1} and suppose that α' is a 1-simplex of K_{n+1}. If $g_n t_{n+1}|\alpha'$ is constant, define $t_{n+1}'|\alpha' = t_{n+1}|\alpha'$. Otherwise let α denote the 1-simplex $t_{n+1}(\alpha')$ of T_{n+1} and recall that $g_n|\alpha$ is a homeomorphism. So define $t_{n+1}'|\alpha'\colon \alpha' \to \alpha$ by $t_{n+1}'|\alpha' = (g_n|\alpha)^{-1} t_n' s|\alpha'$. Thus t_{n+1}' is defined on every 1-simplex of K_{n+1}.

THEOREM 2.5. *Suppose M is the limit of the uniformly simplicial inverse sequence (G, T, g) on graphs and for each i, f_i is a piecewise monotone map from G_{i+1} onto G_i relative to (T_{i+1}, T_i) which is similar to g_i relative to T_{i+1}. Then M is homeomorphic to* $\lim(G, f)$.

Proof. We first show that for each i, f_i is a uniform limit of maps f_i' similar to f_i relative to T_{i+1} with property (*):

(*) f_i' restricted to every simplex of T_{i+1} is either constant or a homeomorphism onto a simplex of T_i.

Suppose $\varepsilon>0$, and α is a 1-simplex of T_{i+1}. If $f_i|\alpha$ is constant, define $f_i'|\alpha = f_i|\alpha$. Otherwise, $f_i|\alpha$ is a monotone map onto a 1-simplex β of T_i, and there is a homeomorphism h from α onto β so that $d(h, f_i|\alpha)<\varepsilon$ (see [2], p. 478, footnote 2). Define $f_i'|\alpha = h$. Thus f_i' is defined on every 1-simplex of T_{i+1}, and clearly f_i' has property (*).

It now follows from Brown's approximation theorem ([2], Theorem 3), that there is an inverse sequence (G, f') such that for each i, f_i' is similar to f_i and has property (*) and such that $\lim(G, f')$ is homeomorphic to $\lim(G, f)$. Hence, by Theorem 2.4, there is a sequence T' of triangulations of the graphs $G_1, G_2, \ldots$ so that for each i, T_i' is similar to T_i and (G, T', f') is uniformly simplicial. So the hypothesis of Theorem 2.3 is satisfied and $\lim(G, T', f')$ is homeomorphic to $\lim(G, T, g)$, completing the proof of the theorem.

3. Obtaining uniformly simplicial inverse sequences. In this section, we develop a type of defining sequence which will allow us to construct simpler inverse sequences than those obtained in Section 1. While not every one-dimensional continuum will have such a defining sequence (it can be shown that any continuum with such a defining sequence contains an arc; hence the methods of this section will not provide us with an inverse sequence for a pseudo-arc, for example) the methods of this section apply to many of the one-dimensional continua commonly found in the literature. The inverse sequences finally obtained in this section will be uniformly simplicial; for more discussion of the continua which are limits of such sequences, see [5] and [6].

A *chain* is a (possibly degenerate) sequence $\alpha = c_1, \ldots, c_n$ of open sets (called *links*) such that two of them intersect if and only if they are adjacent in the sequence. A *subchain* of a chain α is a subsequence of α which is also a chain. If $\alpha = c_1, \ldots, c_n$ is a chain, then α^{-1} denotes the chain $c_n, \ldots, c_1$. The chain α *goes straight through* the chain β if and only if

(1) α is properly embedded in β,

(2) the first (resp. last) link of α intersects only the first (resp. last) link of β, and

(3) if a and b are two links of α which intersect a link c of β, then every link of α between a and b intersects c.

An *order preserving subdivision* of a chain α is a sequence $\alpha_1, \ldots, \alpha_n$ of subchains of α each having at least three links such that

(1) The first (resp. last) link of α_1 (resp. α_n) is the first (resp. last) link of α, and

(2) if $1 \leqslant i < n$ then the last link of α_i is the first link of α_{i+1}.

The chain α is said to *go straight through* the sequence $\beta_1, \ldots, \beta_n$ of chains if and only if there is an order preserving subdivision $\alpha_1, \ldots, \alpha_n$ of α such that α_i goes straight throught β_i if $1 \leqslant i \leqslant n$.

A *chain structure* is a finite collection D of chains, each with at least three links, such that if two elements, α and β, of D intersect then $\alpha \cap \beta$ consists of only one link, and it is an endlink of both α and β. The collection of all endlinks of chains in D is denoted by $V(D)$. If D is a chain structure, then D' is a *subdivision* of D if and only if

(1) D' is a chain structure,

(2) each chain in D' is a subchain of a chain in D, and

(3) each link of each chain in D is a link of some chain in D'.

If D is a chain structure, then (1) D^* denotes the collection of all the links of all the chains of D; hence D^* is a finite collection of open sets, and (2) D^{-1} denotes the collection of chains α^{-1} for all chains α in D.

Definition 3.1. (D, h) is a *chain structure sequence* for a one-dimensional continuum M if and only if D is a sequence $D_1, D_2, \ldots$ and h is a sequence $h_1, h_2, \ldots$ such that $D_1^*, D_2^*, \ldots$ is a defining sequence for M and for each i

(1) D_i is a chain structure and

(2) h_i is a function defined on D_{i+1} such that for each element β of D_{i+1}, $h_i(\beta)$ is a sequence $\beta_1, \ldots, \beta_n$ of chains in $V(D_i) \cup D_i \cup D_i^{-1}$ such that β goes straight through $\beta_1, \ldots, \beta_n$ and no link of β intersects any element of D_i^* not in one of the chains $\beta_1, \ldots, \beta_n$.

We will need some similar terminology concerning maps on arcs.

An *order preserving subdivision* of an ordered arc α is a sequence $\alpha_1, \ldots, \alpha_n$ of subarcs of α with the induced order such that

(1) the first (resp. last) point of α_1 (resp. α_n) is the first (resp. last) point of α, and

(2) if $1 \leqslant i < n$, then the last point of α_i is the first point of α_{i+1}.

If f is a map defined on a directed arc α, then f *goes straight through* the sequence $\beta_1, \ldots, \beta_n$ if and only if

(1) for each i, β_i is either a point or an ordered arc, and

(2) there is an order-preserving subdivision $\alpha_1, \ldots, \alpha_n$ of α such that if $1 \leqslant i \leqslant n$, then (a) $f|\alpha_i$ is an order-preserving homeomorphism onto β_i if β_i is an arc and (b) $f(\alpha_i) = \beta_i$ if β_i is a point.

Definition 3.2. An inverse sequence (G, g) on graphs *follows the pattern* of a chain structure sequence (D, h) for a one-dimensional con inuum M if and only if there are sequences $T = T_1, T_2, \ldots$ and $\varphi = \varphi_1, \varphi_2, \ldots$ such that for each i,

(1) $T_i = (t_i, K_i)$ is an oriented triangulation of G_i,

(2) φ_i is a one-to-one function from $V(D_i) \cup D_i$ onto the collection of all simplexes of T_i such that if β is a chain in D with first endlink c and last endlink d, then $\varphi_i(\beta)$ is an oriented 1-simplex of T_i with first vertex $\varphi_i(c)$ and last vertex $\varphi_i(d)$, and

(3) if β is a chain in D_{i+1} and $h_i(\beta) = \beta_1, \ldots, \beta_n$, then $g_i|\varphi_{i+1}(\beta)$ goes straight through $\varphi_i(\beta_1), \ldots, \varphi_i(\beta_n)$.

Theorem 3.3. *If the inverse sequence* (G, g) *on graphs follows the pattern of the chain structure sequence* (D, h) *for a continuum* M, *then* M *is homeomorphic to* $\lim(G, g)$.

For the proof of this theorem, we need three lemmas.

LEMMA 3.4. *Suppose the chain* $\beta = c_1, \ldots, c_n$ *goes straight through the chain* $\beta' = d_1, \ldots, d_m$ *and* $p\colon \{1, \ldots, n\} \to \{1, 1\frac{1}{2}, \ldots, m-\frac{1}{2}, m\}$ *is defined so that* (1) *if the link* c_i *of* β *intersects only the link* d_j *of* β', *then* $p(i) = j$ *and* (2) *if the link* c_i *of* β *intersects the links* d_j *and* d_{j+1} *of* β', *then* $p(i) = j+\frac{1}{2}$. *Then* p *is non-decreasing and surjective.*

Proof. That p is surjective follows quickly from Definition 1.1 (2), and the fact that β and β' are chains.

Suppose that p fails to be non-decreasing, i.e. there are integers $i_1 < i_2$ so that $p(i_1) > p(i_2)$. Then there are integers $k_2 < k_1$ such that c_{i_e} intersects d_{k_e} for $e = 1, 2$. Since c_1 (resp. c_n) intersects d_1 (resp. d_n) there exist integers j_1 and j_2 such that $1 \leqslant j_2 < i_1 < i_2 < j_1 \leqslant n$ and c_{j_e} intersects d_{k_e} for $e = 1, 2$. Since β goes straight through β', $j_2 < i_1 < i_2$ and both c_{j_2} and c_{i_2} intersect d_{k_2}, c_{i_1} must also intersect d_{k_2}. Since c_{i_1} already intersects d_{k_1} it follows from Definition 1.1 (2) that $k_1 = k_2+1$. Similarly, c_{i_2} intersects both d_{k_2} and d_{k_2+1}. But now $p(i_1) = k_2+\frac{1}{2} = p(i_2)$, a contradiction.

LEMMA 3.5. *Given the hypothesis of Theorem* 3.3, *if for each* i *and each element* β *of* D_{i+1}, $h(\beta)$ *has only one term, then* M *is homeomorphic to* $\lim(G, g)$.

Proof. Let $T_1, T_2, \ldots$ and $\varphi_1, \varphi_2, \ldots$ be as given in Definition 3.2. Suppose α is a 1-simplex of T_{i+1}. Then for some $\beta \in D_{i+1}$, $\alpha = \varphi_{i+1}(\beta)$, and under the hypothesis of this lemma, $\alpha' = g_i(\alpha) = \varphi_i h(\beta)$ is either a vertex of T_i (in which case $g_i|\alpha$ is constant), or a 1-simplex of T_i (in which case $g_i|\alpha$ is a homeomorphism onto α'). Clearly g_i maps each vertex of T_{i+1} onto a vertex of T_i. Hence the hypothesis of Theorem 2.4 is satisfied and there is a sequence $T_1', T_2', \ldots$ similar to $T_1, T_2, \ldots$ so that (G, T', g) is uniformly simplicial. Since $T_1', T_2', \ldots$ now has the properties of Definition 3.1, we may assume for the rest of the proof that (G, T, g) is uniformly simplicial.

Now, let (X, f) denote the inverse sequence associated with $H = D_1^*, D_2^*, \ldots$ as in Definition 1.4. For each i, there is a homeomorphism u_i from G_i onto X_i such that (1) u_i maps the 1-simplex α of T_i, where $\alpha = \varphi_i(\beta)$ for some $\beta \in D_i$, onto the arc $|N(\beta)|$, considered as a subset of $X_i = |N(D_i^*)|$, and (2) if $v \in V(D_i)$, then $u_i\varphi_i(v)$ is the vertex v of $N(D_i^*)$.

We next show that $f_i' = u_i^{-1} f_i u_{i+1}$ is a piecewise monotone map from G_{i+1} onto G_i relative to (T_{i+1}, T_i) and that f_i' is similar to g_i relative to T_{i+1}, so that the hypothesis of Theorem 2.5 is satisfied. Suppose α is a 1-simplex of T_{i+1}, where $\alpha = \varphi_{i+1}(\beta)$ for some $\beta = (c_1, \ldots, c_n) \in D_{i+1}$. Denote $h(\beta)$ by $\beta' = d_1, \ldots, d_m$ and $g_i(\alpha) = \varphi_i(\beta')$ by α'. Since, by Definition 3.1 (2), no link of β intersects any element of D_i^* other than those in β', it follows from Definition 1.4 (2) that f_i maps each vertex of $N(\beta) \subseteq X_{i+1}$ into a simplex of $N(\beta') \subseteq X_i$; hence

$$f_i[u_{i+1}(\alpha)] = f_i[|N(\beta)|] = |N(\beta')| = u_i(\varphi_i(\beta')) = u_i(\alpha')\,.$$

So $g_i(\alpha) = \alpha' = u_i^{-1} f_i u_{i+1} = f_i'(\alpha)$. It follows that g_i and f_i' are similar relative to T_{i+1}. To show that $f_i'|\alpha$ is monotone, it suffices to show that $f_i|\,|N(\beta)|$ is monotone, and this follows directly from Definition 1.4 (2) and Lemma 3.4.

This completes the proof that the hypothesis of Theorem 2.5 is satisfied, so $\lim(G, g) = \lim(G, T, g)$ is homeomorphic to $\lim(G, f')$. But for each i, $u_i f_i' = f_i u_{i+1}$, so that the sequence $u_1, u_2, \ldots$ of homeomorphisms induces a homeomorphism from $\lim(G, f')$ onto $\lim(X, f)$ which is in turn homeomorphic to M by Theorem 1.6.

The next lemma is the inductive step in the proof of Theorem 3.3.

LEMMA 3.6. *Suppose* n *is a positive integer*, (G, g) *is an inverse sequence on graphs following the pattern of the chain structure sequence* (D^n, h^n) *for the one-dimensional continuum* M, *and* T^n *and* φ^n *are sequences satisfying Definition* 3.2. *Then there exist a chain structure sequence* (D^{n+1}, h^{n+1}) *for* M, *and sequences* T^{n+1} *and* φ^{n+1} *satisfying the requirements of Definition* 3.2 *for* (G, g) *and* (D^{n+1}, h^{n+1}) *such that*

(1) *if* $i \neq n+1$, *then* $D_i^{n+1} = D_i^n$, $T_i^{n+1} = T_i^n$, *and* $\varphi_i^{n+1} = \varphi_i^n$,

(2) *if* $n \neq i \neq n+1$, *then* $h_i^{n+1} = h_i^n$,

(3) D_{n+1}^{n+1} *and* T_{n+1}^{n+1} *are subdivisions of* D_{n+1}^n *and* T_{n+1}^n, *respectively, and*

(4) *for each element* β *of* D_{n+1}^{n+1}, $h_n^{n+1}(\beta)$ *consists of only one chain.*

Proof. Suppose β is an element of D_{n+1}^n. Since β goes straight through $h_n^n(\beta) = \beta_1, \ldots, \beta_k$, there exist order preserving subdivisions $\alpha_1, \ldots, \alpha_k$ of β and $\varphi_{n+1}^{n+1}(\alpha_1), \ldots, \varphi_{n+1}^{n+1}(\alpha_k)$ of the arc $\varphi_{n+1}^n(\beta)$ such that if $1 \leqslant i \leqslant k$, then α_i goes straight through β_i and (a) $g_n|\varphi_{n+1}^{n+1}(\alpha_i)$ is an order preserving homeomorphism onto $\varphi_n^n(\beta_i)$ if $\varphi_n^n(\beta_i)$ is an arc and (b) $g_n\varphi_{n+1}^{n+1}(\alpha_i) = \varphi_n^n(\beta_i)$ if $\varphi_n^n(\beta_i)$ is a point. Subdivide the other chains in D_{n+1}^n and the other 1-simplexes in T_{n+1}^n similarly, and extend the definition of φ_{n+1}^{n+1}; denote the collection of chains so obtained by D_{n+1}^{n+1} and the subdivision of T_{n+1}^n so obtained by T_{n+1}^{n+1}. For each chain α in D_{n+1}^{n+1}, let $h_n^{n+1}(\alpha)$ consist of the element of D_n^n that α goes straight through.

Now, if $\beta' \in D_{n+2}^n$, then $h_{n+1}^n(\beta')$ is a sequence $\beta_1', \ldots, \beta_j'$, and there is an order preserving subdivision $\lambda^1, \ldots, \lambda^j$ of β' so that λ^i goes straight through β_i' for each i. But $\beta_i' = d_1, \ldots, d_{k_i}$ has been replaced by an order preserving subdivision $\alpha_1^i, \ldots, \alpha_{n_i}^i$ so that for $1 \leqslant e \leqslant n_i$, α_e^i has endlinks $d_{a_{e-1}}$ and d_{a_e}. Let $\lambda^i = c_1, \ldots, c_{m_i}$ and $p\colon \{1, \ldots, m_i\} \to \{1, 1\frac{1}{2}, \ldots, k_i-\frac{1}{2}, k_i\}$ denote the function for λ^i and β_i' described in Lemma 3.4. Let $r(0) = 1$, $r(n_i) = m_i$, and for $1 \leqslant e < n_i$, pick $r(e)$ so that $pr(e) = a_e$, and for $1 \leqslant e \leqslant n_i$, let λ_e^i denote the subchain of λ^i from $c_{r(e-1)}$ to $c_{r(e)}$. It follows directly from Lemma 3.4 that λ_e^i goes straight through α_e^i for $1 \leqslant e \leqslant n_i$. Hence β' goes straight through $\alpha_1^1, \ldots, \alpha_{n_1}^1, \ldots, \alpha_1^j, \ldots, \alpha_{n_j}^j$, which we denote by $h_{n+1}^{n+1}(\beta')$. Similarly, $g_{n+1}|\varphi_{n+2}^n(\beta')$ goes straight through $\varphi_{n+1}^{n+1}(\alpha_1^1), \ldots, \varphi_{n+1}^{n+1}(\alpha_{n_j}^j)$. Finally, let $D_i^{n+1} = D_i^n$, $T_i^{n+1} = T_i^n$, and $\varphi_i^{n+1} = \varphi_i^n$ if $i \neq n+1$ and $h_i^{n+1} = h_i^n$ if $n \neq i \neq n+1$.

Proof of Theorem 3.3. Let T and φ denote sequences as given by Definition 3.2. We construct a chain structure sequence (D', h') and sequences T' and φ' satisfying the additional hypothesis of Lemma 3.5, from which this theorem follows.

Let $(D^1, h^1) = (D, h)$, $T^1 = T$, and $\varphi^1 = \varphi$. Assuming that (D^n, h^n), T^n, and φ^n are defined, let (D^{n+1}, h^{n+1}), T^{n+1}, and φ^{n+1} be as given by Lemma 3.6. For each n, let $D_n' = D_n^n$, $T_n' = T_n^n$, $\varphi_n' = \varphi_n^n$, and $h_n' = h_n^{n+1}$. By Lemma 3.6 (4), $h_n'(\beta)$ consists of only one chain for each $\beta \in D_{n+1}'$, and by properties (1) and (2) of Lemma 3.6, $D_{n+1}' = D_{n+1}^e$, $T_{n+1}' = T_{n+1}^e$, $\varphi_{n+1}' = \varphi_{n+1}^e$, and $h_n' = h_n^e$ for each

$e \geqslant n+1$. It follows quickly that (D', h'), T' and φ' satisfy the requirements of Lemma 3.5.

In applying Theorem 3.5, the main problem is to find a workable chain structure sequence for a given one-dimensional continuum M. An inverse sequence for M can then easily be obtained by simply following the pattern, as indicated by the following example.

EXAMPLE 3.7. Let M be a spiral around a circle, e.g. the union of the unit circle in the plane and the graph, in polar coordinates, of the equation $r = 1+e^{-\theta}$ $(\theta \geqslant 0)$. There is a chain structure sequence (D, h) for M such that for each i,

(1) $D_i = \{\beta_1^i, \beta_2^i, \beta_3^i, \beta_4^i\}$,

(2) if $j<4$, then the last link of β_j^i is the first link of β_{j+1}^i,

(3) the last link of β_4^i is the last link of β_1^i,

(4) $h_i(\beta_1^{i+1}) = \beta_1^i, \beta_2^i, \beta_3^i, \beta_4^i$, and

(5) $h_i(\beta_j^{i+1}) = \beta_j^i$ if $j \neq 1$.

To follow this pattern, we let G_1 be the union of four arcs $e_1, \ldots, e_4$ such that the last point of e_i is the first point of e_{i+1} if $i<4$, and the last point of e_4 is the last point of e_1. Define $f_1\colon G_1 \to G_1$ so that $f_1|e_1$ goes straight through e_1, e_2, e_3, e_4, and if $i \neq 1$, $f_1|e_i$ is the identity on e_i. Then if for each i, $G_i = G_1$ and $f_i = f_1$, the inverse sequence (G, f) follows the pattern of (D, h). So M is homeomorphic to the limit of this inverse sequence with a single bonding map on a triangle-with-a-sticker.

4. Patterns for chainable continua. We can give a more concise notion of "pattern" for chainable continua (for the basic results on chainable, or snake-like, continua, see [1]).

DEFINITION 4.1. A sequence $s_1, s_2, \ldots$ is a *pattern sequence* for the chainable continuum M if and only if there exists a defining sequence $\{C_i = c_1^i, \ldots, c_{m_i}^i\}_{i=1}^{\infty}$ of chains for M and a sequence $\{C_i' = d_1^i, \ldots, d_{k_i}^i\}_{i=1}^{\infty}$ such that for each i,

(1) C_i' is a subsequence of C_i, no two terms of which intersect,

(2) $d_1^i = c_1^i$ and $d_{k_i}^i = c_{m_i}^i$,

(3) s_i is a sequence $s_i(1), \ldots, s_i(k_{i+1})$ of integers such that if $1 \leqslant j < k_{i+1}$, then the subchain β of C_{i+1} from d_j^{i+1} to d_{j+1}^{i+1} goes straight through the chain β' in C_i from $d_{s_i(j)}^i$ to $d_{s_i(j+1)}^i$, and no link of β intersects any element of $C_i - \beta'$.

DEFINITION 4.2. The inverse sequence (G, f) *follows the pattern sequence* $s_1, s_2, \ldots$ for the chainable continuum M if and only if for each i (continuing the notation of Definition 4.1),

(1) G_i is a number interval $[y_i, z_i]$,

(2) there is an increasing sequence $y_i = a_1^i, \ldots, a_{k_i}^i = z_i$ such that if $1 \leqslant j \leqslant k_{i+1}$, then $f_i(a_j^{i+1}) = a_{s_i(j)}^i$ and f_i is linear on $[a_j^{i+1}, a_{j+1}^{i+1}]$.

THEOREM 4.3. *The limit of any inverse sequence that follows a pattern sequence $s_1, s_2, \ldots$ for a chainable continuum M is homeomorphic to M.*

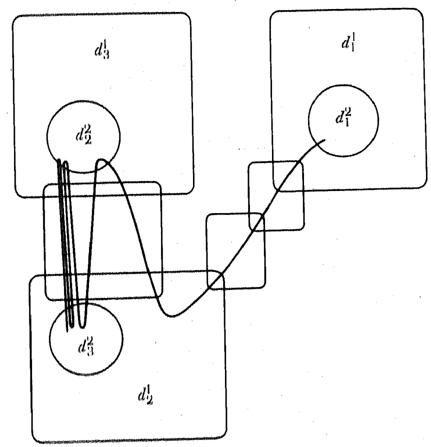
Proof. Suppose (G, f) follows the pattern sequence $s_1, s_2, \ldots$ for M and $C_1, C_2, \ldots$ and $C_1', C_2', \ldots$ are as given by Definition 4.1. For each i, let D_i denote the chain structure consisting of the subchains of C_i from d_j^i to d_{j+1}^i for $1 \leqslant j < k_i$. If β denotes the subchain of C_{i+1} from d_j^{i+1} to d_{j+1}^{i+1}, for $1 \leqslant j < k_{i+1}$, then β goes straight through the chain β' in C_i from $d_{s_i(j)}^i$ to $d_{s_i(j+1)}^i$. Using Lemma 3.4, construct an order preserving subdivision of β' whose elements are chains in D_i, and denote the result by $h_i(\beta)$. Then (D, h) is a chain structure sequence and (G, f) follows the pattern of (D, h); an application of Theorem 3.3 completes the proof.

We conclude with several examples of pattern sequences for well-known chainable continua. In each of these examples it is possible to give a pattern sequence $s_1, s_2, \ldots$ where $s_1 = s_2 = \ldots$, in which case we will call s_1 itself a *pattern* for the continuum. By following this single pattern we will be able to find an inverse sequence (G, f) on $[0, 1] = G_1 = G_2 = \ldots$ with a single bonding map $f_1 = f_2 = \ldots$ whose limit is homeomorphic to the continuum.

EXAMPLE 4.4 (The $\sin(1/x)$-continuum). Let

$$M = \{(x, y) \mid x = 0 \text{ and } -1 \leqslant y \leqslant 1\} \cup \{(x, y) \mid y = \sin(1/x) \text{ and } 0 < x \leqslant 2/\pi\}.$$

Then $(1, 3, 2)$ is a pattern for M (see the figure, where the chain C_1 and the sequences $C_1' = d_1^1, d_2^1, d_3^1$ and $C_2' = d_1^2, d_2^2, d_3^2$ are indicated). Hence if for each i, $G_i = [0, 1]$, $0 = a_1 < a_2 < a_3 = 1$, $f_i(a_1) = a_1$, $f_i(a_2) = a_3$, $f_i(a_3) = a_2$, and f_i is linear on both $[a_1, a_2]$ and $[a_2, a_3]$, then the limit of (G, f) is homeomorphic to M.



In the rest of the examples, we give only a pattern. Both a picture of the continuum and an inverse sequence with a single bonding map can easily be constructed from the pattern.

EXAMPLE 4.5. Let M' denote the reflection in the y-axis of M in the last example, and let $H = M \cup M'$ (the double $\sin(1/x)$-continuum). A pattern is $(1, 3, 2, 4)$.

EXAMPLE 4.6. Let M' denote the reflection of M in Example 4.4 in the line $x = 2/\pi$, and let $H = M \cup M'$. A pattern for H is $(2, 1, 4, 3)$.

EXAMPLE 4.7. $(1, 3, 1)$ is a pattern for a well-known indecomposable continuum with only one endpoint (see [4], p. 332, Figure 8–6).

EXAMPLE 4.8. The union of two copies of Example 4.7 joined at their endpoints is used by Bing as an example of a chainable continuum with no endpoint ([1], p. 662, Example 7). A pattern is $(3, 1, 3, 5, 3)$.

EXAMPLE 4.9. $(2, 3, 1)$ is a pattern for an indecomposable continuum with three endpoints, which is chainable (and hence irreducible) between any two of them (compare with [4], p. 142).

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Extensions, retracts, and absolute neighborhood retracts in proper shape theory

by

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Abstract. The notion of an *extension* of a proper fundamental net is defined and studied. Various results concerning this notion are obtained; these include a homotopy extension theorem and results relating the idea of extension to the concept of proper fundamental retraction. We also define absolute neighborhood proper shape retract (ANPSR), and show that the property of being an ANPSR is a hereditary proper shape invariant.

1. Introduction. In [5] Borsuk introduced the notions of *fundamental retract*, the *extension* of a fundamental sequence, *fundamental absolute retract* (FAR), and *fundamental absolute neighborhood retract* (FANR) for compacta in the Hilbert cube Q. These ideas were later studied by Mardešić [13] for compact Hausdorff spaces using the ANR-system approach to shape theory developed by Mardešić and Segal [14]. In [15], Patkowska proved the important homotopy extension theorem for fundamental sequences on compacta in Q, and this result was then used by Borsuk [6] to show that the property of being an FANR-space is a hereditary shape invariant. Results similar to these have recently been obtained for the shape theory due to Fox [9] by Godlewski ([10], [11], [12]). In a seminar at the University of Georgia during the spring of 1974, Godlewski presented an example to show that similar results do not hold in the theory of shape for metrizable spaces described by Borsuk in [7], [8]. (It was this example and its implications which, to a degree, stimulated the ideas that led to this paper.)

In [1], Ball introduced the notions of *proper fundamental retract* and *absolute proper shape retract* (APSR), which are in some sense the natural analogues in *proper* shape theory ([2], [3]) of Borsuk's fundamental retract and FAR. It is our purpose in the present paper to introduce and study the concepts of *extension* of a proper fundamental net and of *absolute neighborhood proper shape retract* (ANPSR). Perhaps it should be now noted that the notion of extension studied here is not an exact word-for-word carry over into the proper shape theory of the extension of a fundamental sequence; indeed, as noted in Section 2, the precise carry over would not yield the main results here established, notably the (proper) homotopy extension theorem (Theorem 4.1) which yields the fact that the property of being an ANPSR is a hereditary proper shape invariant (Theorem 6.5). Theorems relating the ideas of proper fundamental retraction and the extension of a proper fundamental net