An algebraic approach to the Heyting-Brouwer predicate calculus

by

Cecylia Rauszer (Warszawa)

Abstract. The main purposes of this paper are the construction of an example of semi-Boolean algebras with infinite joins and meets which permit the construction of Kripke models for the Heyting–Brouwer predicate calculus, and the proof of a representation theorem for those algebras.

The Heyting-Brouwer propositional logic is an extension of the intuitionistic propositional logic with two additional operations $\dot{-}$ and $\ulcorner$, which are dual to the intuitionistic implication and negation, respectively. An algebraic study of this logic is made in [3]. An important role in these considerations is played by so called semi-Boolean algebras. The role of those algebras is the same as that of Boolean algebras for classical logic. In [4] semi-Boolean algebras with infinite joins and meets are examined and a representation theorem — a weaker analogue of the Rasiowa–Sikorski lemma — is proved. In proving that theorem the properties of Boolean algebras are used, but that theorem is not sufficient to investigate the H-B predicate calculus. However, those algebras appear to be so strong that using only the properties of semi-Boolean algebras we can prove the following representation theorem:

For every semi-Boolean algebra there exist an order topology $\mathcal{O}(G)$ and a monomorphism h from $\mathfrak{A}$ to $\mathcal{O}(G)$ preserving all infinite joins $\bigcup_{a\in A_{2n}} a$ and meets $\bigcap_{b\in B_{2n+1}} b$, where $n \in \omega$.

The main difficulty of the proof of this theorem is to find enough Q-filters in the sense that the function assigning to every $a \in A$ the set of all Q-filters containing a is injective and preserves the semi-Boolean operations $\Rightarrow$, $\dot{-}$, $\urcorner$, $\ulcorner$. This is done in § 1 and § 2. In § 3 we can find some notions which show applications of the above results to the model theory of the Heyting–Brouwer predicate calculus.

§ 1. Semi-Boolean algebras. We shall say that an abstract algebra $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ is a *semi-Boolean algebra* provided that

(i) $(A, \cup, \cap, \Rightarrow)$ is a relatively pseudo-complement lattice,

(ii) $\dot{-}$ is a binary operation which satisfies the following condition:

$$a \dot{-} b \leqslant x \quad \text{if and only if} \quad a \leqslant b \cup x \quad \text{for any } a, b, x \in A .$$

The operation $\dot{-}$ will be called the *pseudo-difference*. This operation is dual to the relative pseudo-complement $\Rightarrow$.

In [5] it was proved that the definition of a semi-Boolean algebra given above is equivalent to the following one: An abstract algebra $(A, \cup, \cap, \Rightarrow, \dot{-}, \neg, \ulcorner)$ will be called a semi-Boolean algebra provided that $(A, \cup, \cap, \Rightarrow, \neg)$ is a pseudo-Boolean algebra [6] and $(A, \cup, \cap, \dot{-}, \ulcorner)$ is a Brouwerian algebra (2). A more detailed exposition of the properties of semi-Boolean algebras is given in [3].

An example of a semi-Boolean algebra can be constructed in the following way.

By a *quasi-ordered* set we mean an ordered pair $\mathcal{G} = \langle G, \leqslant \rangle$ — where G is a non-empty set and $\leqslant$ is a transitive and reflexive relation in G. Now, let $B \subset G$. We call B *open* if, whenever $x \in B$ and $x \leqslant y$, then $y \in B$.

For $\mathcal{O}(\mathcal{G})$ we take the collection of all open subsets of $\mathcal{G}$ and for the ordering relation $\leqslant$ we take set inclusion. We note that the algebra $(\mathcal{O}(\mathcal{G}), \cup, \cap)$ — where the operations $\cup$ and $\cap$ are the ordinary union and intersection, respectively — is a distributive lattice with the unit element G and the zero element $\varnothing$ ($\varnothing$ — the empty set). Now, let $\Rightarrow$, $\dot{-}$ be two new operations in $\mathcal{O}(\mathcal{G})$ defined by the formulas:

(1) $B \Rightarrow C = \{x \in G\colon$ for every $y \in G$ if $x \leqslant y$ and $y \in B$ then $y \in C\}$,

(2) $B \dot{-} C = \{x \in G\colon$ there exists a $y \in G$ such that $y \leqslant x$ and $y \in B$ and $y \notin C\}$,

for every $B, C \in \mathcal{O}(\mathcal{G})$.

By an easy verification we can prove the following:

1.1. *The algebra* $\mathcal{O}(\mathcal{G}) = (\mathcal{O}(\mathcal{G}), \cup, \cap, \Rightarrow, \dot{-})$ — *where* $\mathcal{O}(\mathcal{G})$ *is the family of all open sets of a quasi-ordered set* $\mathcal{G}$, *the operations* $\cup$ *and* $\cap$ *are the set-theoretical union and intersection, respectively, the operations* $\Rightarrow$ *and* $\dot{-}$ *are defined by* (1) *and* (2), *respectively* — *is a complete semi-Boolean algebra.*

Theorem 1.1 yields an important example of a semi-Boolean algebra. In the sequel this algebra will be called an *order topology*. This example of semi-Boolean algebra is typical because we have the following representation theorem:

THEOREM 1. *For every semi-Boolean algebra* $\mathfrak{A}$ *there exists an order topology* $\mathcal{O}(\mathcal{G})$ *and a monomorphism* h *from* $\mathfrak{A}$ *to* $\mathcal{O}(\mathcal{G})$.

Proof. We denote by G the set of all prime filters of a semi-Boolean algebra $\mathfrak{A}$ and let h be defined as usual, namely

$$h(a) = \{\nabla \in G\colon a \in \nabla\} \quad \text{for every } a \in A .$$

It is obvious that the system $\langle G, \leqslant \rangle$ — where $\leqslant$ is the set-theoretical inclusion — is a quasi-ordered set, and $h(a)$ is open for every $a \in A$. We denote by $\mathcal{O}(\mathcal{G})$ the class of all open sets of $\langle G, \leqslant \rangle$. It is well known that h preserves the join $\cup$ and the meet $\cap$ and, for every $\nabla \in G$, $a, b \in A$,

(3) $a \Rightarrow b \in \nabla$ iff for every $\nabla_1 \in G$ if $\nabla \subset \nabla_1$ and $a \in \nabla_1$ then $b \in \nabla_1$.

The proof of this condition may be found in [1] and it says that

(4) $$h(a \Rightarrow b) = h(a) \Rightarrow h(b) ,$$

where the sign $\Rightarrow$ on the right side of (4) is defined by (1). To show that h is the required monomorphism it is sufficient to prove that h preserves $\dot{-}$.

For this purpose we denote by $\tilde{G}$ the class of all prime ideals of the semi-Boolean algebra $\mathfrak{A}$. It is obvious that

(5) $$\nabla \in G \quad \text{iff} \quad \Delta \in \tilde{G}, \quad \text{where} \quad \Delta = A - \nabla .$$

We can prove, using the same method as in proving (3), that for every $\Delta \in \tilde{G}$

(6) $a \dot{-} b \in \Delta$ iff for every $\Delta_1 \in \tilde{G}$ if $\Delta \subset \Delta_1$ and $b \in \Delta_1$ then $a \in \Delta_1$ for any $a, b \in A$.

On account of (6) and (5) we infer that for any $\nabla \in G$ and for every $a, b \in A$,

(7) $a \dot{-} b \in \nabla$ iff there exists a $\nabla_1 \in G$ such that $\nabla_1 \subset \nabla$, $a \in \nabla_1$ and $b \in \nabla_1$.

Condition (7) proves that

(8 $$h(a \dot{-} b) = h(a) \dot{-} h(b) ,$$

where the sign $\dot{-}$ on the right side of (8) is defined by (2). This completes the proof of Theorem 1.

The following statement follows from Theorem 1 and [1]

1.2. *For every pseudo-Boolean algebra* $\mathfrak{A}$ *there exists a complete semi-Boolean algebra* $\mathfrak{A}'$ *and a monomorphism* g *from* $\mathfrak{A}$ *to* $\mathfrak{A}'$.

1.3. *For every Brouwerian algebra* $\mathfrak{A}$ *there exists a complete semi-Boolean algebra* $\mathfrak{A}'$ *and a monomorphism* h *from* $\mathfrak{A}$ *to* $\mathfrak{A}'$.

The proof of 1.3 is similar to the proof of 1.2. In this case the notion of prime filters is replaced by the notion of prime ideals. It was proved in [3] that

1.4. *Given a semi-Boolean algebra* $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ *if the infinite join* $\bigcup_{t\in T} a_t$ *exists in* $\mathfrak{A}$, *then for every* $a, b \in A$ *the joins* $\bigcup_{t\in T} a_t \cap a$, $\bigcup_{t\in T} (a_t \dot{-} a)$ *and* $\bigcup_{t\in T} ((b \cap a_t) \dot{-})$ *exist and the meets* $\bigcap_{t\in T} (a_t \Rightarrow a)$ *and* $\bigcap_{t\in T} ((b \cap a_t) \Rightarrow a)$ *also exist and*

(9) $$a \cap \bigcup_{t\in T} a_t = \bigcup_{t\in T} a_t \cap a,$$

(10) $$\Big(\bigcup_{t\in T} a_t\Big) \Rightarrow a = \bigcap_{t\in T} (a_t \Rightarrow a),$$

(11) $$\Big(\bigcup_{t\in T} a_t\Big) \dot{-} a = \bigcup_{t\in T} (a_t \dot{-} a),$$

(12) $$\Big(b \cap \bigcup_{t\in T} a_t\Big) \Rightarrow a = \bigcap_{t\in T} ((b \cap a_t) \dot{-} a),$$

(13) $$\Big(b \cap \bigcup_{t\in T} a_t\Big) \dot{-} a = \bigcup_{t\in T} ((b \cap a_t) \dot{-} a);$$

and if the infinite meet $\bigcap_{t\in T} b_t$ *exists in* $\mathfrak{A}$, *then for every* $a, b \in A$ *the joins* $\bigcup_{t\in T} (b \dot{-} a_t)$ *and* $\bigcup_{t\in T} (b \dot{-} (a_t \cup a))$ *exist and the meets* $\bigcap_{t\in T} (a \cup b_t)$, $\bigcap_{t\in T} (a \Rightarrow b_t)$ *and* $\bigcap_{t\in T} ((a \Rightarrow (b_t \cup b))$ *also exist and*

(14) $$a \cup \bigcap_{t\in T} b_t = \bigcap_{t\in T} (a \cup b_t),$$

(15) $$a \Rightarrow \bigcap_{t\in T} b_t = \bigcap_{t\in T} (a \Rightarrow b_t),$$

(16) $$a \dot{-} \bigcap_{t\in T} b_t = \bigcup_{t\in T} (b \dot{-} a_t),$$

(17) $$a \Rightarrow \big(\bigcap_{t\in T} b_t \cup b\big) = \bigcap_{t\in T} \big(a \Rightarrow (b_t \cup b)\big),$$

(18) $$a \dot{-} \big(\bigcap_{t\in T} b_t \cup b\big) = \bigcup_{t\in T} \big(a \dot{-} (b_t \cup b)\big).$$

By DPBA [4] we denote a pseudo-Boolean algebra such that condition (14) is satisfied and by DBA we denote a Brouwerian algebra such that condition (9) is fulfilled. By 1.4 we infer that

1.5. *Every semi-Boolean algebra is a* DPBA.

1.6. *Every semi-Boolean algebra is a* DBA.

§ 2. Q-filters and Q-ideals in semi-Boolean algebras. Let $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ be a semi-Boolean algebra and let (Q) be a set of infinite joins and meets in $\mathfrak{A}$

(Q) $$a_{2n} = \bigcup_{a\in A_{2n}} a \qquad (n\in\omega),$$
$$b_{2n+1} = \bigcap_{b\in B_{2n+1}} b \qquad (n\in\omega).$$

A prime filter ∇ is said to be a *Q-filter* provided that

(f_1) for every $n\in\omega$, if $a_{2n}\in\nabla$ then $A_{2n}\cap\nabla\neq\emptyset$,

(f_2) for every $n\in\omega$, if $B_{2n+1}\subset\nabla$ then $b_{2n+1}\in\nabla$.

Sometimes we say that ∇ preserves joins and meets in Q if it satisfies (f_1) and (f_2). A prime ideal Δ is said to be a *Q-ideal* provided that

(i_1) for every $n\in\omega$, if $A_{2n}\subset\Delta$ then $a_{2n}\in\Delta$,

(i_2) for every $n\in\omega$, if $b_{2n+1}\in\Delta$ then $B_{2n+1}\cap\Delta\neq\emptyset$.

Sometimes we say that Δ preserves joins and meets in (Q) if it satisfies (i_1) and (i_2). We denote by $(x]$ ($[x)$) the set of all elements $y\in A$ such that $y\leqslant x$ ($x\leqslant y$), i.e.,

$$(x] = \{y\in A:\ y\leqslant x\},$$
$$[x) = \{y\in A:\ x\leqslant y\}.$$

The next theorem is analogous to the Rasiowa–Sikorski lemma:

THEOREM 2. *Let $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ be a semi-Boolean algebra and let the set (Q) be defined as above. Let x, y be the elements of A such that the relation $x\leqslant y$ does not hold. Then there exists a Q-filter (a Q-ideal) such that $x\in\nabla$ and $y\notin\nabla$, ($x\notin\Delta$ and $y\in\Delta$).*

Proof. The proof of this theorem is preceded by the following remark:

(R) In every semi-Boolean algebra $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ for any $a, b\in A$, $a\leqslant b$ iff there exists a $c\in A$ such that $a\dot{-}b\in(c]$ and $a\Rightarrow b\in[c)$.

The proof of (R) is by an easy verification.

Now, we define two sequences $\langle\alpha_n:\ n\in\omega\rangle$ and $\langle\beta_n:\ n\in\omega\rangle$ of the elements of A such that

(i) $\alpha_0 = y$, $\beta_0 = x$,

(ii) $\alpha_{n-1}\leqslant\alpha_n$ and $\beta_{n-1}\geqslant\beta_n$ for $n>0$,

(iii) either $\beta_{2n+1}\leqslant b_{2n+1}$ or there exists a $b\in B_{2n+1}$ such that $b\leqslant\alpha_{2n+1}$ for every $n\in\omega$, either there exists an $a\in A_{2n}$ such that $\beta_{2n}\leqslant a$ or $a_{2n}\leqslant\alpha_{2n}$ for every $n\in\omega$,

(iv) the relation $\beta_n\leqslant\alpha_n$ does not hold for any $n\in\omega$.

Suppose that, for $k\in\omega$, $\alpha_0, \ldots, \alpha_{2k}$ and $\beta_0, \ldots, \beta_{2k}$ are constructed so that (i)-(iv) are fulfilled. On account of (iv) we infer that the relation $\beta_{2k}\leqslant\alpha_{2k}$ does not hold. By (R) we infer that for every $c\in A$

$$\text{either}\quad \beta_{2k}\dot{-}\alpha_{2k}\notin(c] \quad\text{or}\quad \beta_{2k}\Rightarrow\alpha_{2k}\notin[c).$$

Putting $c = b_{2k+1}$, we infer that

$$\text{either}\ \beta_{2k}\dot{-}\alpha_{2k}\notin(b_{2k+1}] \quad\text{or}\quad \beta_{2k}\Rightarrow\alpha_{2k}\notin[b_{2k+1}).$$

Suppose that $\beta_{2k}\dot{-}\alpha_{2k}\notin(b_{2k+1}]$. The condition $\beta_{2k}\dot{-}\alpha_{2k}\in(b_{2k+1}]$ is equal to the following one: for every $b\in B_{2k+1}$, $\beta_{2k}\dot{-}\alpha_{2k}\in(b]$. Thus, by our assumption we conclude that there exists a $b\in B_{2k+1}$ such that $\beta_{2k}\dot{-}\alpha_{2k}\notin(b]$. In this case we put $\beta_{2k+1} = \beta_{2k}$ and $\alpha_{2k+1} = \alpha_{2k}\cup b$. It is not difficult to check that β_{2k+1} and α_{2k+1} defined in way satisfy (ii)-(iv).

Now, suppose that $\beta_{2k}\Rightarrow\alpha_{2k}\notin[b_{2k+1})$. In this case we put $\beta_{2k+1} = b_{2k+1}\cap\beta_{2k}$ and $\alpha_{2k+1} = \alpha_{2k}$. Then (ii)-(iv) clearly hold for $n = 2k+1$.

We construct β_{2k+2} and α_{2k+2} in a similar way. By condition (iv) we infer that the relation $\beta_{2k+1}\leqslant\alpha_{2k+1}$ does not hold. Using (R), we can assume that, for every $c\in A$,

$$\text{either}\quad \beta_{2k+1}\dot{-}\alpha_{2k+1}\notin(c] \quad\text{or}\quad \beta_{2k+1}\Rightarrow\alpha_{2k+1}\notin[c).$$

Assume that $c = a_{2k+2}$ and that the first case is true, i.e., that

$$\beta_{2k+1}\dot{-}\alpha_{2k+1}\notin(a_{2k+2}].$$

Then $\beta_{2k+2} = \beta_{2k+1}$ and $\alpha_{2k+2} = \alpha_{2k+1}\cup a_{2k+2}$ satisfy (ii)-(iv).

Now, we observe that the condition $\beta_{2k+1}\Rightarrow\alpha_{2k+1}\in[a_{2k+2})$ is equal to the following one: for every $a\in A_{2k+2}$, $\beta_{2k+1}\Rightarrow\alpha_{2k+1}\in[a)$. Thus the condition $\beta_{2k+1}\Rightarrow\alpha_{2k+1}\notin[a_{2k+2})$ shows that there exists an $a\in A_{2k+2}$ such that $\beta_{2k+1}\Rightarrow\alpha_{2k+1}\notin[a)$. Putting $\beta_{2k+2} = a\cap\beta_{2k+1}$ and $\alpha_{2k+2} = \alpha_{2k+1}$, we find that (ii)-(iv) hold for $n = 2k+2$. Thus α_n and β_n are defined for all $n\in\omega$.

Let I be the ideal generated by the sequence $\langle\alpha_n\colon n\in\omega\rangle$ and let F be the filter generated by $\langle\beta_n\colon n\in\omega\rangle$. Then by (iv) I and F are disjoint and

(v) either $b_{2n+1}\in F$ or there exists a $b\in B_{2n+1}$ such that $b\in I$, for $n\in\omega$,

(vi) either there exists an $a\in A_{2n}$ such that $a\in F$ or $a_{2n}\in I$ for any $n\in\omega$.

It is well known that, in a distributive lattice, every filter can be separated from an ideal disjoint from it by a prime filter, and every ideal can be separated from a filter disjoint from it by a prime ideal. Let ∇ be a prime filter containing F such that ∇ is disjoint from I. Similarly, let Δ be a prime ideal such that Δ contains I and Δ is disjoint from ∇. It is obvious that $x\in\nabla$, $y\in\nabla$ as well as $x\notin\Delta$ and $y\in\Delta$. By (v) and (vi) ∇ is the required Q-filter and Δ is the required Q-ideal, which completes the proof of Theorem 2.

Now, we observe that by the definition of (Q) if, for every $n\in\omega$, $a_{2n}=\bigcup\limits_{a\in A_{2n}}a$ and $b_{2n+1}=\bigcap\limits_{b\in B_{2n+1}}b$ exist in $\mathfrak{A}$, then they belong to (Q). Thus and from Lemma 1.4, we infer that the joins

$$(*)\qquad \begin{array}{ll}\bigcup\limits_{a\in A_{2n}}(a\dot{-}c), & \bigcup\limits_{a\in A_{2n}}((d\cap a)\dot{-}c),\\ \bigcup\limits_{b\in B_{2n+1}}(c\dot{-}b), & \bigcup\limits_{b\in B_{2n+1}}(c\dot{-}(b\cup d))\end{array}$$

exist for every $n\in\omega$ and $c,d\in A$ but they need not be in (Q). Similarly, the meets

$$(**)\qquad \begin{array}{ll}\bigcap\limits_{a\in A_{2n}}(a\Rightarrow c), & \bigcap\limits_{a\in A_{2n}}((a\cap\ )\Rightarrow c),\\ \bigcap\limits_{b\in B_{2n+1}}(c\Rightarrow b), & \bigcap\limits_{b\in B_{2n+1}}(c\Rightarrow(b\cup a))\end{array}$$

exist for every $n\in\omega$ and $c,d\in A$ but they need not be in (Q).

We will impose some properties on the set A_{2n}, B_{2n+1} such that if a prime filter ∇ (a prime ideal Δ) preserves a_{2n}, b_{2k+1}, then, for any $c,d\in A$, it preserves infinite joins and meets given in $(*)$ $((**))$.

First we note that the following two statements result from [5] and Theorem 1.

2.1. *Let* $\mathfrak{A}=(A,\cup,\cap,\Rightarrow,\neg)$ *be a* DPBA. *For every* $n\in\omega$, *take* A_{2n}, $B_{2n+1}\subset A$, *such that*

(i) $a_{2n}=\bigcup\limits_{a\in A_{2n}}a$ *and* $b_{2n+1}=\bigcap\limits_{b\in B_{2n+1}}b$ *exist,*

(ii) *for any* $c\in A$

$$\{a\Rightarrow c|\ a\in A_{2n}\}\in\{B_{2k+1}|\ k\in\omega\},\qquad \{c\Rightarrow b|\ b\in B_{2n+1}\}\in\{B_{2k+1}|\ k\in\omega\},$$

(iii) *for any* $c,d\in A$

$$\{c\Rightarrow(b\cup d)|\ b\in B_{2n+1}\}\in\{B_{2k+1}|\ k\in\omega\}.$$

Then there exists a complete semi-Boolean algebra $\mathfrak{A}'$ *and a monomorphism* h *from* $\mathfrak{A}$ *to* $\mathfrak{A}'$ *preserving all infinite joins* a_{2n} *and meets* b_{2n+1}, $n\in\omega$.

2.2. *Let* $\mathfrak{A}=(A,\cup,\cap,\dot{-},\neg)$ *a* DBA, *and suppose that, for every* $n\in\omega$, A_{2n}, $B_{2n+1}\subset A$ *and*

(i) $a_{2n}=\bigcup\limits_{a\in A_{2n}}a$ *and* $b_{2n+1}=\bigcap\limits_{b\in B_{2n+1}}b$ *exist,*

(iv) *for any* $c\in A$,

$$\{a\dot{-}c|\ a\in A_{2n}\}\in\{A_{2k}|\ k\in\omega\},\qquad \{c\dot{-}b|\ b\in B_{2n+1}\}\in\{A_{2k}|\ k\in\omega\},$$

(v) *for any* $c,d\in A$

$$\{c\dot{-}(b\cup d)|\ b\in B_{2n+1}\}\in\{A_{2k}|\ k\in\omega\}.$$

Then there exists a complete semi-Boolean algebra $\mathfrak{A}'$ *and a monomorphism* h *from* $\mathfrak{A}$ *to* $\mathfrak{A}'$ *preserving all infinite joins and meets in* (Q).

2.3. *Let* $\mathfrak{A}=(A,\cup,\cap,\Rightarrow,\dot{-})$ *be a semi-Boolean algebra. For every* $n\in\omega$ *suppose that we have* A_{2n}, $B_{2n+1}\subset A$ *such that conditions* (i)-(v) *from* 2.1 *and* 2.2 *are satisfied.*

Then for every Q*-filter* ∇ *in* A *such that* $a\Rightarrow b\notin\nabla$ *there exists a* Q*-filter* ∇' *such that* $a\in\nabla'$, $b\notin\nabla'$ *and* $\nabla\subset\nabla'$.

Let $\mathfrak{A}=(A,\cup,\cap,\Rightarrow,\dot{-})$ be a semi-Boolean algebra and let be a Q-filter such that $a\Rightarrow b\notin\nabla$. We take the quotient algebra $\mathfrak{A}/\nabla$. On account of 2.7 [3] the algebra $\mathfrak{A}/\nabla$ need not be a semi-Boolean algebra but it is a DPBA [5] and, for every $n\in\omega$, $c\in A$ $|c|\cup\bigcap\limits_{b\in B_{2n+1}}|b|=\bigcap\limits_{b\in B_{2n+1}}(|c|\cup|b|)$. By 2.1 this algebra can be extended to a semi-Boolean algebra $\mathfrak{A}'$. More precisely: there exist a semi-Boolean algebra $\mathfrak{A}'$ and a monomorphism h from $\mathfrak{A}/\nabla$ to $\mathfrak{A}'$ preserving all infinite joins and meets in (Q).

Now, we note that the relation $h(|a|)\leqslant h(|b|)$ does not hold. This follows from the fact that $a\Rightarrow b\notin\nabla$. By Theorem 2 there exists a Q-filter $\tilde{\nabla}$ such that $h(|a|)\in\tilde{\nabla}$ and $h(|b|)\notin\tilde{\nabla}$. Let us set

$$\nabla'=\{x\in A\colon h(|x|)\in\tilde{\nabla}\}.$$

It is obvious that ∇' is a filter. Moreover, ∇' is a Q-filter as $\tilde{\nabla}$ is a Q-filter and h preserves all infinite joins and meets in (Q). We observe that $a\in\nabla'$ and $b\notin\nabla'$. Now, let $x\in\nabla$, then $|x|=V_{\mathfrak{A}/\nabla}$ and $h(|x|)=V_{\mathfrak{A}'}$. Thus $h(|x|)\in\tilde{\nabla}$ and this gives $x\in\nabla$, which proves that $\nabla\subset\nabla'$, i.e., ∇' is the required Q-filter.

In the sequel we assume that A_{2n}, B_{2n+1} always satisfy (i)-(v) of 2.1 and 2.2. We denote by G the set of all Q-filters of a semi-Boolean algebra $\mathfrak{A}$. We take $\nabla\in G$.

2.4. $a\Rightarrow b\in\nabla$ *iff, for every* $\nabla'\in G$ *such that* $\nabla\subset\nabla'$, *if* $a\in\nabla'$ *then* $b\in\nabla'$.

If $a\Rightarrow b\in\nabla$ then the lemma is obvious. On the other hand, suppose that $a\Rightarrow b\in\nabla$. On account of 2.3 we can construct a Q-filter ∇' such that $a\in\nabla'$ and $b\in\nabla'$ and $\nabla\subset\nabla'$.

Using analogous methods to those used in the proof of 2.3, we can prove the following lemma:

2.5. *Let* $\mathfrak{A}$ *be a semi-Boolean algebra and let* Δ *be a* Q*-ideal such that* $a\dot{-}b\notin\Delta$. *Then there exists a* Q*-ideal* Δ' *such that* $a\notin\Delta$, $b\in\Delta$ *and* $\Delta\subset\Delta'$.

The proof of 2.5 is analogous to the proof of 2.3, but in this case the quotient algebra $\mathfrak{A}/\Delta$ is a DBA. On account of 2.2 the algebra $\mathfrak{A}/\Delta$ can be extended to a semi-Boolean algebra $\mathfrak{A}'$. We take for the required Q-ideal Δ' the set

$$\{x \in A:\ h(|x|) \in \tilde{\Delta}\},$$

where Δ' is a Q-ideal such that $h(|a|) \notin \tilde{\Delta}$ and $h(|b|) \in \Delta$ and h is a function embedding $\mathfrak{A}/\Delta$ in $\mathfrak{A}'$ preserving all infinite joins and meets in (Q).

We denote by $\tilde{G}$ the set of all Q-ideals of a semi-Boolean algebra $\mathfrak{A}$ and let $\Delta \in \tilde{G}$. It follows from the above theorem that

2.6. *$a \dot{-} b \in \Delta$ iff for every $\Delta_1 \in \tilde{G}$ such that $\Delta \subset \Delta_1$ if $b \in \Delta_1$ then $a \in \Delta_1$.*

We observe that if ∇ is a Q-filter in a semi-Boolean algebra $\mathfrak{A}$ then $A - \nabla$ is a Q-ideal and

(1) $$\nabla \in G \quad \text{iff} \quad A - \nabla = \Delta \in \tilde{G}.$$

Let $\nabla \in G$.

2.7. *$a \dot{-} b \in \nabla$ iff there exists a $\nabla_1 \in G$ such that $\nabla_1 \subset \nabla$, $a \in \nabla_1$ and $b \notin \nabla_1$.*

Indeed, suppose that $a \dot{-} b \in \nabla$. By (1) $a \dot{-} b \notin \Delta$. Using 2.5 we infer that there exists a $\Delta_1 \in \tilde{G}$ such that $\Delta \subset \Delta_1$, $b \in \Delta_1$ and $a \notin \Delta_1$. On account of (1) we infer that there exists a $\nabla_1 = A - \Delta_1$ such that $\nabla_1 \subset \nabla \Rightarrow A - \Delta$ and $b \notin \nabla_1$ and $a \in \nabla_1$. On the other hand, the proof is similar.

THEOREM 3. *Let $\mathfrak{A}$ be a semi-Boolean algebra and let the set* (Q) *be defined as usual, and assume that A_{2n} and B_{2n+1} satisfy* (i)-(v) *of* 2.1 *and* 2.2. *Then there exists a monomorphism h from $\mathfrak{A}$ to an order topology preserving all infinite joins and meets in* (Q).

This theorem follows from 1.1, Theorem 2, 2.4 and 2.7.

§ 3. Notes on semantic models for the Heyting-Brouwer predicate calculus. Let $\mathscr{S} = \{L, C\}$ be the Heyting–Brouwer predicate calculus (briefly H–BPC) described in [3]. By a semantic model structure (H-B s.m.s) we will understand a triple $\langle \mathscr{G}, D, \Vdash \rangle$ — where $\mathscr{G}$ is a quasi-ordered set, $D \neq \emptyset$ and $\Vdash$ is a relation which satisfies the conditions (Q_0)-(Q_7) from [1] and the following ones:

(Q_8) $x \Vdash_v (\alpha \dot{-} \beta)$ iff there exists a $y \in G$, such that $y \leqslant x$ and $y \Vdash_v \alpha$ and y not $\Vdash_v \beta$,

(Q_9) $x \Vdash_v \urcorner\alpha$ iff there exists a $y \in G$ such that $y \leqslant x$ and y not $\Vdash_v \alpha$,

(Q_{10}) $x \Vdash_v \forall \xi \beta(u/\xi)$ iff for every $c \in D$, $x \Vdash_{v'} \beta(u)$, where the valuation v' is defined as follows:

$$v'(u') = \begin{cases} v(u) & \text{if} \quad u \neq u', \\ c & \text{if} \quad u = u'. \end{cases}$$

Conditions (Q_8) and (Q_{10}) give an intuitive interpretation for the new logical operations $\dot{-}$ and $\ulcorner$. Namely, let $\langle \mathscr{G}, D, \Vdash \rangle$ be a H-B s.m.s. The set G is intended to be a collection of states of our knowledge. Thus $x \in G$ may be considered as a collection of physical facts known at a particular time. If we have enough information to prove a formula α at the point of time x, we say that x forces α. So, condition (Q_8) says: to assert $(\alpha \dot{-} \beta)$ at a point of time x we need to know that there exists an earlier time y such that our information or state of knowledge at that time is not sufficient to assert that

if we get a proof of α we can also get a proof of β.

By (Q_8) this means that at every time earlier than y we can perhaps get a proof of α but we can get no proof of β.

In the same way we interpret condition (Q_9). To assert $\ulcorner\alpha$ at a point of time x we need to know that there exists an earlier point of time y such that

our information about α is not sufficient to prove α at the time y.

This means that at no time earlier than y can we get a proof of α.

The notions of an algebraic model (A-model) and semantic model (S-model) we introduce in usual way.

On account of 1.1, Theorem 2 and conditions (Q_0)-(Q_{10}) it is not difficult to prove the following theorems:

THEOREM 4. *Let $M_s = \langle G, D, \Vdash \rangle$ be a* H-B s.m.s. *Then there exists an algebraic model M_a such that, for every $\alpha \in F$, M_s is a semantic model for α if and only if M_a is an algebraic model for α.*

THEOREM 5. *Let M_a be an algebraic model. Then there exists a semantic model M_s such that, for every $\alpha \in F$, M_s is a semantic model for α if and only if M_a is an algebraic model for α.*

Now, by Theorem 4, Theorem 5, Theorem 3 and appropriate definitions we have:

3.1. *The following conditions are equivalent for any* H-B *theory $\mathscr{T}$:*

(i) *$\mathscr{T}$ is consistent,*

(ii) *there exists an A-model for $\mathscr{T}$,*

(iii) *there exists an S-model for $\mathscr{T}$.*

References

[1] M. C. Fitting, *Intuitionistic Logic, Model Theory and Forcing*, Amsterdam–London 1969.

[2] J. C. C. McKinsey and A. Tarski, *On closed elements in closure algebras*, Ann. of Math. 47 (1946), pp. 122–162.

[3] C. Rauszer, *Semi-Boolean algebras and their applications to intuitionistic logic with dual operations*, Fund. Math. 83 (1974), pp. 219–249.

[4] — *Representation theorems of semi-Boolean algebras II*, Bull. Acad. Polon. Sci. 10 (1971), pp. 889–892.

[5] — and B. Sabalski, *Remarks on distributive pseudo-Boolean algebras*, Bull. Acad. Polon. Sci. 2 (1975), pp. 123–129.

[6] H. Rasiowa and R. Sikorski, *The Mathematics of the Metamathematics*, Warszawa 1963.

Accepté par la Rédaction le 28. 5. 1975