On the division by the circle

by

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Abstract. The cartesian product leads to number-theoretical considerations within the category of topological spaces. We investigate whether or not the circle is a prime space in this context.

1. If X and Y are spaces we say that $X|Y$ if there exists a space Z such that $Y = X\times Z$ where $=$ means equality of homotopy types. We say that X is a *prime space* if whenever $X|(Y\times Z)$ then either $X|Y$ or $X|Z$. This suggests that an analogue of number theory may be attempted in the collection of homotopy types of spaces with respect to the cartesian product as multiplication. Work along this direction has been initiated by Sieradski, see [6] and [7]. In particular, in [7], be shows that neither S^3 nor $RP(3)$ is a prime space.

A space X can only be divided by S^1 if its fundamental group $\pi_1 X$ admits Z, the infinite cyclic group, as a direct factor. Hence, the study of S^1 as a direct factor has to be preceded by algebraic considerations. The group-theoretic results of Section 2 allow us to conclude that S^1 is not prime, either among finite-dimensional countable CW-complexes (and thus finite-dimensional polyhedra as well) or among finite-dimensional manifolds. Based on results about fibrations with cross-sections, we see that $Z|\pi_1 X$ implies $\Omega S^1|\Omega X$ if $\pi_1 X$ is abelian. In particular, if X admits an H-space structure, then $Z|\pi_1 X$ implies that $S^1|X$. Among its consequences we find that S^1 is prime among spaces with H-space structures. This last section is rather elementary and the results may therefore be partially known, although most of them have not been published. We assume that our spaces are of the homotopy type of pointed connected CW-complexes with homology and cohomology of finite type.

2. The object of our study are collections which admit a binary operation which is associative, commutative and which has a unit element e. Among the numerous examples, we mention the following ones:

(T): The operation induced by the cartesian product on the collection of classes of homotopy equivalent spaces.

(G): The operation induced by the direct product on the collection of isomorphism classes of groups.

(K): On the collection of positively graded families of isomorphism classes of finitely generated abelian groups, we have the operation induced by the "Kunneth-product"

$$(A_* \circ B_*)_n := \sum_{s\geqslant 0}(A_s \oplus B_{n-s}) \oplus \sum_{t\geqslant 0} \operatorname{Tor}(A_t, B_{n-1-t}) .$$

(G) has substructures (G; f.g.) and (G; ab), involving finitely generated and abelian groups respectively. We have homomorphisms fundamental group π_1: (T)$\to$(G),
higher homotopy group π_k: (T)$\to$(G; ab), $k\geqslant 2$,
homology H_*: (T)$\to$(K),
cohomology H^*: (T)$\to$(K).

All the given examples feature the fact that there is no zero-element around, that is, there is no element o in (Γ, Π) such that $\Pi(a, o) =: a\cdot o = o$ for all $a \in \Gamma$. Let us restrict henceforth to those structures (Γ, Π) which do not have a zero-element.

We define division in Γ with respect to Π by putting

$$a|b \text{ if there exists } x \in \Gamma \text{ such that } a\cdot x = b .$$

Division gives rise to the following sets:

Units $U := \{u \in \Gamma |\ u|e\}$,

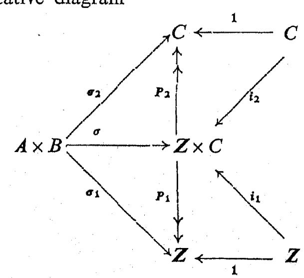
Irreducible elements $Q := \{a \in \Gamma|$ if $b|a$, $b \neq e$, then $b = a\cdot u$ for some $u \in U\}$,

Prime elements $P := \{a \in \Gamma|$ if $a|(x\cdot y)$ then either $a|x$ or $a|y\}$.

A main structure theorem in (Γ, Π) would be the existence and uniqueness (up to U) of the factorization of any element into a product of finitely many irreducible ones. For example, this theorem is true in (G; ab, f.g.). However, (G: f.g.) does not have this property (see Lemma 2 below), and the same is true for (T): In either case, there are for instance irreducible elements which are not prime (for (T), see [7]).

We start with an analysis of Z within (G). Obviously, Z is irreducible in (G), and therefore, Z is prime in (G; ab, f.g.). This result can be improved.

Let A, B, C be groups (we omit the particular structures) such that there is an isomorphism $\sigma\colon A\times B \approx Z\times C$. It is uniquely determined by its factors σ_1 and σ_2, and we have a commutative diagram



Clearly, σ_1 and σ_2 are epimorphisms, and so, we have short exact sequences

$$1 \to \ker\sigma_1 \xrightarrow{j_1} A\times B \xrightarrow{\sigma_1} Z \to 1 ,$$

$$1 \to \ker\sigma_2 \xrightarrow{j_2} A\times B \xrightarrow{\sigma_2} C \to 1 .$$

Let $n \in Z$, and let $(a, b) \in A\times B$ be the element such that $\sigma(a, b) = i_1(n)$. Since $p_2 i_1(n) = e_C$, we see that $(a, b) \in \ker\sigma_2$. Thus, the homomorphism $\tilde{\sigma}_1 := \sigma_1 j_2\colon \ker\sigma_2 \to Z$ is surjective, for

$$\tilde{\sigma}_1(a, b) = p_1\sigma j_2(a, b) = p_1\cdot i_1(n) = n .$$

We claim that $\tilde{\sigma}_1$ is actually an isomorphism. In fact, if $\tilde{\sigma}_1(a, b) = 0$, it follows that $p_1\sigma j_2(a, b) = 0$, and thus, $\sigma j_2(a, b) = i_2(c)$ for some $c \in C$. Notice that σ, j_2 and i_2 are all injective. Since $p_2 i_2(c) = c$, we get

$$c = p_2\sigma j_2(a, b) .$$

On the other hand, we have

$$p_2\sigma j_2 = \sigma_2 j_2 ,$$

which is the zero-homomorphism, as indicated by the exact sequence of σ_2. Therefore, we conclude that $c = e_C$, and then from $i_2(e_C) = \sigma j_2(a, b)$ that (a, b) is the neutral element of $\ker\sigma_2$.

Similarly, one shows that $\tilde{\sigma}_2 := \sigma_2 j_1\colon \ker\sigma_1 \to C$ is an isomorphism. Let then (a_0, b_0) be the element in $\ker\sigma_2$ such that $\tilde{\sigma}_1(a_0, b_0) = 1$, the generator of Z. Obviously, not both a_0 and b_0 can have finite order. If $\sigma_1(a_0, e_B) = s$ and $\sigma_1(e_A, b_0) = t$, then we find $1 = \sigma_1(a_0, b_0) = s+t$. In the following we will assume that $\sigma_1(a_0, e_B)$ is not of finite order. Consider then the homomorphism $\alpha\colon A \to Z$ defined by $\alpha := \sigma_1 l_1$, where $l_1\colon A \to A\times B$ is the inclusion $\{1_A, o\}$. From the above we see that $\sigma_1(a_0, e_B) = \sigma_1 l_1(a_0) = \alpha(a_0) \neq 0$, that is, α is not the trivial homomorphism. Thus, $\operatorname{im}\alpha = nZ$ for some integer $n \neq 0$.

Restricting ourselves now to (G; ab), we find that the exact sequence of α splits because im α is free. Moreover, we have then

$$\ker\alpha \overset{l_1}{\approx} \ker\sigma_1 \cap l_1 A \overset{\sigma_2}{\approx} C \cap \sigma_2 l_1 A ,$$

that is,

$$A \approx Z\times C_A,$$

where C_A is a subgroup of C.

We have shown

LEMMA 1. *Z is prime in* (G; ab).

However, this result cannot be extended to (G):

LEMMA 2. *Z is not prime in* (G; f.g.).

Proof. We give a counterexample, found by Kurosh (see [5]), in a more modern set-up.

Let A, B, C be finitely generated groups, given as follows:

$$A = \langle a_1, a_2;\ a_1^2 = a_2^2\rangle,$$
$$B = \langle b_1, b_2;\ b_1^3 = b_2^3\rangle,$$
$$C = \langle c_1, c_2, c_3, c_4;\ c_i^2 = c_j^3,\ c_ic_j = c_jc_i,\ i = 1, 2,\ j = 3, 4\rangle.$$

Let z generate Z. A homomorphism $\sigma\colon A\times B\to Z\times C$ is given by its factors σ_1 and σ_2, which are defined on the generators in the following way:

$$\sigma_1(a_i, e_B) := z =: \sigma_1(e_A, b_i),$$
$$\sigma_2(a_i, e_B) := c_i, \quad \sigma_2(e_A, b_{j-2}) := c_j.$$

One easily checks that σ_1 and σ_2 give rise to homomorphisms. Similarly, we find a homomorphism $\tau = \{\tau_1, \tau_2\}\colon Z\times C\to A\times B$ from the following data:

$$\tau_1(z, e_C) := a^{-1}, \quad \tau_1(0, c_i) := a\cdot a_i, \quad \tau_1(0, c_j) := a,$$
$$\tau_2(z, e_C) := b, \quad \tau_2(0, c_i) := b^{-1}, \quad \tau_2(0, c_j) := b^{-1}b_{j-2},$$

here, $a := a_1^2$ $(= a_2^2)$ and $b := b_1^3$ $(= b_2^3)$.

One easily checks that $\sigma\tau = 1$ and $\tau\sigma = 1$, that is, σ and τ are inverse isomorphisms.

We claim that neither A nor B can be divided by Z. Both A and B have presentations of the type $G = \langle g_1, g_2;\ g_1^n = g_2^n\rangle$ where $n\geqslant 2$. Such a group G is the pushout of the inclusions of $\langle g\rangle$ into $\langle g_1\rangle$ and $\langle g_2\rangle$, where $g := g_1^n$ $(= g_2^n)$. It follows from standard theory that the center $Z(G)$ of G is just $\langle g\rangle$, and e_G is the only element of finite order in G. Moreover, we can apply the Barr-Beck theorem to find the cohomological dimension $\mathrm{cd}(G)$ of G (see ([3]). Since $\langle g\rangle$ is free, it follows that $\mathrm{cd}\,G\leqslant 2$. The subgroup H of G generated by g and g_1g_2 is then abelian, and this is the only non-trivial relation between these generators. Therefore, $H\approx Z\times Z$ and thus, $\mathrm{cd}(H) = 2$, which implies that $\mathrm{cd}(G) = 2$.

Assume now that $Z|G$, that is, that there is a group K such that $G\approx Z\times K$. Since $\mathrm{cd}(G) = 2$, it follows that $\mathrm{cd}(K) = 1$. Consequently, K is free. Since G is not abelian, K must have more than one generator, and thus, its center $Z(K)$ must be trivial. But $G/Z(G)\approx Z_n * Z_n$, while $Z(A\times B) = Z(A)\times Z(B)$ implies that $(Z\times K)/Z(Z\times K)\approx K$, a plain contradiction to $G\approx Z\times K$.

3. The homomorphisms from (T) to (G), (G; ab) and (K) respectively, mentioned earlier, can be used to analyze the irreducibility of a space. For example, using H_* and π_1, one easily finds that S^n is irreducible for all $n\geqslant 1$. To decide whether or not S^1 is a prime space, one has to solve two problems. First, we have to study the existence of spaces X, Y and Z such that $X\times Y = S^1\times Z$ together with $X\neq S^1$ and $Y\neq S^1$. If such spaces do exist, it remains to check whether or not there is a space W such that $S^1\times W = X$ or $S^1\times W = Y$.

Obviously, if S^1 divides a space X, then Z is a direct factor of π_1X. The proof Lemma 2 leads therefore to a homotopy equivalence between products of Eilenberg–Maclane spaces, namely $K(A, 1)\times K(B, 1) = S^1\times K(C, 1)$, where A, B and C

are the groups defined above. A closer examination of these spaces entails the following result.

THEOREM 1. *S^1 is not prime among finite-dimensional countable* CW-*complexes.*

Proof. Since $K(G, 1)$ has a contractible universal covering space, we can apply Wall's results (see Theorem 3 in [4]) to see that $K(A, 1)$, $K(B, 1)$ and $K(C, 1)$ can be chosen to be countable CW-complexes. Indeed, condition (Cn) for countability (see [9] or [4]) is satisfied for all n. If $\mathrm{cd}(G) = m$, then condition (Dm) for finite dimensionality is satisfied as well. In our case, $\mathrm{cd}(A) = 2$, $\mathrm{cd}(B) = 2$, and thus, $\mathrm{cd}(C)\leqslant 3$, imply that $K(A, 1)$, $K(B, 1)$ and $K(C, 1)$ can be chosen to be 3-dimensional countable CW-complexes.

Now a d-dimensional countable CW-complex is homotopy equivalent to a d-dimensional locally compact polyhedron, and such a polyhedron can be embedded in R^{2d} (see [4]). Taking then a homotopy equivalent neighborhood, one finally obtains in our case manifolds of type $K(G, 1)$, where $d = \max(3, \mathrm{cd}(G))$:

COROLLARY 1. *S^1 is not prime among finite-dimensional manifolds.*

4. From now on we restrict ourselves to spaces with abelian fundamental group. For example, if $E\xrightarrow{p}B$ is a fibration in this class of spaces which admits a cross-section s, it follows from the exact homotopy sequence that $i_\#\colon \pi_kF\to\pi_kE$ is injective for all k, and thus, π_1F is abelian since $[\pi_1F, \pi_1F]$ is in the kernel of $i_\#$. Here, $i\colon F\to E$ is the inclusion of the fibre. Similarly, π_1B is abelian since $s_\#\colon \pi_1B\to\pi_1E$ is mono. We continue with the analysis of this example and find an isomorphism

$$\langle s_\#, i_\#\rangle\colon \pi_kB\times\pi_kF\to\pi_kE$$

for all k, which is in general not induced by a geometric map. But there is at least one case where $\langle s_\#, i_\#\rangle$ does come from a map.

Let M be the class of spaces which admit an H-space structure. As is well-known, any multiplication on such a space induces the standard structure on the homotopy groups, and its fundamental group is abelian.

In our case, if $E\in M$, it follows that $\langle s_\#, i_\#\rangle$ is just the homomorphism induced by $m(s\times i)\colon B\times F\to E$, where m is any multiplication on E. Thus, we have

LEMMA 3. *Let $E\in M$ be the total space of a fibration $F\xrightarrow{i}E\xrightarrow{p}B$ which has a cross-section s. Then we have a homotopy equivalence $m(s\times i)\colon B\times F\to E$, where m is any H-space structure on E.*

Of course, also $B\in M$ since it is dominated by E, and Lemma 3 shows then that also $F\in M$.

Remark. We would be able to deduce that E is homotopy equivalent to $B\times F$ in the general case if i admitted a homotopy left-inverse $j\colon E\to F$. In fact, if we have an exact sequence of abelian groups $o\to B\xrightarrow{\alpha}A\xrightarrow{\gamma}C\to o$ together with homomorphisms $\gamma'\colon C\to A$ and $\alpha'\colon A\to B$ such that $\gamma\gamma' = 1$ and $\alpha'\alpha = 1$, one finds an isomorphism $\{\gamma, \alpha'\}\colon A\to C\times B$: Obviously, it is injective, and $(c, b) = \{\gamma, \alpha'\}\big(\gamma'(c)+\alpha(b-\alpha'\gamma'(c))\big)$. Therefore, the map $\{p, j\}\colon E\to B\times F$ would be a homotopy equivalence.

Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration with a cross-section s. We can apply Lemma 3 to the fibration Ωp: $\Omega E \to \Omega B$, for it admits a cross-section, namely Ωs, and since E and consequently B and F have abelian fundamental groups, we do not encounter any difficulties for π_0:

LEMMA 4. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration with a cross-section s, and assume that E has an abelian fundamental group. Then there is a homotopy equivalence*

$$n(\Omega s \times \Omega i) \colon \Omega B \times \Omega F \to \Omega E,$$

where n is any multiplication on ΩE.

Note. We can obtain another homotopy equivalence $\Omega B \times \Omega F \to \Omega E$ if we use the mapping sequence of s, converted into a fibration. In fact, in the sequence $\ldots \to \Omega B \xrightarrow{\Omega s} \Omega E \xrightarrow{\partial} G \xrightarrow{\varrho} B \xrightarrow{s} E$ the fibration ∂: $\Omega E \to G$ has a cross-section φ: $G \to \Omega E$, for $\varrho \simeq ps\varrho \simeq o$. Since Ωs, the inclusion of the fibre, has a homotopy left-inverse, namely Ωp, we find a homotopy equivalence $\{\partial, \Omega p\}$: $\Omega E \to G \times \Omega B$. The composition

$$f \colon \{\Omega p, \partial\} \cdot n(\Omega s \times \Omega i) \colon \Omega B \times \Omega F \to \Omega B \times G$$

leads therefore to isomorphisms in homotopy which are given by

$$f_{\#}(\alpha, \beta) = \big(\alpha, \partial_{\#}(\Omega s)_{\#}(\alpha) + \partial_{\#}(\Omega i)_{\#}(\beta)\big).$$

Thus, we find a homotopy equivalence $p_2 \cdot fj_2$, where j_2, p_2 respectively is the obvious inclusion, projection respectively, between ΩF and the fibre of s. As the definition of f indicates, $p_2 fj_2 = \partial \cdot \Omega i$: $\Omega F \to G$. Lemma 3 gives a homotopy equivalence $n(\Omega s \times \varphi)$: $\Omega B \times G \to \Omega E$, and so, we obtain a homotopy equivalence

$$n(\Omega s \times \varphi \partial \Omega i) \colon \Omega B \times \Omega F \to \Omega E.$$

We now return to the original problems. Suppose that $Z | \pi_1 X$, where $\pi_1 X$ is abelian. Let $K(\pi_1 X, 1) \underset{s_1}{\overset{p_1}{\rightleftarrows}} S^1$ be those maps which induce the projection and inclusion of $Z | \pi_1 X$. The map s_1: $S^1 \to K(\pi_1 X, 1)$ can be stepwise lifted through a Postnikov system of X, for its induced fibrations are classified by elements in the trivial groups $H^{k+2}(S^1; \pi_{k+1} X)$. Thus, we obtain a fibration $F \xrightarrow{i} X \xrightarrow{p} S^1$ with cross section s.

Lemma 4 gives therefore rise to

PROPOSITION 1. *If $Z | \pi_1 X$, where $\pi_1 X$ is abelian, then $\Omega S^1 | \Omega X$.*

Remark. $Z | \pi_1 X$ does not imply that $S^1 | X$, even if $\pi_1 X \approx Z$. Counterexamples are provided by the generalized Klein bottles K_n, $n \geqslant 3$, total spaces of bundles $S^{n-1} \to K_n \to S^1$ (see [8]). In fact, a splitting of S^1 would contradict that K_n is not orientable.

Of course, if $S^1 | (X \times Y)$, then Z is a direct factor of either $\pi_1 X$ or $\pi_1 Y$ (or both):

COROLLARY 2. *If $S^1 | (X \times Y)$, where X and Y have abelian fundamental groups, then $\Omega S^1 | \Omega X$ or $\Omega S^1 | \Omega Y$.*

As usual, if we are dealing with spaces in $\boldsymbol{M}$, we can "deloop once more". In fact, if $X \in \boldsymbol{M}$ and $Z | \pi_1 X$, then Lemma 3 can be applied to the fibration $F \to X \to S^1$ above and yields

PROPOSITION 2. *Let $X \in \boldsymbol{M}$, and let $Z | \pi_1 X$. Then $S^1 | X$.*

Suppose now that $S^1 | (X \times Y)$, where X and Y are both spaces in $\boldsymbol{M}$. It follows that either $Z | \pi_1 X$ or $Z | \pi_1 Y$ (or both). Hence, we can apply Proposition 2 and find that either $S^1 | X$ or $S^1 | Y$ (or both). Moreover, if $S^1 | X$, then the space F such that $S^1 \times F = X$ is also in $\boldsymbol{M}$, that is, division by S^1 of spaces in $\boldsymbol{M}$ does not lead out of $\boldsymbol{M}$. In other words, we have

COROLLARY 3. *S^1 is prime in $\boldsymbol{M}$.*

Proposition 2 also implies the following partially well-known result.

COROLLARY 4. *Let $X \in \boldsymbol{M}$ have rational type $(1, n_1, n_2, \ldots)$. Then $X = S^1 \times Y$, where $Y \in \boldsymbol{M}$ has rational type $(n_1, n_2, \ldots)$.*

Proof. Since $H^1 X$ is torsionfree, $H^1(X; Q) \neq o$ implies that $H^1 X \neq o$. But $H^1 X$ is just the free part of $H_1 X$, which in turn is isomorphic to $\pi_1 X$ for $X \in \boldsymbol{M}$. Since we assume X to have homology of finite type, we see that $Z | \pi_1 X$. Hence we can apply Proposition 2. The Künneth formula takes care of the remaining part.

For example, if $X \in \boldsymbol{M}$ has rational type $(1, n)$, then Y is a rational cohomology n-sphere which admits an H-space structure. In case of finite CW-complexes, such spaces have been classified by Browder (see Theorem 5.2 in [1]). Using this list, we deduce

COROLLARY 5. *Let $X \in \boldsymbol{M}$-finite have rational type $(1, n)$. Then X has the homotopy type of $S^1 \times S^n$, $n = 1, 3, 7$, or $S^1 \times \boldsymbol{RP}(n)$, $n = 3, 7$.*

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Accepté par la Rédaction le 9. 9. 1974