A subset theorem in dimension theory *

by

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Abstract. A new class of spaces larger than the totally normal spaces is defined and the subset theorem for Ind and dim is proved for this class.

In [4] V. V. Filippov gave an example of a hereditarily normal Hausdorff space for which the subset theorem for dim and Ind fails. Thus a problem of C. H. Dowker [2] has been resolved. The example is clearly not a totally normal space since Dowker showed that the subset theorem does hold for totally normal spaces ([2] and [3]). The space of ordinal numbers not exceeding the first uncountable ordinal number is not totally normal [2]. In the present note we define a new class of spaces called super normal which is larger than the class of totally normal spaces and which includes this ordinal space. The subset theorem is proved for Ind and dim in this class of spaces.

1. Super normal spaces. The spaces under consideration need not be Hausdorff spaces.

A set U is called *D-open in a space X* if U is the union of a collection of cozero sets of X which is locally finite in U.

A space X is called *totally normal* if X is normal and each open set of X is D-open in X.

A space X is *super normal* if for each pair of separated sets A and B of X there are disjoint sets U and V D-open in X with $A\subset U$ and $B\subset V$.

THEOREM 1. *Each totally normal space is super normal. Each super normal space is hereditarily normal.*

Proof. Since a D-open set in a normal space is also normal, each open subspace of a totally normal space is normal. Thus a totally normal space is hereditarily normal. Consequently, if A and B are separated sets in a totally normal space then there are disjoint open sets U and V with $A\subset U$ and $B\subset V$. That is, a totally normal space is super normal.

The fact that a super normal space is hereditarily normal is obvious.

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EXAMPLE 1. Let X be the space of ordinal numbers not exceeding the first uncountable ordinal number Ω. It was shown in [2] that X is not totally normal. We show X is super normal.

Let A and B be separated sets in X. We consider three cases. Suppose $\sup A < \Omega$. Then $I = \{x \in X \mid x \leqslant \sup A\}$ and $J = \{x \in X \mid x > \sup A\}$ are disjoint cozero sets in X. The required D-open sets are easily found since I is a metrizable closed and open subset of X. Symmetrically, we have the case $\sup B < \Omega$. Finally, suppose $\sup A = \Omega = \sup B$. Then $\bar{A} \cap \bar{B}$ has Ω as a limit point. $X \setminus \bar{A} \cap \bar{B}$ is the union of disjoint open intervals each of which has a least upper bound less than Ω. Since A and B are separated, $A \cup B \subset X \setminus \bar{A} \cap \bar{B}$. If I is an open interval of X with $I \subset X \setminus \bar{A} \cap \bar{B}$, then I is metrizable and hence one can find disjoint D-open sets in X separating $A \cap I$ and $B \cap I$. Consequently, there are disjoint sets U and V D-open in X and contained in $X \setminus \bar{A} \cap \bar{B}$ such that $A \subset U$ and $B \subset V$. The proof is now completed.

THEOREM 2. *If X is super normal and $Y \subset X$ then Y is super normal.*

Proof. The proof is immediate.

2. The subset theorems. We now prove the subset theorems. According to C. H. Dowker a hereditarily normal space X may satisfy one of the following conditions.

(β_n): If $G \subset Y \subset X$ with G open in Y and $\operatorname{Ind} Y \leqslant n$, then $\operatorname{Ind} G \leqslant n$.

(γ_n): If $Y = \bigcup_{i=1}^{\infty} F_i \subset X$ with each F_i closed in Y and $\operatorname{Ind} F_i \leqslant n$, then $\operatorname{Ind} Y \leqslant n$.

We define a third condition.

(δ_n): If $G \subset Y \subset X$ with G D-open in Y and $\operatorname{Ind} Y \leqslant n$, then $\operatorname{Ind} G \leqslant n$.

LEMMA 3. *Let X be a super normal space.*

A. *If X satisfies condition (γ_{n-1}) then it satisfies condition (δ_n).*

B. *If X satisfies conditions (β_{n-1}) and (δ_n) then it satisfies condition (β_n).*

Proof. The proof of statement A is a modification of that of C. H. Dowker [2]. (See [6] Lemmas 11-3 and 11-4, p. 60.)

We prove statement B. Let Y be a subset of X with $\operatorname{Ind} Y \leqslant n$ and G be open in Y. It will be evident from the proof that we may assume $Y = X$. Let A and B be two disjoint sets closed in G. Then A and $C = [(X \setminus G) \cup B] \setminus \bar{A}$ are separated in X. Hence there are disjoint sets U and V which are D-open in X with $A \subset U$ and $C \subset V$. Since $U \cap \bar{V} = \emptyset$, we have $U \cap \bar{B} = \emptyset$. Let $D = \bar{A} \setminus U$. Then D is closed in X and $\bar{A} \setminus D$ and $(X \setminus U) \setminus D$ are disjoint sets closed in $X \setminus D$. Hence there are disjoint sets S and T open in $X \setminus D$ such that $\bar{A} \setminus D \subset S$ and $(X \setminus U) \setminus D \subset T$. Clearly, S and T are open in X. Obviously, $S \subset U$. Since U is D-open in X, $\operatorname{Ind} U \leqslant n$ by condition (δ_n). So there is a set W open in U such that $\bar{A} \setminus D = \bar{A} \cap U \subset W \subset S$ and $\operatorname{Ind} B_U(W) \leqslant n-1$. W is also open in X. Now, $\bar{W} \subset \bar{S} \subset X \setminus T \subset U \cup D$. Hence $\bar{W} \setminus D \subset U$. Finally, $B \cap W \subset B \cap U = \emptyset$ and $X \setminus D \supset X \setminus (\bar{A} \setminus A) \supset G \cap [X \setminus (\bar{A} \setminus A)] = G \setminus (\bar{A} \setminus A) = G$. Since $W \subset U \subset X \setminus D = \bar{A} \cap U \subset W \subset X \setminus B$ we get $A \subset W \cap G \subset G \setminus B$. Since $B_G(W \cap G)$ $= B_{X \setminus D}(W) \cap G$ is open in $B_{X \setminus D}(W)$, condition (β_{n-1}) gives $\operatorname{Ind} B_G(W \cap G) \leqslant n-1$. Thus we have shown $\operatorname{Ind} G \leqslant n$.

THEOREM 4. *Let X be super normal and $Y \subset X$. Then* $\operatorname{Ind} Y \leqslant \operatorname{Ind} X$.

Proof. It is known that for a hereditarily normal space X, condition (β_n) implies condition (γ_n). (See [2] or [6], Lemma 11-2, p. 60.) Also, condition (β_n) implies condition (δ_n). Conditions (γ_{-1}) and (β_{-1}) are trivially satisfied. With the aid of Lemma 3, one can prove conditions (γ_n) and (β_n) are valid for every n by induction. Consequently, $\operatorname{Ind} Y \leqslant \operatorname{Ind} X$ for each open set Y of X. We complete the proof for arbitrary subsets of X by referring to a lemma of C. H. Dowker [6], Lemma 11-2, p. 60.

LEMMA 5. *Let X be normal and Y be D-open in X. Then* $\dim Y \leqslant \dim X$.

Proof. Being D-open in X, Y is the union of a collection $\mathcal{U}$ of cozero sets of X which is locally finite in Y. Y is normal and each member $U \in \mathcal{U}$ has $\dim U \leqslant \dim X$. There is a locally finite closed covering $\mathcal{V}$ of Y which shrinks $\mathcal{U}$. By the locally finite sum theorem for covering dimension [7], we have $\dim Y \leqslant \dim X$.

THEOREM 6. *Let X be super normal and $Y \subset X$. Then* $\dim Y \leqslant \dim X$.

Proof. Let $\mathcal{V} = \{V_1, V_2, \ldots, V_k\}$ be a finite cover of Y by sets open in Y. There is a collection $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ of sets closed in Y such that $F_i \subset V_i$ $(i = 1, 2, \ldots, k)$ which covers Y. Now F_i and $Y \setminus V_i$ are separated sets in X. Hence there are disjoint sets U_i, W_i, D-open in X, such that $F_i \subset U_i$ and $Y \setminus V_i \subset W_i$ $(i = 1, 2, \ldots, k)$. Obviously, $\{U_i \cap Y \mid i = 1, 2, \ldots, k\}$ is a refinement of $\mathcal{V}$ and covers Y. Let $Z = \bigcup \{U_i \mid i = 1, 2, \ldots, k\}$. Z is normal and $\dim U_i \leqslant \dim X$ $(i = 1, 2, \ldots, k)$. Hence $\dim Z \leqslant \dim X$. The open cover $\{U_i \mid i = 1, 2, \ldots, k\}$ of Z has an open cover refinement $\mathcal{M}$ with order $\mathcal{M} \leqslant \dim X + 1$. $\mathcal{M} | Y$ is an open cover refinement of $\mathcal{V}$ and order $\mathcal{M} | Y \leqslant \dim X + 1$. Thereby, we have shown $\dim Y \leqslant \dim X$.

3. Remarks. The class of super normal spaces can be enlarged to one in which the subset theorems remain valid as follows.

Let $\mathcal{K}$ be the class hereditarily normal spaces which are the union of a σ-locally finite closed cover of super normal subspaces.

THEOREM 7. *If X is hereditarily normal and is the union of a σ-locally finite closed cover $\mathcal{F} \subset \mathcal{K}$ then $X \in \mathcal{K}$.*

Proof. The proof is straight-forward.

LEMMA 8. *Let X be hereditarily normal and $\mathcal{F}$ be a σ-locally finite closed cover of X by super normal subspaces with $\operatorname{Ind} F \leqslant n$ for each $F \in \mathcal{F}$. Then $\operatorname{Ind} X \leqslant n$.*

Proof. It is clear that one needs to prove a countable sum theorem and a locally finite sum theorem. The countable sum theorem follows from a lemma of Dowker ([6], Lemma 11-2, p. 60) and Theorem 4 above. To prove the locally finite sum theorem, one can easily adapt the proof of [5] with the aid of Theorem 4 above.

THEOREM 9. *For the class $\mathscr{K}$ and the dimensions* Ind *and* dim *the following are true.*

1. *The countable sum theorem.*
2. *The locally finite sum theorem.*
3. *The subset theorem.*

Proof. The case of Ind: The two sum theorems are valid due to Theorem 7 and Lemma 8. The subset theorem follows from Theorems 4 and 7 and Lemma 8.

The case of dim: The two sum theorems are known to be valid under more general conditions. (See [6], Theorem 9–10, p. 53, and [7], Theorem 2, p. 212.) The subset theorem follows from the sum theorems and Theorems 6 and 7.

EXAMPLE 2. The following is an example a space in the class $\mathscr{K}$ which is not super normal.

Let X be the one-point compactification of the countable disjoint topological sum of the space of ordinal numbers less than the first uncountable ordinal number. The verification is left to the reader.

4. The huge inductive dimension. In [1], Aarts defined a new dimension on the class of hereditarily normal spaces called the huge inductive dimension, Hind. The proof of a sum theorem for Hind is extremely simple and Hind$\geqslant$Ind. The subset theorem for Hind also fails in the class of hereditarily normal spaces. In [1], it is shown that $\mathrm{Hind}X = \mathrm{Ind}X$ for totally normal spaces X. We remark that the equality also holds for spaces in our class.

After the completion of the present paper, Professor R. Engelking made the following two observations to the author through a correspondence. We include them with his permission.

THEOREM. *A space X is super normal if and only if X is hereditarily normal and every regular open set is D-open in X. The hereditarily normal condition cannot be deleted.* (*A regular open set U is a set of the form $U = \mathrm{Int}\,\overline{U}$.*)

Lifanov and Pasynkov in Vestnik MGU No. 3 (1970), pp. 33–37, have defined a Dowker space to be a space X which is hereditarily normal and every open set $U\subset X$ is the union of a point-finite family of open F_σ-sets in X. We could modify our definition of D-open sets to be a set U in X which is the union of a point-finite family of open F_σ-sets in X. Then we could define super normality using these D-open sets. The subset theorem for Ind will now remain valid for these super normal spaces.

References

[1] J. M. Aarts, *A new dimension function*, Proc. Amer. Math. Soc. (to appear).
[2] C. H. Dowker, *Inductive dimension of completely normal spaces*, Quart. J. Math. Oxford (2) 4 (1953), pp. 267–281.
[3] — *Local dimension of normal spaces*, Quart. J. Math. Oxford (2), 6 (1955), pp. 101–120.

[4] V. V. Filippov, *On the dimension of normal spaces*, Soviet Math. Dokl. 14 (1973), pp. 547–550.
[5] N. Kimura, *On a sum theorem in dimension theory*, Proc. Japan Acad. 43 (1967), pp. 98–102.
[6] K. Nagami, *Dimension Theory*, Pure and Appl. Math. Ser. No. 37, New York 1970.
[7] P. A. Ostrand, *Covering dimension in general spaces*, Gen. Topology and its Appl. 1 (1971), pp. 209–221.

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