In particular, we have now established

**Theorem 3.** Suppose $\mathfrak{B}$ is $(P^*)^\kappa$-generic. Then $\mathfrak{B}$ has Scott height at most $\text{o}(P^*)$.

An absoluteness argument shows that it is not really necessary to assume that $\mathfrak{A}$ is countable.

The example of the previous section shows that a $(P^*)^\kappa$-generic structure need not be $P^\kappa$-generic, nor even of height $\leq \text{o}(P^*)$. If we assume that our original theory $T$ is complete, it is then clear that all $(P^*)^\kappa$-generic structures have the same Scott height.

References


Homogeneity, universality and saturatedness of limit reduced powers III

by

Leszek Pachowski (Wrocław)

Abstract. Let $\mathcal{F}$ be an ultrafilter on $J$ and $\mathcal{G}$ a filter over $I \times J$. The paper gives a characterization of those pairs $(\mathcal{F}, \mathcal{G})$ which have the property that for every relational structure $\mathfrak{B}$ the limit ultrapower $\mathfrak{B}_{\mathcal{G}}/\mathcal{G}$ is $\kappa^\mu$-saturated. The notion used to obtain this characterization is a natural extension of Keisler’s notion of a $\kappa^\mu$-good filter.

A property $P$ of a relational structure $\mathfrak{A}$ is a compactness type property if there is a definition of $P$ which is of the form: for every set $\Sigma$ of formulae (of some language connected with $\mathfrak{A}$), $\mathfrak{A}$ can be satisfied in $\mathfrak{A}$ if and only if every finite subset of $\Sigma$ can be satisfied in $\mathfrak{A}$. The saturatedness, universality and homogeneity of relational structures can be considered as properties of the compactness type. Various other properties of the compactness type have been investigated by several authors (e.g. atomic compactness [6], [11], positive compactness [11]). Here we restrict ourselves to saturatedness, homogeneity and universality.

By the classical results of Keisler ([3], [4]) ultraproducts can be used to obtain structures with a given compactness type property. For example, if $\mathcal{F}$ is $(\alpha, \nu)$-regular, then for every relational structure $\mathfrak{B}$ with $[L(\mathfrak{B})]^{\kappa^\nu} \leq \kappa$, the ultrapower $\mathfrak{B}_{\mathcal{F}}/\mathcal{F}$ is $\kappa^\mu$-universal. If $\mathcal{F}$ is $\kappa^\mu$-good, then for every family $\{\mathfrak{B}_i : i \in I\}$ of similar relational structures with $[L(\mathfrak{B}_i)]^{\kappa^\nu} \leq \kappa$, the ultraproduct $\prod_{i \in I} \mathfrak{B}_i/\mathcal{F}$ is $\kappa^\mu$-saturated.

The results of Keisler have been extended by Shelah and the present author to the case of products which are not necessarily maximal (see [7] and [10]). Another application of reduced products to compactness can be found in [8]. For the generalization of Keisler’s results to Boolean ultrapowers see [5].

The problem of homogeneity of reduced products had not been extensively investigated. By a recent result of Wierzejewski [13] if the ultrapower $\mathfrak{B}_{\mathcal{F}}/\mathcal{F}$ is $\kappa^\mu$-homogeneous for every structure $\mathfrak{A}$, then for every $\mathfrak{B}$ the ultrapower $\mathfrak{B}_{\mathcal{G}}/\mathcal{G}$ is $\kappa^\mu$-saturated.

In the present paper we investigate the problem of compactness of limit ultrapowers. We give a characterization of pairs $(\mathcal{F}, \mathcal{G})$ which have the property that for every relational structure $\mathfrak{A}$ such that $[L(\mathfrak{A})]^{\kappa^\nu} \leq \kappa$, the limit ultrapower $\mathfrak{A}_{\mathcal{G}}/\mathcal{G}$ is...
$\kappa^+$-saturated ($\mathcal{F}$ is an ultrafilter on $I$ and $\mathcal{G}$ is a filter over $I \times I$). We also deal with limit ultrapowers which are $\kappa^+$-universal.

This paper is a by product of an attempt to answer the question of J. Wierzejewski (see the introduction in [13]) whether it is possible to give for homogeneity a characterization similar to that given in [7] (cf. also [10]) for saturatedness.

I would like to mention that the results below would not have been obtained without the encouragement of all the members of the seminar on model theory in Wrocław, especially by B. Węglorz.

An extension of the results below to the case of filters which are not necessarily maximal will be published in [9].

0. Our terminology is standard and coincides with the terminology of [1]. Let $I$ be a non-empty set. Then $E(I)$ denote the set of all equivalence relations on $I$. Let $f: I \to A$ be a function. Then $\text{eq}(f) = \{(i, j) \in I \times I : f(i) = f(j)\}$. Of course $\text{eq}(f) \subseteq E(I)$. Let $\mathcal{G}$ be a filter in $E(I)$ (i.e., $\mathcal{G} \subseteq E(I)$; if $\varphi_1, \varphi_2 \in \mathcal{G}$, then $\varphi_1 \cap \varphi_2 \in \mathcal{G}$; if $\varphi_1 \subseteq \varphi_2 \in E(I)$, $\varphi_1 \in \mathcal{G}$, then $\varphi_2 \in \mathcal{G}$). If $A \neq \emptyset$, then by $A^\mathcal{G}$ we denote $\{f \in A^I : \text{eq}(f) \in \mathcal{G}\}$. In particular, $2^I[\mathcal{G}]$ is the algebra of subsets of $I$ which can be composed of the equivalence classes of a relation in $\mathcal{G}$. If $\varphi \in E(I)$, then $\mathcal{G}[\varphi]$ denotes the algebra of subsets which are unions of equivalence classes of $\varphi$. Let $\mathcal{G}$ and $\mathcal{A}$ be as above and let $\mathcal{F}$ be a filter over $I$. If $A^\mathcal{G}[\mathcal{F}]$, then $f[\mathcal{F}] = \{g \in A^I : \{i \in I : f(i) = g(i)\} \in \mathcal{F}\}$. We put $A^\mathcal{G}[\mathcal{F}] = (f[\mathcal{F}] : f \in A^\mathcal{G}[\mathcal{F}])$. If $\mathcal{H}$ is a relational structure, then $\mathcal{H}[\mathcal{G}]$ is a substructure of $\mathcal{H}[\mathcal{F}]$ with the universe $A^\mathcal{G}[\mathcal{F}]$ (see [2]). If $I$ is a set, $i \in I$ and $g \in E(I)$, then $1/g = \{j : j \in I, (j, i) \in g\}$ and $1_i = \{j : i \in I, (j, i) \in g\}$.

Let $X$ be a set; then $S(X)$ denotes the set of all subsets of $X$ and $S_T(X)$ is the set of all finite subsets of $X$. Let $f: S_T(X) \to \mathcal{S}(I)$. We say that $f$ is monotonic if $\forall x \in X$ implies $f(\{x\}) \subseteq f(x)$. A function $f$ is additive if $(x \cup y) = (x) \cap (y)$ for every $x, y \in S_T(X)$. Let $g: S_T(X) \to \mathcal{S}(I)$. We write $f \models g$ to denote that $f(x) \subseteq f(y)$ for every $x, y \in S_T(X)$. The image of $x$ in $f$ by $g$ is denoted by $f^*x$. If $A$ is a set, then $|A|$ is the cardinality of $A$, and $\times$ is always an infinite cardinal. By $(\mathcal{L})$ we denote the language of $\mathcal{A}$. For other definitions consult [1].

1. Recall that a filter $\mathcal{F}$ is $(\omega, \times)$-regular if and only if there is an $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $|\mathcal{F}_0| = \kappa^+$ and $\cap \mathcal{F}_0 = \emptyset$, and $\mathcal{F}$ is $\kappa^+$-good if and only if $\mathcal{F}$ is $(\omega, \times)$-regular and for every monotonic function $f: S_T(X) \to \mathcal{S}(I)$, there is an additive function $g: S_T(X) \to \mathcal{S}(I)$ such that $g \models f$. If $\mathcal{F}$ is a $\kappa^+$-good filter, then $\mathcal{F}$ is $(\omega, \times)$-regular (see [4]). It is possible to give a definition of $\kappa^+$-goodness in which $(\omega, \times)$-regularity is explicitly stated.

**Definition 1.1**

1. Let $h: S_T(X) \to \mathcal{S}(I)$. We say that $h$ is a partition function if $x \neq i$ implies $h(i) = h(t) = 0$ and moreover $\bigcup_{x \in S_T(X)} h(x) = I$.

2. Let $h: S_T(X) \to \mathcal{S}(I)$. Then $g$ is a union function of $h$ if $g(i) = \bigcup_{x \in S_T(X)} h(x)$. The union function of $h$ is denoted by $u_h$.

3. If $\mathcal{F}$ is $(\omega, \times)$-regular* if there is a partition function $h: S_T(X) \to \mathcal{S}(I)$ such that for $x \in S_T(X)$, $u_h(x) \in \mathcal{F}$.

4. $\mathcal{F}$ is $\kappa^+$-good* if for every monotonic function $f: S_T(X) \to \mathcal{S}(I)$ there is a partition function $h: S_T(X) \to \mathcal{S}(I)$ such that $u_h(x) \in \mathcal{F}$ for $x \in S_T(X)$ and $u_h \in \mathcal{F}$.

It is obvious that if $\mathcal{F}$ is $\kappa^+$-good*, then it is also $(\omega, \times)$-regular*.

**Proposition 1.2**

1. $\mathcal{F}$ is $(\omega, \times)$-regular if and only if $\mathcal{F}$ is $(\omega, \times)$-regular*.

2. $\mathcal{F}$ is $\kappa^+$-good if and only if $\mathcal{F}$ is $\kappa^+$-good*.

**Proof.** 1. Assume that $\mathcal{F}$ is $(\omega, \times)$-regular. Then there is a family $\{I_i : i \in I\}$ of distinct elements of $\mathcal{F}$ such that for every infinite subset $X$ of $\kappa$ we have $\bigcap_{i \in I} I_i = \emptyset$.

We put $h(i) = \{i : i \in I_i \Rightarrow i \in I\}$. It is obvious that $h$ is a partition function. Moreover $u_h(i) = \bigcap_{i \in I_i} I_i$, whence $u_h(i) \in \mathcal{F}$.

Now assume that $\mathcal{F}$ is $(\omega, \times)$-regular*. Since $u_h(i) \in \mathcal{F}$ for every $s \in S_T(X)$, it remains to prove that if $X$ is an infinite subset of $S_T(X)$, then $\bigcap_{i \in I_i} I_i = \emptyset$.

In fact, assume that $i \in \cap u_h(i)$; then there is an $s \in S_T(X)$ such that $i \in h(s)$ and $\exists x \neq \emptyset$.

2. Assume that $\mathcal{F}$ is $\kappa^+$-good. Then by a theorem of Keisler [3] $\mathcal{F}$ is $(\omega, \times)$-regular. Let $\{I_i : x \in S_T(X)\}$ be a family of distinct elements of $\mathcal{F}$ such that $\cap I_i$ is empty for every infinite $X, X \subseteq S_T(X)$. Let $f: S_T(X) \to \mathcal{S}(I)$ and let $g: S_T(X) \to \mathcal{S}(I)$ be such that $g \models f$. We put $u(i) = g(i) \cap I_i$ and $h(i) = \{i : i \in u(i) \Rightarrow i \in I\}$. It is a matter of simple computation to check that $u = u_h$ and consequently $u_h \in \mathcal{F}$ and $u_h \in \mathcal{F}$ for $x \in S_T(X)$.

**Definition 1.3** Let $\mathcal{F} \subseteq I(I)$ and $\mathcal{G} \subseteq E(I)$ be arbitrary filters.

1. We say that the pair $(\mathcal{F}, \mathcal{G})$ is $(\omega, \times)$-regular if there is a $g \in \mathcal{G}$ and a function $h: S_T(X) \to I(I)$ such that $h$ is a partition function and $u_h(\mathcal{G}) \subseteq \mathcal{G}$ for every $x \in S_T(X)$.

2. Let $h: S_T(X) \to \mathcal{S}(I)$ be a partition function and let $F: S_T(X) \to E(I)$. Then $h \models F$ is an equivalence relation on $I$ such that $(i, f) \in h \models F$ if and only if $i \neq h(t) \equiv f$. $h \models F$ holds for every $x \in S_T(X)$ and moreover if $i \in h(t)$ for some $t \in S_T(X)$, then $(i, f) \in F(t)$.

3. The pair $(\mathcal{F}, \mathcal{G})$ is $\kappa^+$-good* if and only if for every additive function $F: S_T(X) \to \mathcal{G}$ and every monotonic function $f: S_T(X) \to \mathcal{S}(I)$ such that $\mathcal{F} \subseteq E(I)$.

   \[ f(S_T(I)) \subseteq 2^I[\mathcal{G}] \]
there is a partition function \( h: S_\lambda(x) \rightarrow S(I) \) such that

\[
\text{(1.1)} & \quad u_\lambda \leq f, \\
\text{(2.2)} & \quad u_\lambda(x) \in \mathcal{F} \quad \text{for every } x \in S_\lambda(x) \\
\text{and} & \\
\text{(3.3)} & \quad h \ast F \in \mathcal{F}.
\]

It follows from the definition that if a pair \((\mathcal{F}, \mathcal{G})\) is \(\kappa\)-good*, then it is \(\langle 0, \kappa\rangle\)-regular. To check this take \(F(x) = 1 \ast x \ast F\) and \(f(x) = f(x)\).

It is possible to give a definition of \(\kappa\)-goodness of a pair of filters which is more similar to the original definition of \(\kappa\)-goodness of a filter.

**Definition 1.4.** The pair \((\mathcal{F}, \mathcal{G})\) is \(\kappa\)-good if and only if it is \((\alpha, \alpha)\)-regular and for every monotonic function \( f: S_\lambda(x) \rightarrow \mathcal{F} \) and every additive function \( F: S_\lambda(x) \rightarrow \mathcal{G} \) such that \( f \ast S_\lambda(x) \leq 2 \ast \mathcal{G} \) there is a \( g \in \mathcal{G} \) and an additive function \( g: S_\lambda(x) \rightarrow \mathcal{G} \) such that

\[
\text{(4.1)} & \quad g \leq f \\
\text{(4.2)} & \quad g \ast S_\lambda(x) \leq 2 \ast \mathcal{G} \\
\text{and} & \\
\text{(4.3)} & \quad \text{for every } x \in I \ast x \text{ and every } i, j \in x \text{ if } x \in g(x) \text{, then } (i, j) \in F(x) \text{ (i.e., relation } g \text{ is on } g(x) \text{ finer than } F(x)) \in \mathcal{G}.
\]

**Lemma 1.5.** If \((\mathcal{F}, \mathcal{G})\) is \(\kappa\)-good, then \((\mathcal{F}', \mathcal{G}')\) is \(\kappa\)-good*.

**Proof.** Let \( h: S_\lambda(x) \rightarrow S(I) \) be a function whose existence follows from the \((\alpha, \alpha)\)-regularity of \((\mathcal{F}, \mathcal{G})\). Let \( \mathcal{G}_0 \in \mathcal{F}' \) be such that \( \mathcal{G}_0(S_\lambda(x)) = S_\lambda(x) \). Let \( F = F \) and \( f' = f \) be functions as in the definition of \(\kappa\)-goodness*. For \( s \in S_\lambda(x) \) we put \( f'(s) = f(s) \ast \mathcal{G}_0(x) \), where \( i = 0, 1, 2, \ldots, \lceil |x| - 1 \rceil \). Since \( \mathcal{G}_0 \) and \( f' \) satisfy the hypotheses of Definition 1.4, there is \( g \in \mathcal{G} \) and \( g: S_\lambda(x) \rightarrow \mathcal{G}_0 \) such that (4.1)-(4.3) hold with \((f, \mathcal{G})\) replaced by \((f', \mathcal{G}_0)\). Let

\[
h(t) = \{ i, j \in x : t \in g(x) \}.
\]

Of course \( h \) is a partition function. Now let \( i \in g(x) \). We claim that there is a \( t \) such that \( i \in g(t) \) and \( i \in g'(r) \) for every \( r \neq i \). Suppose not; then there is an infinite sequence \( \{ t_i \}_{i \in \mathcal{I}} \) such that \( i \in g(t_i) \) and \( t_i \neq t_j \) for every \( n \neq m \). Since \( g(t_i) \leq f(x), \) we have \( i \in s \cup f(x) \), whence \( i \in \mathcal{G}_0(x) \), which is impossible. We have just proved that for every \( i \in g(x) \) there is a \( t \) such that \( i \in g(t) \), whence \( h(x) \). This proves that \( g \) is \(\mathcal{G}_0 \ast \mathcal{G} \). But \( g \in \mathcal{F} \), and consequently \( u_\lambda(x) \in \mathcal{G} \). Moreover, since \( g(x) \subseteq f(x) \), we have \( u_\lambda(x) \subseteq f(x) \). This proves that \( h \) is \(\mathcal{G}_0 \ast \mathcal{G} \).

It remains to prove that \( h \ast \mathcal{G} \). In fact, let \( i, j \in \mathcal{G} \). Then for every \( s \in S_\lambda(x) \) we have \( h(s) \ast \mathcal{G} \), whence for arbitrary \( s \in S_\lambda(x) \) every equivalence class of \( s \) is disjoint with \( h(s) \) or included in \( h(s) \). Now let \( i, j \in x \) and \( x \in I \ast i \). Then there is an \( s \in S_\lambda(x) \) such that \( x \in s \in h(s) \).

in particular \( x \subseteq h(s) \). Consequently by the definition of \(\kappa\)-goodness we have \((i, j) \in F(x) \), which finishes the proof that \((\mathcal{F}, \mathcal{G})\) is \(\kappa\)-good*. 

**Theorem 1.6.** \((\mathcal{F}, \mathcal{G})\) is \(\kappa\)-good if and only if \((\mathcal{F}, \mathcal{G})\) is \(\kappa\)-good*.

**Proof.** If \((\mathcal{F}, \mathcal{G})\) is \(\kappa\)-good, then it is \(\kappa\)-good* by Lemma 1.5. If \((\mathcal{F}, \mathcal{G})\) is \(\kappa\)-good*, then putting \( g = h \ast F \) and \( g(x) = u_\lambda(x) \), we obtain an equivalence relation and a function which satisfies properties (g.1)-(g.3).

2. Now we are ready to state and prove the main results of the paper. The necessity of the assumptions of \(\kappa\)-goodness in the theorem below will be proved in the next section.

**Theorem 2.1.** If \( \mathcal{F} \subseteq S(I) \) is an ultrafilter and \( \mathcal{G} \subseteq S(I) \) is a filter such that \((\mathcal{F}, \mathcal{G})\) is \(\kappa\)-good, then for every relational structure \( \mathcal{U} \) with \(|L(\mathcal{U})| \leq \kappa \) the limit ultrapower \( \mathcal{U}_\mathcal{F}^\mathcal{G} \) is \(\kappa^\ast\)-saturated.

**Proof.** Let \( \mathcal{U} \) be a relational structure with \(|L(\mathcal{U})| \leq \kappa \). Let \( \langle a_i : i < \kappa \rangle \) be a sequence of elements of \( A \) and finally let \( \Sigma \) denote a set of (power \( \kappa \)) elements of \( L(\mathcal{U}) \) with one free variable \( x \). We assume that \( \Sigma \) is finitely satisfiable in \( \mathcal{U} = \langle [a_i]_{i < \kappa} \rangle \). For \( x \in S_\lambda(x) \) and \( i, j \in x \) we put \( F(x) = \bigcap \{ \{i \in x : x \subseteq [a_i] \} \} \). It is obvious that \( F \) is an additive function and \( F \) is a monotonic function. Moreover, for every \( s \in S_\lambda(x) \) we have \( \mathcal{U} \models \Sigma \subseteq \lambda \); consequently \( f(x) \in \mathcal{F} \). Finally (g.9) follows from the fact that \( f(x) \in \mathcal{F} \).

Consequently, there is an \( a \in \mathcal{U} \) such that for every \( i \in x \), \( [a_i]_{i < \kappa} \) holds (if \( x = h(0) \) then \( a_0 \) is an arbitrary element of \( A \)). We put \( a(x) = a_0 \), where \( x \) is an element of \( I(h \ast F) \) such that \( i \in x \). By definition \( a \) is constant on every equivalence class of \( h \ast F \), whence \( a \in A \). We claim that \( \mathcal{U} = \langle [a_i]_{i < \kappa} \rangle \). Then

\[
E_\mathcal{U} = \bigcup_{x \in I(h \ast F)} h(x) = u_\lambda(x) \in \mathcal{F}.
\]

But it follows from the definition of \( a \) that \( i \in g(x) \), whence \( a(x) \). 

**Theorem 2.2.** If \( \mathcal{F} \subseteq S(I) \) is an ultrafilter and \( \mathcal{G} \subseteq S(I) \) is a filter such that \((\mathcal{F}, \mathcal{G})\) is \((\alpha, \alpha)\)-regular, then for every structure \( \mathcal{U} \) with \(|L(\mathcal{U})| \leq \kappa \) the limit ultrapower \( \mathcal{U}_\mathcal{F}^\mathcal{G} \) is \(\kappa^\ast\)-universal.

**Proof.** We proceed almost exactly as in the proof of Theorem 1.5 in [4]. The only difference is that for \( s \in S_\lambda(x) \) we defined functions constant on \( h(x) \).
Theorem 2.2 was obtained independently by B. Węglorz as a corollary to his embedding theorem (see [12]).

Now we shall give an application of Theorem 2.2 to the problem of homogeneity of limit ultrapowers.

**Theorem 2.3.** For every relational structure $A$ with $|L(A)| \leq \omega$ the limit reduced power $\mathcal{U}_\omega A$ is $\kappa^+$-homogeneous, then the pair $(\mathcal{F}, \mathcal{G})$ is $(\omega, \kappa)$-regular.

**Proof.** Let $B_\xi = S_{\kappa^+}(\kappa^+ + \xi) \cup S_{\kappa^+}(\kappa^+ + \xi)$, $C_\xi = S_{\kappa^+}(\kappa^+ + \xi) \times \{0\}$ and $A_\xi = B_\xi \cup C_\xi$ (here $\cdot$ denotes the addition of ordinals). For $\kappa < \kappa^+$ we put $a_\xi = \{a\}$, $b_\xi = \kappa^+ + \kappa^+ - (\kappa + \kappa) = (\kappa, 0)$ and $d_\xi = (\kappa^+ + \kappa^+ - (\kappa + \kappa), \kappa)$. Moreover, $c_\xi = (\kappa, 0)$. Now assume that for every relational structure $A$ the limit reduced power $\mathcal{U}_\omega A$ is $\kappa^+$-homogeneous. Hence in particular $\mathcal{U}_\omega A$ is $\kappa^+$-homogeneous, where $A = \langle A_\xi : \xi \in \kappa^+ \rangle$. If $a \in A_\xi$, then by $a$ we denote the element of $(A_\xi)^\mathcal{G}$ such that $\bar{a} = a$ for $i \in I$. Since every infinite atomic Boolean algebras are elementary equivalent, we have

$$\langle \mathcal{U}_\omega A, \mathcal{U}_\omega F, \mathcal{U}_\omega (\mathcal{F}, \mathcal{G}) \rangle_{\kappa^+} = \langle \mathcal{U}_\omega A, \mathcal{U}_\omega F, \mathcal{U}_\omega (\mathcal{F}, \mathcal{G}) \rangle_{\kappa^+} .$$

Now since $\mathcal{U}_\omega A$ is $\kappa^+$-homogeneous, there is an $a \in (A_\xi)^\mathcal{G}$ such that

(1) \[ \langle \mathcal{U}_\omega A, \mathcal{U}_\omega F, \mathcal{U}_\omega (\mathcal{F}, \mathcal{G}) \rangle_{\kappa^+} = \langle \mathcal{U}_\omega A, \mathcal{U}_\omega F, \mathcal{U}_\omega (\mathcal{F}, \mathcal{G}) \rangle_{\kappa^+} . \]

Let

$$h(\xi) = \{ \xi : h(\xi) = \{ \xi : h(\xi) = \{ \xi : a(\xi) = b(\xi) = c(\xi) \} \} \} .$$

Finally $h(\xi) = h(\xi) \cup h(\xi)$. Of course $h$ is a partition function and $h(\xi) = \{ \xi : a(\xi) = a(\xi) \}$, whence by (1) $\mathcal{U}_\omega F$ is $\kappa^+$-homogeneous. Moreover, since $a \in (A_\xi)^\mathcal{G}$ and all $a(\xi)$ and $b(\xi)$ are constant, the equivalence relation $\equiv$ defined by $I_{\equiv} = \{ (h(\xi), h(\xi)) : \xi \in \kappa^+ \}$ is an element of $\mathcal{G}$.

As a corollary to Theorems 2.2 and 2.3 we get the following theorem of J. Wiciakowski [13].

**Theorem 2.4.** For any relational structure $A$ with $|L(A)| \leq \omega$ the limit reduced power $\mathcal{U}_\omega A$ is $\kappa^+$-homogeneous, then for every $A$ such that $|L(A)| = \kappa$ the limit ultrapower $\mathcal{U}_\kappa A$ is $\kappa^+$-saturated.

**Proof.** By Theorem 2.3 $(\mathcal{F}, \mathcal{G})$ is $(\omega, \kappa)$-regular, whence by Theorem 2.1 $\mathcal{U}_\omega A$ is $\kappa^+$-universal; whence $\kappa^+$-saturated.

**Theorem 2.5.** If for every $A$ with $|L(A)| = \omega$, $\mathcal{U}_\omega A$ is $\omega$-homogeneous, then $(\mathcal{F}, \mathcal{G})$ is $(\omega, \omega)$-regular.

**Proof.** Let $A$ denote the set of rationals. We consider the structure $A = \langle A, <, 1/n : n \in \omega \rangle$. Let $a(i) = 0$ and $b(i) = -1$ for $i \in I$. Then $(\mathcal{U}_\omega A, \mathcal{U}_\omega F, $ $\mathcal{U}_\omega (\mathcal{F}, \mathcal{G}))$ is an element of $\mathcal{G}$ and since $\mathcal{U}_\omega A$ is $\omega$-homogeneous, there is a $c \in A^\mathcal{G}$ such that

$$\langle \mathcal{U}_\omega A, \mathcal{U}_\omega F, c(\mathcal{F}, \mathcal{G}) \rangle = \langle \mathcal{U}_\omega A, \mathcal{U}_\omega F, c(\mathcal{F}, \mathcal{G}) \rangle .$$

We put $h(\xi) = \{ \xi : c(\xi) = c(\xi) \}$ and then proceed as in the proof of Theorem 2.4.

3. Now we shall prove that the assumption in Theorems 2.1 and 2.2 are necessary.

**Theorem 3.1.** For every $A$ with $|L(A)| = \omega$ the limit ultrapower $\mathcal{U}_\omega A$ is $\kappa^+$-universal, then the pair $(\mathcal{F}, \mathcal{G})$ is $(\omega, \kappa)$-regular.

**Proof.** Let $A = \langle A, <, 1/n : n \in \omega \rangle$ and let $\Sigma = \{ (x) \in \omega \omega \}$. Then $\Sigma$ is finitely satisfiable in $\mathcal{U}_\omega A$ and since $\mathcal{U}_\omega A$ is $\kappa^+$-universal, there is an element $a \in A^\mathcal{G}$ such that $a(\mathcal{F})$ satisfies $\Sigma$ in $\mathcal{U}_\omega A$. We put $h(\xi) = \{ \xi : a(\xi) = a \}$. The proof above is a slight modification of a proof given by Keisler in [4].

Also the proof of the theorem below is based on an idea of Keisler’s (see [3]).

**Theorem 3.2.** For every $A$ such that $|L(A)| = \kappa$ the limit reduced power $\mathcal{U}_\kappa A$ is $\kappa^+$-saturated, then $(\mathcal{F}, \mathcal{G})$ is \( \kappa \times \kappa \)-good.

**Proof.** Proof the problem we are going to present is divided into several steps. From now on we assume that for every $A$ with $|L(A)| = \kappa$ the limit reduced power $\mathcal{U}_\kappa A$ is $\kappa^+$-saturated. Let $\mathcal{F}$ and $\mathcal{G}$ be arbitrary but fixed functions which satisfy the hypotheses of the definition of $\kappa$-goodness.

3.2.1. There is a $a \in \mathcal{G}$ and a function $d: \kappa \rightarrow \mathcal{G}$ such that

(1.a) $d(\xi) \in \mathcal{G} \quad (1.b) \quad \xi \in \mathcal{G}$. \( \kappa \times \kappa \)-good.

**Proof.** Let $\mathcal{B} = \langle B, <, 1/n : n \in \omega \rangle$, where $A = \langle A, <, 1/n : n \in \omega \rangle$ and $\mathcal{G}$ is $\kappa^+$-saturated, whence there is an element $b$ of $B^\mathcal{G}$ such that $b(\mathcal{F})$ satisfies $\Sigma$ in $\mathcal{U}_\omega A$. We put $a(\xi) = a(\xi)$ and $b(\xi) = b(\xi)$. Of course $a(\xi) \in \mathcal{G}$ and $a(\xi) \in \mathcal{G}$. Now let $a(\xi) = a(\xi) \in \mathcal{G}(a(\xi))$. Since $b(\xi)$ is constant on every equivalence class of $a(\xi)$ and $b(\xi)$ is constant on every equivalence class of $a(\xi)$, the logical value of $0 \neq b(\xi) \neq b(\xi)$ is constant on every equivalence class of $a(\xi)$. But $a(\xi) \in \mathcal{G}$, whence (1.a) holds. Now assume that $a(\xi) \in \mathcal{G}$ and $i, j \in a(\xi) \cap a(\xi)$, then, since $i, j \in a(\xi)$, we have $b(\xi) = b(\xi)$; moreover, since $i, j \in a(\xi)$, we have $b(\xi) = b(\xi)$ and $b(\xi) = b(\xi)$. This proves that $b(\xi)$ and $b(\xi)$ are equivalence classes of $F(a(\xi))$ and have non-empty intersection. Hence $b(\xi) = b(\xi)$ and consequently (1.b) holds.

3.2.2. Assume that $f_0: S_\omega(x) \rightarrow \mathcal{F}$ is a monotonous function such that $f_0(\xi) \in \mathcal{G}$ for every $\xi \in S_\omega(x)$. Then there is a $a \in \mathcal{G}$ and $f_0: S_\omega(x) \rightarrow \mathcal{F}$ such that, for every $\xi \in S_\omega(x)$,

(2.a) $f_0(\xi) \in \mathcal{G}(a(\xi))$ and

(2.b) $\xi \in S_\omega(x) \cap \mathcal{G}(a(\xi)) \Rightarrow f_0(\xi) \in \mathcal{G}$. 

Then there is a $a \in \mathcal{G}$ and $f_0: S_\omega(x) \rightarrow \mathcal{F}$ such that, for every $\xi \in S_\omega(x)$,
Proof. Let $\emptyset \subseteq \langle S_n(S_n(x)) \cup \{S_n(y)\} \subseteq \neq \rangle$. For $s \in S_n(x)$ and $t \in I$ we put
\[
c_t(s) = \begin{cases} 
1 & \text{if } s \in t, \\
0 & \text{if } s \notin f_t.
\end{cases}
\]
and
\[
c_t(S_n(y)) = S_n(y).
\]
\[\Sigma = \{v \in c_t[f_t(S_n(y))] \mid v \neq c_t(S_n(y))\}.\]
We claim that $\Sigma$ is finitely satisfiable in $\mathcal{U}_x^v$. 
In fact, if $\Sigma_0$ is a finite subset of $\Sigma$ and $t = \bigcup \{c : c \in \Sigma_0\}$, then putting $c_f(S_n(y)) = S(S_n(y))$ we obtain an element of $\Sigma_f \cap \mathcal{U}_x^v$.

Now, since $\mathcal{U}_x^v$ is $\mathcal{U}_x^v$-saturated, there is an element $c \in \mathcal{U}_x^v$ such that $c_t \subseteq \Sigma_f$. Let $t \in \Sigma_f \cap \mathcal{U}_x^v$. Of course $c_t \subseteq \mathcal{U}_x^v$. Since $c_t \subseteq \Sigma_f \cap \mathcal{U}_x^v$, we have
\[
E_n = \{t \in I : c_{f_t}(S_n(y)) \neq c_{f_t}(S_n(x)) \} \subseteq \mathcal{U}_x^v.
\]
But
\[
E_n = \bigcup \{t \in I : c_{f_t}(S_n(y)) \neq c_{f_t}(S_n(x)) \} \subseteq \mathcal{U}_x^v.
\]
Moreover, if $t \in E_n$, then in particular $c_{f_t}(S_n(y)) \subseteq S_n(x)$, whence $i \in f_t(s)$, i.e.,
\[E_n \subseteq f_t(s).
\]
Let $f_t(s) = E_n$, then by (2.1), (2.2) and (2.3), $f_t(s)$ satisfies (2.1) and (2.2).

3.2.3. There exists a monotonic function $f_t : S_n(x) \to \mathcal{U}_x^v$ and equivalence relations $\approx_1, \approx_2, \approx_3$ such that
\[f_t \subseteq f_t(S_n(y)) \subseteq \mathcal{U}_x^v \text{ and } f_t \subseteq f_t(S_n(x)) \subseteq \mathcal{U}_x^v.
\]
Assume that $x \in f_t(s)$ and $i, j \in \approx_1 \cap f_t(s)$, then $i \in f_t(s)$.

Proof. Recall that $f_t : S_n(x) \to \mathcal{U}_x^v$ is monotonic and $F : S_n(x) \to \mathcal{U}_x^v$ is additive. Moreover $f_t(S_n(x)) \subseteq \mathcal{U}_x^v$. By 3.2.1 there is a $t \in \emptyset$ add $d : \emptyset \to \mathcal{U}_x^v$ such that (1.a) and (1.b) hold. For $s \in S_n(x)$ we put $f_t(s) = d(s) \cap f(S_n(y))$. Of course $f_t(s)$ is a
monotonic function and $f_t(S_n(x)) \subseteq \mathcal{U}_x^v$. Let $s \in S_n(x)$; then $f_t(s) \subseteq f_t(S_n(x))$ and $d(s) \subseteq f_t(S_n(x)) \subseteq \mathcal{U}_x^v$ for every $s \in s$, whence $f_t(s) \subseteq f_t(S_n(x))$. We proved that $f_t(S_n(x)) \subseteq \mathcal{U}_x^v$.

3.2.4. There is a function $g : S_n(x) \to \mathcal{U}_x^v$ and $g \subseteq S_n(x)$ such that $g \subseteq f_t$ and $g \subseteq f_t(S_n(y)) \subseteq \mathcal{U}_x^v$. 

References

Note on decompositions of metrizable spaces I

by

R. Pol (Warszawa)

Abstract. In this note we investigate, in the class of metrizable spaces, the property of being $\sigma$-locally of weight $<\gamma$, introduced by A. H. Stone in his theory of non-separable absolutely Borel spaces [8], and we prove some facts related to the questions raised in [8].

In this note we investigate, in the class of metrizable spaces, the property of being $\sigma$-locally of weight $<\gamma$, introduced by A. H. Stone in his theory of non-separable absolutely Borel spaces [8], and we prove some facts related to the questions raised in [8].

Our topological terminology and notation is from [2] and [5]; our set — theoretical terminology will follow [4]. All of our spaces are assumed to be metrizable. For a given space $X$ we say that $\varphi$ is a metric on $X$ if $\varphi$ is any metric compatible with the topology of $X$. For a metric $\varphi$, a set $A\subseteq X$ and $\varepsilon > 0$, we write $B(A, \varepsilon) = \{x \in X : \varphi(x, A) < \varepsilon\}$. The symbol $\pi(X)$ denotes the weight of a space $X$ and $|S|$ the cardinality of a set $S$. The set of all ordinals less than a given ordinal $\lambda$ is denoted by $\omega(\lambda)$. For an initial ordinal $\omega$ of a regular cardinal $\tau$ we call a set $\Gamma \subseteq \omega(\lambda)$ stationary if and only if for every function $\Phi : \Gamma \rightarrow \omega(\lambda)$ with $\Phi(\Gamma) \subseteq \gamma$, there exists $\omega < \lambda$ such that $\Phi^{-1}(\omega) = \tau$. The successor of a cardinal number $\lambda$ is denoted by $\lambda^+$. We say that a space $X$ is $\delta$-locally of weight $<\tau$ (in symbols, $X \in \delta \Lw(<\tau)$; see [8], 2.1) provided $X = \bigsqcup \{X_\alpha : \alpha \in A\}$, where $|A| \leq \delta$ and each $X_\alpha$ is locally of weight $<\tau$. It is easy to verify (cf. [8], 2.1) that for a metric $\varphi$ on $X$ this is equivalent to the following condition: there are families $\mathcal{F}_s$ of subsets of $X$ of weight $<\tau$ and $\varepsilon_s > 0$ for $s \in S$, where $|S| \leq \delta$, such that

(1) $\Gamma = \bigsqcup \{\mathcal{F}_s : s \in S\}$ and $\varphi(F', F'') \leq \varepsilon_s$ for different $F', F'' \in \mathcal{F}_s$,

For $\delta = \omega_\alpha$, we write $X \in \omega \Lw(<\omega_\alpha)$ if $X \in \omega \Lw(<\omega_\alpha)$ we say that $X$ is $\omega$-discrete.

Proposition (cf. [8], Theorem 3). Suppose that $\tau$ is a regular or sequential cardinal and $\delta < \tau$. If $X \in \delta \Lw(<\tau)$, then $X \in \sigma \Lw(<\tau)$. 
