cushioned pair with $\mathscr{U}^*$ cushioned in $\mathscr{V}^*$, let $\mathscr{U}\mathscr{V} = \{U \in \mathscr{U}^*\colon U \to V$ where $V \in \mathscr{V}^*$ and $V \subset W$ for some $W \in \mathscr{W}\}$. Then $\mathscr{U}\mathscr{V}$ is cushioned in $\mathscr{W}$ under the correspondence $U \to W$, there are countably many such collections, and we will be done if we can show that $\bigcup \mathscr{U}\mathscr{V}$ refines $\mathscr{W}$. This is easy to do, for given $x \in X$, there is a $W \in \mathscr{W}$ for which $x \in \bigcap_i U(n, \alpha_i, a_i, t_i) \subset W$. Choose rationals b_i, s_i with $a_i < b_i < f_{n\alpha_i}(x) < s_i < t_i$ and note that the set $\bigcap_i U(n, \alpha_i, b_i, s_i)$ contains x and is a member of some $\mathscr{U}\mathscr{V}$. Thus, $\mathscr{W}$ does indeed have an open σ-cushioned refinement, and the proof is complete.

Thus, Theorem 9 is more general than Theorem 7. Furthermore, Example 4 shows that there are relatively complete collections satisfying the hypotheses of Theorem 9 that are not locally finite partitions of unity.

References

[1] J. A. Guthrie and M. Henry, *Metrization and paracompactness in terms of real functions*, Bull. Amer. Math. Soc. 80 (1974), pp. 720.
[2] E. Michael, *A note on paracompact spaces*, Proc. Amer. Math. Soc. 4 (1953), pp. 831–838.
[3] — *Yet another note on paracompact spaces*, Proc. Amer. Math. Soc. 10 (1959), pp. 309–314.
[4] J. Nagata, *A contribution to the theory of metrization*, J. Inst. Polytech., Osaka City University, 8 (1957), pp. 185–192.

UNIVERSITY OF TEXAS AT EL PASO
WEST VIRGINIA UNIVERSITY

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Locally well-behaved paracompacta in shape theory

by

G. Kozlowski and **J. Segal** * (Seattle, Wash.)

Abstract. We generalize the classical notion of ANR to paracompacta in shape theory to obtain the notion of absolute neighborhood shape extensor (ANSE). Although the corresponding classical statement is false for compacta we have the theorem: Any LC^n paracompactum of dimension $\leqslant n$ is an ANSE. We also generalize the various notions of movability to arbitrary topological spaces.

THEOREM. *Every* LC^{n-1} *paracompactum of dimension* $\leqslant n$ *is uniformly movable.*

Contents

1. Introduction. In Section 3 we give a categorical description of shape theory for arbitrary topological spaces based on the concept of natural transformations of homotopy classes of maps into polyhedra. We show precisely in what sense this theory agrees with the Mardešić–Segal ANR-systems approach to shape theory on compacta.

K. Borsuk [2] introduced the notion of movability for metric compacta as a generalization of ANR's, and S. Mardešić and J. Segal [16] extended this property to compacta by means of ANR-systems. Movability appears to be the most interesting shape invariant discovered so far. It occurs as the hypothesis under which many classical theorems of algebraic topology generalize to shape theory, for example, [9] and [19]. In Section 4 we present a definition of movability for arbitrary

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topological spaces which is consistent with the original description and can be interpreted categorically, but differs from the usual definition in systems.

In Section 5 we generalize the notion of extensor to the theory of shape for paracompacta. We also generalize the notions of FANR [1] and ANSR [14] to paracompacta. The starting point is the generalization of the neighborhood extension of maps to the neighborhood extension of shape morphisms. The universal quantification of this property gives the concept of ANSE.

In Section 7 we show that any LC^n paracompactum X of dimension $\leqslant n$ is shape dominated by some polyhedron of the nerve of an open cover of X. Moreover, such an X is an ANSE and therefore an ANSR. An interesting feature of this domination is the fact that it is induced by maps.

2. Complexes. We will now give a brief summary of the definitions and main results needed in our consideration of complexes. The reader is referred to [22, Chapter 3] for a more detailed account.

DEFINITION 2.1. A *complex* K is a (possibly infinite) set of objects, called vertices, $\{v^\alpha\}$, and a set of finite subsets of the vertices, called simplexes; the simplexes satisfy the condition that any subset of a simplex of K is also a simplex of K. If K is a complex, a subcollection K_0 of its simplexes is called a *subcomplex* of K if it is a complex. The *dimension* of a simplex s of K is the number of vertices of s minus one. K^n denotes the subcomplex of K consisting of those simplexes of dimension $\leqslant n$.

We now define a topological space $|K|$ called the *polyhedron* of K. First let E be the vector space consisting of all real-valued functions defined on the vertices of K with operations defined pointwise. Identify the vertex v with the function which is 1 on v and 0 elsewhere. For each simplex s of K let $|s|$ be the set of all functions p in E such that

(i) $p(v) = 0$ if $v \notin s$,

(ii) $p(v) \geqslant 0$ for all v,

(iii) $\sum_{v\in K} p(v) = 1$.

The *open star* of the vertex v is the set of all p in $|K|$ with $p(v)>0$.

There is a metric on $|K|$ defined by

$$\varrho(p, q) = [\sum_{v\in K}(p(v)-q(v))^2]^{1/2}$$

and the topology on $|K|$ defined by this metric is called the *metric topology*. The metric ϱ also induces a topology on $|s|$ which makes the space $|s|$ homeomorphic to a geometric simplex of the same dimension lying in a finite dimensional Euclidean space. The *weak topology* on $|K|$ consists precisely of those subsets U of $|K|$ for which $U \cap |s|$ is open in $|s|$ (in the metric topology) for each simplex s of K. If K is a locally finite complex (i.e., every vertex of K belongs to only finitely many simplexes of K), then these two topologies are identical.

We mention two facts we will need later. First, a function $f\colon |K|\to X$ from $|K|$ into any space X is continuous, if the restriction $f\,|\,|s|$ is continuous for all $s\in K$. Second, a function $H\colon |K|\times I\to X$ is continuous, if $H\,|\,|s|\times I$ is continuous for all $s\in K$.

Finally, let X be a topological space and let $\mathcal{U}$ be a covering of X by a collection of non-empty open sets. We define a complex $K(\mathcal{U})$ in which the vertices are members of $\mathcal{U}$ and the simplexes are the subcollections of members of $\mathcal{U}$ with non-empty intersection. This complex is called the *nerve* of $\mathcal{U}$. If $\mathcal{U} = \{U\}$ is an open covering of a space X and $K(\mathcal{U})$ is its nerve, a *barycentric map* $f\colon X\to |K(\mathcal{U})|$ is a continuous map such that, for every $x\in X$, each vertex of the smallest simplex $s\in K(\mathcal{U})$ with $f(x)\in |s|$ contains x. A barycentric map is often called a canonical map. If $\mathcal{U}$ and $\mathcal{V}$ are open coverings of X, with $\mathcal{V}$ a refinement of $\mathcal{U}$, a *projection* from $\mathcal{V}$ to $\mathcal{U}$ is a function π (or $\pi_{\mathcal{U}\mathcal{V}}$) which assigns to each $V\in\mathcal{V}$ an element $\pi(V)\in\mathcal{U}$ such that $V\subset\pi(V)$. A projection defines a simplical map $K(\mathcal{V})\to K(\mathcal{U})$ which in turn defines a continuous map $|K(\mathcal{V})|\to|K(\mathcal{U})|$ both of which are also denoted by π.

Recall that a paracompactum is a Hausdorff spaces every open cover of which has an open locally finite refinement. Notice that if X is paracompact, then for any open cover $\mathcal{U}$ of X there is a barycentric map $u\colon X\to|K(\mathcal{U})|$, because $\mathcal{U}$ has a locally finite refinement $\mathcal{V}$ and we may take u as the composition of a barycentric map $v\colon X\to|K(\mathcal{V})|$ (whose existence is a standard fact) followed by a projection $\pi\colon |K(\mathcal{V})|\to|K(\mathcal{U})|$.

If $\mathcal{U}$ is an open cover of a space X, then a simplical map $\psi\colon K(\mathcal{U})\to K$ is said to be *barycentrically induced* by a map $p\colon X\to|K|$, provided the set $p(U)$ is contained in the open star of the vertex $\psi(U)$ for each $U\in\mathcal{U}$. Simplical maps barycentrically induced by the same map $X\to|K|$ are contiguous and therefore homotopic. If $\mathcal{V}$ refines $\mathcal{U}$, then for any projection $\pi\colon K(\mathcal{V})\to K(\mathcal{U})$ the composition $\psi\pi$ is also barycentrically induced by p.

A *barycentric factorization* of a map $p\colon X\to|K|$ is a pair

$$(u\colon X\to|K(\mathcal{U})|,\ p_u\colon |K(\mathcal{U})|\to|K|)$$

in which $\mathcal{U}$ is an open cover of X, u is a barycentric map, and p_u is a simplical map barycentrically induced by p. For any $x\in X$ each vertex U of the smallest simplex containing $u(x)$ contains x and $p(U)$ is contained in the open star of $p_u(U)$, for each such U; thus $p(x)$, $p_u u(x)\in|s|$ for some simplex $s\in K$ and, consequently, $p\simeq p_u u$.

3. Shape theory for topological spaces. Let $\mathcal{P}$ be the category of polyhedra and homotopy classes of continuous maps between them. If X is a (topological) space, then Π_X is the functor from $\mathcal{P}$ to the category of sets and functions which assigns to a polyhedron P the set $\Pi_X(P) = [X, P]$ of all homotopy classes of maps of X into P and which assigns to any homotopy class $\varphi\colon P\to Q$ between polyhedra the induced function $\varphi_{\#}\colon [X, P]\to[X, Q]$ which maps the homotopy class $f\colon X\to P$

into the composition $\varphi f = \varphi_{\#}(f)$ of the homotopy classes of f and φ. A natural transformation Ψ of the functor Π_X into the functor Π_Y assigns to each homotopy class $f\colon X \to P$ a homotopy class $\Psi(f)\colon Y \to P$ in such a way that for all homotopy classes $f\colon X \to P$, $g\colon X \to Q$, and $\varphi\colon P \to Q$ such that $\varphi f = g$ we have $\varphi\Psi(f) = \Psi(g)$. If $f\colon X \to Y$ is a map, then there is a natural transformation $f^{\#}\colon \Pi_Y \to \Pi_X$ which assigns to the homotopy class $\varphi\colon Y \to P$ the composition $\varphi[f] = f^{\#}(\varphi)$ of the homotopy class $[f]$ of f with φ. A natural transformation from Π_Y to Π_X will be called a shape class from X to Y.

Given two spaces X and Y we say that X *shape dominates* Y if and only if there are natural transformations $\Phi\colon \Pi_Y \to \Pi_X$ and $\Psi\colon \Pi_X \to \Pi_Y$ such that $\Psi\Phi = 1_Y^{\#}$. If, in addition, $\Phi\Psi = 1_X^{\#}$, then X and Y are said to be of the *same shape*. In other words, X and Y have the same shape if and only if there is an invertible natural transformation (i.e., a natural equivalence) of the functors Π_X and Π_Y. This approach is essentially the same as that of Mardešić [12] except that he used shape maps instead of natural transformations. The development of shape theory via natural transformations and its equivalence with Borsuk's theory for metric compacta was obtained independently by Kozlowski [6].

Remark 3.1. For a paracompactum Y the functor Π_Y is represented as the direct limit of the system $\{\Pi = \Pi_{|K(\mathcal{U})|}, \pi^{\#}_{\mathcal{U}\mathcal{V}}, \mathrm{Cov}(Y)\}$ by means of the maps $u^{\#}\colon \Pi_{\mathcal{U}} \to \Pi_{\mathcal{V}}$ induced by barycentric maps $u\colon Y \to |K(\mathcal{U})|$ (see [10]). As a consequence of Remark 3.1 we have the following two remarks.

Remark 3.2. If $P = |K|$, $Q = |L|$ are polyhedra, X paracompact, and $f\colon X \to P$, $g\colon X \to Q$, $\theta\colon P \to Q$ maps such that $\theta f \simeq g$, then for any sufficiently fine open cover $\mathcal{U}$ and for any barycentric factorizations ($u\colon X \to |K(\mathcal{U})|$, $f_u\colon |K(\mathcal{U})| \to |K|$), ($u\colon X \to |K(\mathcal{U})|$, $g_u\colon |K(\mathcal{U})| \to |L|$) of f, g respectively, $g_u \simeq \theta f_u$.

Remark 3.3. For any space X a family $\{\varphi_{\mathcal{U}}\colon X \to |K(\mathcal{U})| \mid \mathcal{U} \in \mathrm{Cov}(Y)\}$ with the property that

(1) $\varphi_{\mathcal{U}} \simeq \pi_{\mathcal{U}\mathcal{V}}\varphi_{\mathcal{V}}$, when $\mathcal{V}$ refines $\mathcal{U}$, defines a natural transformation $\Phi\colon \Pi_Y \to \Pi_X$ which is uniquely specified by the requirements

(2) $\Phi[u] = [\varphi_{\mathcal{U}}]$ for every $\mathcal{U} \in \mathrm{Cov}(Y)$.

For any natural transformation $\Psi\colon \Pi_Q \to \Pi_X$, where Q is a polyhedron, there is a unique homotopy class $g\colon X \to Q$ such that $g^{\#} = \Psi$. In fact, if $g^{\#} = \Psi$, then $g = \Psi(1_Q)$, and taking this as the definition of g we see from the fact that $\Psi(h) = h\Psi(1_Q)$ holds for any homotopy class $h\colon Q \to P$ that $g^{\#}(h) = hg = \Psi(h)$ for any such h.

It is useful to observe that the scope of the natural transformations occurring in this definition of shape may be enlarged. By the theorem of Appendix 2 of [10] any natural transformation of Π_Y into Π_X has a unique extension to a natural transformation of $\tilde{\Pi}_Y$ into $\tilde{\Pi}_X$, where the tilde indicates that the homotopy class functors are considered to be defined on the category of all shapes dominated by polyhedra and homotopy classes of maps. By a theorem of Milnor [17] the class of spaces dominated by polyhedra coincides with the class of spaces dominated by CW-complexes and includes the class of ANR's. Accordingly, a given natural transformation of Π_Y into Π_X will be assumed to give a transformation of homotopy classes of maps of Y into ANR's into such classes of X into ANR's. If $\Phi\colon \Pi_Y \to \Pi_X$ is a natural transformation and $f\colon X \to P$, $g\colon Y \to P$ are maps into a polyhedron P, then the relations $\Phi[g] = [f]$ and $\Phi g^{\#} = f^{\#}$ are equivalent.

The definition of shape for compacta was obtained by Mardešić and Segal [15] in terms of ANR-systems.

An ANR-system is an inverse system $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$ where each X_α is an ANR, i.e., a compact ANR for metric spaces and $p_{\alpha\alpha'}$, $\alpha \leqslant \alpha'$, $\alpha, \alpha' \in A$, are maps from $X_{\alpha'}$ into X_α; $(A, \leqslant)$ is a closure-finite directed set [15]. If $X = \mathrm{Inv\,lim}\,\underline{X}$, we say that $\underline{X}$ is associated with X and we denote by $p_\alpha\colon X \to X_\alpha$ the natural projections. A map of systems $\underline{f}\colon \underline{X} \to \underline{Y} = \{Y_\beta, g_{\beta\beta'}, B\}$ consists of an increasing function $f\colon B \to A$ and of a collection $\{f_\beta, B\}$ of maps $f_\beta\colon X_{f(\beta)} \to Y_\beta$ such that $\beta \leqslant \beta'$ implies the homotopy relation

$$f_\beta p_{f(\beta)f(\beta')} \simeq q_{\beta\beta'} f_{\beta'}\,.$$

The identity map $\underline{1}_X\colon \underline{X} \to \underline{X}$ is given by $1(\alpha) = \alpha$, $1_\alpha = 1_{X_\alpha}$. The composition of maps $\underline{f}\colon \underline{X} \to \underline{Y}$, $\underline{g}\colon \underline{Y} \to \underline{Z} = \{Z_\gamma, r_{\gamma\gamma'}, C\}$ is the map $\underline{h} = \underline{g}\,\underline{f}\colon \underline{X} \to \underline{Z}$ given by $h(\gamma) = fg(\gamma)$ and $h_\gamma = g_\gamma f_{g(\gamma)}$. Two maps of systems $\underline{f}, \underline{g}\colon \underline{X} \to \underline{Y}$ are said to be homotopic, $\underline{f} \simeq \underline{g}$, provided for every $\beta \in B$ there is an index $\alpha \in A$, $\alpha \geqslant f(\beta), g(\beta)$ such that $f_\beta p_{f(\beta)\alpha} \simeq g_\beta p_{g(\beta)\alpha}$. ANR-systems $\underline{X}$ and $\underline{Y}$ are said to be of the same homotopy type, $\underline{X} \simeq \underline{Y}$, provided there exists maps of systems $\underline{f}\colon \underline{X} \to \underline{Y}$, $\underline{g}\colon \underline{Y} \to \underline{X}$, such that $\underline{g}\,\underline{f} \simeq \underline{1}_X$, $\underline{f}\,\underline{g} \simeq \underline{1}_Y$. Two compacta are said to be the same shape if they have associated ANR-systems of the same homotopy type.

The equivalence of the ANR-systems definition of shape and the functorial definition on compacta is established by the next result (which follows from [12, Theorem 6]).

THEOREM 3.1. *For two compacta X and Y and any map of systems $\underline{f}\colon \underline{X} \to \underline{Y}$ there is a natural transformation $\underline{f}^{\#}\colon \Pi_Y \to \Pi_X$ which only depends on the shape class of $\underline{f}$.*

The assignment $\underline{f} \to \underline{f}^{\#}$ has the following properties:

(1) $\underline{1}^{\#}$ = identity and $(\underline{g}\,\underline{f})^{\#} = \underline{f}^{\#}\underline{g}^{\#}$,

(2) if $\underline{f}^{\#} = \underline{g}^{\#}$, then $\underline{f} \simeq \underline{g}$,

and

(3) for any natural transformation $\Phi\colon \Pi_Y \to \Pi_X$, there exists $\underline{f}\colon \underline{X} \to \underline{Y}$ such that $\underline{f}^{\#} = \Phi$.

Remark 3.4. It follows from Theorems 4 and 3 of [10] that if $\underline{Y}$ is any ANR-system associated with Y, then, for any space W dominated by a CW-complex, $\Pi_Y(W)$ is represented as the direct limit of the direct system

$$\{\Pi_\beta(W) = [Y_\beta, W], q^{\#}_{\beta\beta'}, B\}$$

by the functions $q^{\#}_\beta\colon \Pi_\beta(W) \to \Pi_Y(W)$. This means that for any set E and any family of functions $\Phi_\beta\colon \Pi_\beta(W) \to E$ satisfying $\Phi_\beta = \Phi_{\beta'} q^{\#}_{\beta\beta'}$ $(\beta \leqslant \beta')$, there is a unique function $\Phi\colon \Pi_Y(W) \to E$ satisfying $\Phi = \Phi_\beta q^{\#}_\beta$.

4. Movability and n-movability. In [2] and [3] Borsuk introduced the notions of movability and n-movability for metric compacta. In [16] Mardešić and Segal and in [8] Kozlowski and Segal generalized these notions to arbitrary compacta. In this section we give a categorical description of these properties which apply to arbitrary topological spaces and show they are shape invariants. In fact, our version of movability is a generalization of uniform movability [18, Definition 3.1, p. 132].

DEFINITION 4.1. A space X is said to be *uniformly movable* provided, that for each map $f\colon X\to P$ of X into a polyhedron P, there exists a polyhedron Q and natural transformations $\Phi\colon \Pi_X\to\Pi_Q$, $\Psi\colon \Pi_Q\to\Pi_X$ such that $\Psi\Phi[f] = [f]$.

Remark 4.1. If a space is shape dominated by a polyhedron then it is uniformly movable.

Remark 4.2. Since any natural transformation $\Psi\colon \Pi_Q\to\Pi_X$ satisfies $\Psi = g^{\#}$ for a map (unique up to homotopy) $g\colon X\to Q$, the condition of the above definition can be stated: for each map $f\colon X\to P$ there exist a polyhedron Q, maps $g\colon X\to Q$, $\varphi\colon Q\to P$, and a natural transformation $\varphi\colon \Pi_X\to\Pi_Q$ such that $\varphi g\simeq f$ and $\Phi[f] = [\varphi]$.

DEFINITION 4.2. Two maps $f\colon X\to P$, $g\colon Y\to P$ of spaces into the same polyhedron P are *shape-equivalent*, provided there exist natural transformations $\Phi\colon \Pi_Y\to\Pi_X$, $\Psi\colon \Pi_X\to\Pi_Y$ such that $f\in\Phi[g]$ and $g\in\Psi[f]$.

Remark 4.3. We may say then that a space is uniformly movable if and only if any map of it into a polyhedron P is shape equivalent to a map of a polyhedron into P.

DEFINITION 4.3. A space X is said to be *uniformly n-movable*, provided that for any map $f\colon X\to P$ of X into a polyhedron P, there exists a polyhedron Q, maps $g\colon X\to Q$, $\varphi\colon Q\to P$, and a natural transformation $\Phi\colon \Pi_X\to\Pi_{Q^n}$ such that $\varphi g\simeq f$ and $\Phi[f] = [\varphi|Q^n]$.

THEOREM 4.1. *If X is a uniformly movable paracompactum, then for any map $f\colon X\to P$ of X into a polyhedron P and for any sufficiently fine open cover $\mathcal{U}$ of X there exist a barycentric factorization $(u\colon X\to|K(\mathcal{U})|,\ f_u\colon |K(\mathcal{U})|\to P)$ of f and a natural transformation $\Phi\colon \Pi_X\to\Pi_{|K(\mathcal{U})|}$ such that $\Phi[f] = [f_u]$. If X is a uniformly n-movable paracompactum, the above conclusion is modified only in that Φ maps Π_X to $\Pi_{|K^n(\mathcal{U})|}$ and $\Phi[f] = [f_u||K^n(\mathcal{U})|]$.*

Proof. Suppose X is a uniformly movable paracompactum. From Remark 4.2 it follows that there exist a polyhedron Q, a natural transformation $\Psi\colon \Pi_X\to\Pi_Q$ and maps $g\colon X\to Q$, $\varphi\colon Q\to P$ such that $\varphi g\simeq f$ and $\Psi[f] = [\varphi]$.

For any sufficiently fine open cover $\mathcal{U}$ there are barycentric factorizations (u, f_u) and (u, g_u) of f and g respectively, and by Remark 3.2 $\varphi g_u\simeq f_u$. Take $\Phi = g_u^{\#}\Psi$ and note that

$$\Phi[f] = g_u^{\#}\Psi[f] = g_u^{\#}[\varphi] = [\varphi g_u] = [f_u].$$

When X is a uniformly n-movable paracompactum, Ψ now maps Π_X into Π_{Q^n} and $\Psi[f] = [\varphi|Q^n]$. Obtain $\mathcal{U}$, (u, f_u), and (u, g_u) as above, and observe that g_u, being simplical, maps $|K^n(\mathcal{U})|$ into Q^n and thus defines a map $\gamma\colon |K^n(\mathcal{U})|\to Q^n$. Then $\Phi = \gamma^{\#}\Psi$ is the desired natural transformation.

DEFINITION 4.4. A space X is said to be *movable* provided, that for any map $f\colon X\to P$ of X into a polyhedron P there exist a polyhedron Q and maps $g\colon X\to Q$, $\varphi\colon Q\to P$ such that

(i) $f\simeq\varphi g$ and

(ii) for any polyhedron R and any maps $h\colon X\to R$, $\psi\colon R\to P$ such that $\psi h\simeq f$ there is a map $\theta\colon Q\to R$ such that $\psi\theta\simeq\varphi$.

n-movability is defined by replacing condition (ii) by

(ii)$_n$ for any polyhedron R and any maps $h\colon X\to R$, $\psi\colon R\to P$ such that $\psi h\simeq f$ there is a map $\theta\colon Q^n\to R$ such that $\psi\theta\simeq\varphi|Q^n$.

THEOREM 4.2. *Any uniformly movable space is movable. Any uniformly n-movable space is n-movable.*

Proof. Let X be uniformly movable, and let $f\colon X\to P$ be a map into a polyhedron. By Remark 4.2 there exist Q, $g\colon X\to Q$, $\varphi\colon Q\to P$ and $\Phi\colon \Pi_X\to\Pi_Q$ with $f\simeq\varphi g$ and $\Phi[f] = [\varphi]$. If R is a polyhedron and $h\colon X\to R$ and $\psi\colon R\to P$ satisfy $\psi h\simeq f$, then taking $\theta\in\Phi[h]$ gives

$$[\psi\theta] = \psi_{\#}\Phi[h] = \Phi\psi_{\#}[h] = \Phi[f] = [\varphi].$$

Thus X is movable.

If M is uniformly n-movable, and $f\colon X\to P$ is given, choose Q, $g\colon X\to Q$, $\varphi\colon Q\to P$, and $\Phi\colon \Pi_X\to\Pi_{Q^n}$ as in the definition of uniform n-movability, and let R, $h\colon X\to R$, $\psi\colon R\to P$ satisfy $\psi h\simeq f$. Taking $\theta\in\Phi[h]$ we have

$$[\psi\theta] = \psi_{\#}\Phi[h] = \Phi\psi_{\#}[h] = \Phi[f] = [\varphi|Q^n].$$

Thus X is n-movable.

Remark 4.4. The converse of Theorem 4.2 is not valid in general (see our paper, *Movability and shape-connectivity*, Fund. Math. 93 (1976) pp. 145–154).

THEOREM 4.3. *A paracompactum X is movable if and only if for any open cover $\mathcal{U}$ of X, there is an open cover $\mathcal{V}$ refining $\mathcal{U}$ such that for any open cover $\mathcal{W}$ refining $\mathcal{U}$ there is a map $\lambda = \lambda^{\mathcal{V}\mathcal{W}}\colon |K(\mathcal{V})|\to|K(\mathcal{W})|$ such that $\pi_{\mathcal{U}\mathcal{W}}\lambda\simeq\pi_{\mathcal{U}\mathcal{V}}$. A paracompactum X is n-movable if and only if for any open cover $\mathcal{U}$ of X, there is an open cover $\mathcal{V}$ refining $\mathcal{U}$ such that for any open cover $\mathcal{W}$ refining $\mathcal{U}$ there is a map $\lambda||K^n(\mathcal{U})|\to|K(\mathcal{W})|$ such that $\pi_{\mathcal{U}\mathcal{W}}\lambda\simeq\pi_{\mathcal{U}\mathcal{V}}||K^n(\mathcal{V})|$.*

Proof. Suppose X is movable, and let $\mathcal{U}$ be an open cover of X. Let $f\colon X\to|K(\mathcal{U})|$ be a barycentric map, and choose Q, g, φ as in the definition of movability. Let $(v\colon X\to|K(\mathcal{V})|,\ g_v\colon |K(\mathcal{V})|\to Q)$ be a barycentric factorization of g in which $\mathcal{V}$ refines $\mathcal{U}$. Since $\varphi g_v v\simeq\pi_{\mathcal{U}\mathcal{V}}v$, we may assume by Remark 3.2 that $\varphi g_v\simeq\pi_{\mathcal{U}\mathcal{V}}$. Now if $\mathcal{W}$ refines $\mathcal{U}$, there is a barycentric factorization

$$(w\colon X\to|K(\mathcal{W})|,\ \pi_{\mathcal{U}\mathcal{W}}\colon |K(\mathcal{W})|\to|K(\mathcal{U})|)$$

of f; hence there is a map $\theta\colon Q\to|K(\mathscr{W})|$ such that $\pi_{\mathscr{U}\mathscr{W}}\theta\simeq\varphi$ and $\lambda^{\mathscr{V}\mathscr{W}}=\theta g_v$ satisfies the condition of the theorem.

Conversely, assume this condition is satisfied. If $f\colon X\to P$ is a map into a polyhedron, take a barycentric factorization $(u\colon X\to|K(\mathscr{U})|, f_u\colon |K(\mathscr{U})|\to P)$ of f. Then there is $\mathscr{V}$ as in that condition. Take $Q=|K(\mathscr{V})|$, $g\colon X\to Q$ a barycentric map, and

$$\varphi=f_u\pi_{\mathscr{U}\mathscr{V}}.$$

If $h\colon X\to R$, $\psi\colon R\to P$ satisfies $\psi h\simeq f$, choose a barycentric factorization $(w\colon X\to|K(\mathscr{W})|, h_w\colon |K(\mathscr{W})|\to R)$ in which $\mathscr{W}$ refines $\mathscr{U}$. Since $\psi h_w w\simeq f_u\pi_{\mathscr{U}\mathscr{W}}w$, we may assume as above $\psi h_w\simeq f_u\pi_{\mathscr{U}\mathscr{W}}$. Then $\theta=h_w\lambda^{\mathscr{V}\mathscr{W}}$ satisfies $\psi\theta\simeq\varphi$.

For n-movability the argument is similar.

THEOREM 4.4. *If X is uniformly movable and shape dominates Y, then Y is uniformly movable.*

Proof. Since X shape dominates Y there exists natural transformations $\Lambda\colon \Pi_Y\to\Pi_X$ and $\Theta\colon \Pi_X\to\Pi_Y$ such that $\Theta\Lambda=1_{\Pi_Y}$. Consider an arbitrary map $f\colon Y\to P$ of Y into a polyhedron P. Then $[f]\in\Pi_Y(P)$ and $\Lambda[\varphi]\in\Pi_X(P)$. Since X is uniformly movable there exists a polyhedron Q and natural transformations $\Phi\colon \Pi_X\to\Pi_Q$, $\Psi\colon \Pi_Q\to\Pi_X$ such that $\Psi\Phi\Lambda[f]=\Lambda[f]$. Let $\Psi'=\Theta\Psi$ and $\Phi'=\Phi\Lambda$. Then

$$\Psi'\Phi'[f]=\Theta\Psi\Phi\Lambda[f]=\Theta\Lambda[f]=1_{\Pi_Y}[f]=[f].$$

THEOREM 4.5. *If X is movable (n-movable) and shape dominates Y, then Y is movable (n-movable).*

Proof. Let $\Phi\colon \Pi_Y\to\Pi_X$ and $\Psi\colon \Pi_X\to\Pi_Y$ denote natural transformations such that $\Psi\Phi=1_{\Pi_Y}$. Let $f\colon Y\to P$ be a map of Y into a polyhedron P. Then $\Phi[f]$ is a homotopy class of maps of X into P. Since X is movable we have there exists a polyhedron Q and maps $q\colon X\to Q$, $\varphi\colon Q\to P$ such that: (i) $[\varphi g]=\Phi[f]$ and (ii) for any polyhedron R and maps $h\colon X\to R$, $\psi\colon R\to P$ such that $[\psi h]=\Phi[f]$ there is a map $\theta\colon W\to R$ such that

$$\psi\theta\simeq\varphi. \tag{1}$$

Consider any maps $h'\colon Y\to R$ and $\psi'\colon R\to P$ such that

$$\psi'h'\simeq f. \tag{2}$$

Applying the natural transformation Ψ to (i) we get

$$\varphi_{\#}\Psi[g]=[f]. \tag{3}$$

Then applying the natural transformation Φ to (2) we get

$$\psi'_{\#}\Phi[h']=\Phi[f]. \tag{4}$$

Now take $h=\Phi[h']$ and $\psi=\psi'$. Then

$$[\psi h]=\psi'_{\#}[h]=\psi'_{\#}\Phi[h']=\Phi[f].$$

So the hypothesis of (ii) are satisfied. Therefore, there exists $\theta\colon Q\to R$ such that $\psi\theta\simeq\varphi$ and so

$$\psi'\theta\simeq\varphi. \tag{5}$$

Lines (3) and (5) imply conditions (i) and (ii) hold for the map $f\colon Y\to P$. Hence Y is movable.

The proof in case X is n-movable is similar.

The following generalizes the analogous result for compacta in [8].

THEOREM 4.6. *If X is an n-dimensional, uniformly n-movable paracompactum, then X is uniformly movable.*

Proof. Let $f\colon X\to P$ be any map of X into a polyhedron P. Since X is uniformly n-movable there exist a polyhedron Q, maps $g\colon X\to Q$, $\varphi\colon Q\to P$, and a natural transformation $\Phi\colon \Pi_X\to\Pi_{Q^n}$ such that $\varphi g\simeq f$ and $\Phi[f]=[\varphi|Q^n]$. Since X is paracompact Q can be taken as the geometric nerve of a covering $\mathscr{U}$ which is a member of some cofinal family Γ in $\mathrm{Cov}(X)$. Since X is n-dimensional Γ can be taken as a family of coverings of order n, so that Q will be n-dimensional and so $Q^n=Q$. We thus have obtained the desired relation for uniform movability.

THEOREM 4.7. *A compactum is movable in the sense of Mardešić–Segal if and only if it is movable. It is n-movable in the sense of* [8], *if and only if it is n-movable.*

Proof. Throughout the entire proof $\{X_\alpha, p_{\alpha\alpha'}, A\}$ is an ANR-system associated with X in which each X_α is a polyhedron; $p_\alpha\colon X\to X_\alpha$ is a projection of the system. Suppose X is movable. By Theorem 4.1 there is a finite open cover $\mathscr{U}$ and a barycentric factorization $(u\colon X\to|K(\mathscr{U})|, p_{\alpha\cdot u}\colon |K(\mathscr{U})|\to X_\alpha)$ of p_α such that for any polyhedron R and maps $h\colon X\to R$, $\psi\colon R\to X_\alpha$ such that $p_\alpha\simeq\psi h$ there is a map $\theta\colon |K(\mathscr{U})|\to R$ such that $\psi\theta\simeq p_{\alpha\cdot u}$.

By Remark 3.4 there is an index $\beta\geqslant\alpha$ and a map $\eta\colon X_\beta\to|K(\mathscr{U})|$ such that $\eta p_\beta\simeq u$. Since $p_{\alpha\cdot u}\eta p_\beta\simeq p_{\alpha\beta}p_\beta$, by the direct limit property in this remark there is an $\alpha'\geqslant\beta$ such that $p_{\alpha\cdot u}\eta p_{\alpha'\beta}\simeq p_{\alpha\alpha'}$. Set $u'=\eta p_{\alpha'\beta}$. Then $p_{\alpha\cdot u}u'\simeq p_{\alpha\alpha'}$. If $\alpha''\geqslant\alpha$, take $R=X_{\alpha''}$, $h=p_{\alpha''}$, $\psi=p_{\alpha\alpha''}$ to conclude that there is a map $\theta\colon |K(\mathscr{U})|\to X_{\alpha''}$ such that $p_{\alpha\alpha''}\theta\simeq p_{\alpha\cdot u}$. Set $r^{\alpha'\alpha''}=\theta u'$; and observe that $p_{\alpha\alpha''}r^{\alpha'\alpha\alpha''}\simeq p_{\alpha\cdot u}u'\simeq p_{\alpha\alpha'}$. Thus movability implies movability in the sense of Mardešić–Segal.

To prove the converse recall that Π_X is represented as the direct limit of the system $\{\Pi_\alpha=\Pi_{X_\alpha}, p^{\#}_{\alpha\alpha'}, A\}$ by means of the maps $p^{\#}_\alpha\colon \Pi_\alpha\to\Pi_X$. If $f\colon X\to P$ is a map into a polyhedron, it follows that there exist an index α and a map $f_\alpha\colon X_\alpha\to P$ such that $f\simeq f_\alpha p_\alpha$. By the definition of movability in the sense of Mardešić–Segal there is $\alpha'\geqslant\alpha$ such that for any $\alpha''\geqslant\alpha$ there is $r^{\alpha'\alpha\alpha''}\colon X_{\alpha'}\to X_{\alpha''}$ such that $p_{\alpha\alpha''}r^{\alpha'\alpha\alpha''}\simeq p_{\alpha\alpha'}$. Take $Q=X_{\alpha'}$, $g=p_{\alpha'}$, and $\varphi=f_\alpha p_{\alpha\alpha'}$ to satisfy Definition 4.4 as follows:

To verify (ii) consider $h\colon X\to R$, $\psi\colon R\to P$ with $\psi h\simeq f$. As before there exist $\beta\geqslant\alpha$ and a map $h_\beta\colon X_\beta\to R$ such that $h\simeq h_\beta p_\beta$. Since $p^{\#}_\beta[\psi h_\beta]=[f]=p^{\#}_\beta[f_\alpha p_{\alpha\beta}]$, it follows from the representation of Π_X as the direct limit that there is $\alpha''\geqslant\beta$ such that $p^{\#}_{\beta\alpha''}[\psi h_\beta]=p^{\#}_{\beta\alpha''}[f_\alpha p_{\alpha\beta}]$.

Let $\theta=h_\beta p_{\beta\alpha''}r^{\alpha'\alpha''}$ and observe that

$$\psi\theta=\psi h_\beta p_{\beta\alpha''}r^{\alpha'\alpha''}\simeq f_\alpha p_{\alpha\beta}p_{\beta\alpha''}r^{\alpha'\alpha''}\simeq\varphi.$$

Since $\varphi g \simeq f$, it follows that the conditions of Definition 4.4 are satisfied. Thus, movability, in the sense of Mardešić–Segal implies movability. This completes the proof of the first assertion. To prove the second observe that the definition of n-movability in [8] takes by means of cellular approximation [23] an equivalent form when one considers systems of polyhedra associated with X as above: X is n-movable if and only if for each α there is an $\alpha' \geqslant \alpha$ such that for any $\alpha'' \geqslant \alpha$ there is a map $r^{\alpha'\alpha''}\colon X^n_{\alpha'} \to X_{\alpha''}$ with $p_{\alpha\alpha''} r^{\alpha'\alpha''} \simeq p_{\alpha\alpha'}|X^n_{\alpha'}$.

When X is n-movable, the first part of the above proof is changed in that θ and h are defined on $|K^n(\mathcal{U})|$ instead of $|K(\mathcal{U})|$. By cellular approximation we may assume $u'(X^n_{\alpha'}) \subset |K^n(\mathcal{U})|$. Hence, $r^{\alpha'\alpha''} = hu'$ defines the desired map.

When X is n-movable in the sense of [8], the second part of the proof is changed in that $r^{\alpha'\alpha''}$ is defined on $X^n_{\alpha'}$. As above, the map $\theta = h_\beta p_{\beta\alpha''} r^{\alpha'\alpha''}$ gives the desired conclusion.

THEOREM 4.8. *If X is a metric compactum which is movable (n-movable in the sense of* [8]), *then it is uniformly movable (uniformly n-movable).*

Proof. Let $\underline{X} = \{X_m, p_{mm'}\}$ be an ANR-sequence associated with X such that each X_m is a polyhedron. Let $f\colon X \to P$ be any map of X into a polyhedron P.

By Remark 3.4 there exists an index m_0 and a map $f_{m_0}\colon X_{m_0} \to P$ such that $f_{m_0} p_{m_0} \simeq f$. Truncating the sequence and renumbering we have a map $f_1\colon X_1 \to P$ such that

$$f_1 p_1 \simeq f.$$

Now since X is movable $\underline{X}$ is a movable sequence. We define by induction a strictly increasing sequence $m_0 = 1 < m_1 < m_2 < \ldots < m_k < \ldots$ such that m_k is α' if m_{k-1} is α in the definition of movability [8]. Since $m_{k+1} > m_{k-1}$ we have maps $\varphi_1 = p_{12}$, $\varphi_2 = p_{23}\psi_3$ where ψ_3 is obtained by applying Definition 2 of [8] to $1_{X_2}\colon X_2 \to X_2$ to get a map $\psi_3\colon X_2 \to X_3$ such that $p_{13}\psi_3 \simeq p_{12} 1_{X_2}$, $\varphi_3 = p_{34}\psi_4$ where $\psi_4\colon X_2 \to X_4$ satisfies, $p_{24}\psi_4 \simeq p_{23}\psi_3$, and is obtained by definition as above, and so on. Inductively, we have $\varphi_m = p_{m\,m+1}\psi_{m+1}$ where $p_{m-1,m+1}\psi_{m+1} \simeq p_{m-1,m}\psi_m$.

Let $Q = X_2$ in Definition 4.1 and note $\{\varphi_m\}$ defines a map of sequences $\underline{\varphi} = \{\varphi_m\}\colon Q \to \underline{X}$. Now in Definition 4.1 as modified by the remark let $h = f_1 p_{12}\colon Q \to P$ and let $g = p_2\colon X \to Q$. Furthermore, we have $\underline{\varphi}$ determines a natural transformation $\underline{\varphi}^\#$ which we take for $\Phi\colon \Pi_X \to \Pi_Q$. Then we have the relations required for uniform movability, namely,

$$\Phi[f] = \underline{\varphi}^\#[f] = [f_1\varphi_1] = [f_1 p_{12}] = [h]$$

and

$$hg = f_1 p_{12} p_2 = f_1 p_1 \simeq f.$$

For uniform n-movability we proceed as above. However, $\varphi_m\colon Q'' = X^n_2 \to X_m$ is the restriction of the φ_m obtained previously to X^n_2. This determines a map of sequences $\underline{\varphi} = \{\varphi_m\}\colon Q'' \to \underline{X}$. Then $\underline{\varphi}$ determines a natural transformation $\Phi = \underline{\varphi}^\#\colon \Pi_X \to \Pi_{Q''}$. Let $h = f_1 p_{12}\colon Q \to P$ and let $g = p_2\colon X \to Q$. Then

$$\Phi[f] = \underline{\varphi}^\#[f] = [f_1\varphi_1] = [f_1(p_{12}|Q'')] = [(f_1 p_{12})|Q''] = [h|Q'']$$

and

$$hg = f_1 p_{12} p_2 = f_1 p_1 \simeq f.$$

(S. Spież has also obtained the same result in the movable case.)

5. ANSE's and ANSR's. Before we state our generalization of extensor to the shape of paracompacta we recall the classical definition from [5]. The closed subset A of a space X is said to have the *neighborhood extension property* in X with respect to the space Y if and only if every map $f\colon A \to Y$ can be extended to some neighborhood N of A in X. Let $\mathcal{F}$ denote a weakly hereditarily topological class of spaces. By an *absolute neighborhood extensor* for the class $\mathcal{F}$ (written ANE($\mathcal{F}$)) we mean a space Y such that every closed subset A of any space $X \in \mathcal{F}$ has the neighborhood extension property in X with respect to Y.

DEFINITION 5.1. We say Y is an *absolute neighborhood shape extensor* for paracompacta (ANSE) if for any natural transformation $\Phi\colon \Pi_Y \to \Pi_A$, where A is any closed subset of an arbitrary paracompactum X, there is a closed neighborhood N of A and a natural transformation $\Psi\colon \Pi_Y \to \Pi_N$ such that $\varrho\Psi = \Phi$ (where $\varrho\colon \Pi_N \to \Pi_A$ denotes the restriction). In the ANR-systems approach this implies that any compactum Y is an absolute neighborhood shape extensor if any shape map $\underline{f}\colon \underline{A} \to \underline{Y}$ can be extended to a shape map $\underline{F}$ of a closed neighborhood N of A in X. (Here $\underline{F}$ extends $\underline{f}$ means $\underline{F}\,\underline{i} \simeq \underline{f}$ where $\underline{i}$ is a shape map of A into N induced by the inclusion $i\colon A \to N$.)

We will also generalize the following description of absolute neighborhood retracts for compacta in shape theory due to Mardešić [4] to paracompacta. Mardešić's definition was a generalization of Borsuk's [1] fundamental absolute neighborhood retracts (FANR's) to the compact Hausdorff case. Mardešić says that a compactum Y is a *absolute neighborhood shape retract* provided, for every compactum Z, $Y \subset Z$, there exists a closed neighborhood N of Y in Z, such that Y is a shape retract of N (i.e., there is a shape map $\underline{r}\colon N \to Y$ such that $\underline{r}\,\underline{i} \simeq 1_Y$, where $i\colon Y \to N$ is the inclusion map).

DEFINITION 5.2. The paracompactum Y is said to be an *absolute neighborhood shape retract* (ANSR) if, whenever Y is a closed subset of a paracompactum Z, there exist a neighborhood N of Y in Z and a natural transformation $\Psi\colon \Pi_Y \to \Pi_N$ such that $\varrho\Psi = 1_{\Pi_Y}$ (where $\varrho\colon \Pi_N \to \Pi_Y$ is the restriction). Clearly, every compact ANSR is an ANSR (in the sense of Mardešić) by Theorem 3.1.

THEOREM 5.1. *If a paracompactum Y is an* ANSE *then it is an* ANSR.

Proof. In the definition of ANSE take $X = Z$, $A = Y$ and let the natural transformation $\Phi\colon \Pi_Y \to \Pi_A$ be the identity. Then by the definition there is a closed neighborhood N of Y in Z and a natural transformation $\Psi\colon \Pi_Y \to \Pi_N$ such that $\varrho\Psi = 1_{\Pi_Y}$ so that Y is an ANSR.

THEOREM 5.2. *A compactum Y is an* ANSR (*in the sense of Mardešić*) *iff it is an* ANSE.

Proof. Let Y be an ANSR (in the sense of Mardešić). Let $h\colon Y\to Y'\subset I^m$ be an imbedding of Y in a cube (where m is possibly uncountable). Now Y' is a shape retract of a neighborhood Q in I^m and Q can be taken to be the product of a compact polyhedron P with a cube: $Q = P\times I^n\subset I^m$. Let the shape retraction be $\underline{r}\colon \underline{Q}\to \underline{Y}'$. Since $P\times I^n$ is an ANR the shape class $A\to P\times I^n$ defined on a closed subset A of paracompactum X contains a map; this map extends over a neighborhood N of A in X to give the desired shape class extension.

The converse is obvious by Theorem 5.1.

THEOREM 5.3. *Any space shape dominated by an* ANSE *is also an* ANSE.

Proof. Let Y' be an ANSE which shape dominates Y. Then we have natural transformations $\Lambda\colon \Pi_Y\to\Pi_{Y'}$ and $\Theta\colon \Pi_{Y'}\to\Pi_Y$ such that $\Theta\Lambda = 1_{\Pi_Y}$. Now let A be a closed subset of a paracompactum X. Let $\Phi\colon \Pi_Y\to\Pi_A$ be a natural transformation. Since Y' is an ANSE, and $\Phi\Theta\colon \Pi_{Y'}\to\Pi_A$ is a natural transformation, there is a closed neighborhood N of A in X and a natural transformation $\Psi'\colon \Pi_{Y'}\to\Pi_N$ such that $\varrho\Psi' = \Phi\Theta$. Let $\Psi = \Psi'\Lambda$. Then

$$\varrho\Psi = \varrho\Psi'\Lambda = \Phi\Theta\Lambda = \Phi 1_{\Pi_Y} = \Phi$$

and so Y is an ANSE.

The following is a restatement in shape theory of a result of [10].

THEOREM 5.4. *Any polyhedron P is an* ANSE.

Proof. Let A be a closed subset of a paracompactum X. Consider the shape class $A\to P$. It contains a map $A\to P$ and by [10, Corollary 4.8] the map is homotopic to one which can be extended over a neighborhood of A in X.

COROLLARY 5.1. *Any space shape dominated by a polyhedron is an* ANSE.

THEOREM 5.5. *For a metrizable space X the following three properties are equivalent*:

(1) *X is an* ANSR,

(2) *X is an* ANSE,

(3) *X is shape dominated by a polyhedron.*

Proof. It suffices to show that (1) implies (3). Let X be an ANSR and let $h\colon X\to C_0(X)$ be the Kuratowski–Wojdyslawski imbedding of X into the space of all bounded continuous functions on X defined by choosing a bounded metric d on X and setting $h(x) = d_x$ where $d_x(y) = d(x, y)$. Then $h(X)$ is a closed subset of its convex hull H in $C_0(X)$. Hence there exists an open neighborhood N of $h(X)$ in H and a natural transformation $\Psi\colon \Pi_{h(X)}\to\Pi_N$ such that $\varrho\Psi$ = identity (where ϱ is the restriction). Since N is an ANR, it is dominated by a polyhedron P. Let $f\colon N\to P$, $g\colon N\to P$ satisfy $gf\simeq 1_N$. Then $g^{\#}\Psi$ and $\varrho f^{\#}$ give a shape domination of X by P.

6. Partial realizations.

DEFINITION 6.1. A *partial realization* of a complex K in a collection $\mathscr{A}$ of sets in a space X is a map $t\colon |L|\to X$ from the polyhedron of some subcomplex L, which contains all the vertices of K, into X such that for any simplex s of K there is an $A\in\mathscr{A}$ with $t(|s|\cap|L|)\subset A$. A *full realization* of K in $\mathscr{A}$ is a partial realization in $\mathscr{A}$ in which $L = K$.

We now recall a theorem from [5, p. 157, 159]. Note that "LC" indicates homotopy local connectedness.

THEOREM 6.2. *If X is an LC^{n-1} paracompactum, then for any open cover $\mathscr{R}$ there is a cover $\mathscr{P}$ such that any partial realization of an n-complex in $\mathscr{P}$ extends to a full realization in $\mathscr{R}$.*

If $\mathscr{U}$ is an open cover of X and A is any subset of X, then the star of A in $\mathscr{U}$ $\mathrm{St}(A, \mathscr{U}) = \{U\in\mathscr{U}|\ A\cap U\neq\emptyset\}$. When $A\in\mathscr{U}$, we write $A^* = \bigcup\mathrm{St}(A, \mathscr{U})$. An open cover $\mathscr{U}$ is called a *star-refinement* of an open cover $\mathscr{V}$, provided the collection $\mathscr{U}^* = \{U^*|\ U\in\mathscr{U}\}$ refines $\mathscr{V}$. We frequently use the fact that every open cover of a paracompactum has a star-refinement.

LEMMA 6.3. *Suppose that* (1) *$\mathscr{P}$, $\mathscr{R}$ and $\mathscr{U}$ are open cover of a topological space X,* (2) *$\mathscr{P}$ refines $\mathscr{R}$, $\mathscr{R}$ star-refines $\mathscr{U}$, and $\mathscr{U}$ is locally finite, and* (3) *any partial realization in $\mathscr{P}$ of any n-complex has a full realization in $\mathscr{R}$. Then for any star-refinement $\mathscr{V}$ of $\mathscr{P}$ there is a full realization $t\colon |K^n(\mathscr{V})|\to X$ of $K^n(\mathscr{V})$ in $\mathscr{R}$ such that for any projection $\pi\colon K(\mathscr{V})\to K(\mathscr{U})$ and barycentric map $u\colon X\to|K(\mathscr{U})|$ the maps ut and $\pi||K^n(\mathscr{V})|$ are homotopic. Moreover, the map t can be chosen so that for any simplex $s\in K^n(\mathscr{V})$ the image $t(|s|)$ is contained in $\bigcup\mathrm{St}(\bigcup s, \mathscr{R})$.*

Proof. Define a partial realization

$$t^0\colon K^0(\mathscr{V})\to X$$

of $K(\mathscr{V})$ in $\mathscr{V}$ by choosing $t^0(V)\in V$ for each $V\in\mathscr{V}$. If s is any simplex of $K(\mathscr{V})$, then

$$\bigcap s\neq\emptyset;$$

hence there is a $V_s\in\mathscr{V}$ such that

$$s\subset\mathrm{St}(V_s, \mathscr{V}).$$

Thus there is a $P_0\in\mathscr{P}$ such that $t^0(s)\subset\bigcup s\subset P_0$, which shows that t^0 is a partial realization of $K^n(\mathscr{V})$ in $\mathscr{P}$. By hypothesis there is a full realization of $K^n(\mathscr{V})$ in $\mathscr{R}$, i.e., there is a map $t\colon |K^n(\mathscr{V})|\to X$ such that, for any simplex s of $K^n(\mathscr{V})$, there is an $R(s)\in\mathscr{R}$ with $t(|s|)\subset R(s)$.

Let $u\colon X\to|K(\mathscr{U})|$ be any barycentric map of X into $|K(\mathscr{U})|$. If s is any simplex of $K^n(\mathscr{V})$ and $x\in t(|s|)$, then there exists a simplex $s(x)$ of $K(\mathscr{U})$ with $u(x)\in|s(x)|$ and $s(x)\subset\mathrm{St}(x, \mathscr{U})$. Let $C(s)$ be the subcomplex of $K(\mathscr{U})$ consisting of all those simplexes s' such that $t(|s|)\cap(\bigcap s')\neq\emptyset$. For any $p\in|s|$ write $x = t(p)$ and observe that because $ut(p)\in|s(x)|$ and $t(p)\in|s(x)|$, that

$$ut(p)\in|C(s)|\,.$$

Hence $ut(|s|)\subset|C(s)|$ for any simplex s of $K^n(\mathscr{V})$.

Consider a projection $\pi_1\colon K(\mathscr{V})\to K(\mathscr{R})$. If s is any simplex of $K(\mathscr{V})$, then $\pi_1(s)$ is a simplex of $K(\mathscr{R})$ which has the property that any vertex of $\pi_1(s)$ contains a point of $t(|s|)$. Thus $\pi_1(s) \subset \mathrm{St}(R(s), \mathscr{R})$. Let π_2: $K(\mathscr{R}^*) \to K(\mathscr{U})$ be a projection. If s is any simplex of $K(\mathscr{V})$, then for any vertex R of $\pi_1(s)$, $R \cap R(s) \neq \emptyset$; hence

$$\pi_2(R^*) \supset R^* \supset R(s) .$$

Define a projection π_0: $K(\mathscr{R}) \to K(\mathscr{U})$ by $\pi_0(R) = \pi_2(R^*)$. Since any vertex U of $\pi_0\pi_1(s)$ has the form $U = \pi_2(R^*)$ for some vertex R of $\pi_1(s)$, $\pi_0(R(s))$ and the vertices of $\pi_0\pi_1(s)$ comprise a simplex of $K(\mathscr{U})$.

Now let $U(s) = \pi_0(R(s))$ and notice that any simplex of $C(s)$ is contained in a simplex of which $U(s)$ is a vertex. Thus $C(s)$ is a cone with cone vertex $U(s)$ which implies that $|C(s)|$ is contractible. Since any vertex of $\pi_0\pi_1(s)$ contains $t(|s|)$, $C(s)$ contains the simplex $\pi_0\pi_1(s)$. Thus for any simplex s of $K^n(\mathscr{V})$ we have

$$t(|s|) \cup \pi_0\pi_1(s) \subset |C(s)|$$

and if s' is a face of s, then

$$C(s') \subset C(s) .$$

The fact that $ut \simeq \pi_0\pi_1||K^n(\mathscr{V})|$ follows from Lemma 6.4.

LEMMA 6.4. *Two maps f and g of a polyhedron $|K|$ into a space Y are homotopic, provided that there is an assignment of simplexes $s \in K$ to contractible subsets $C(s)$ of Y such that:*

(1) $$f(|s|) \cup g(|s|) \subset C(s)$$

and

(2) *if s' is a face of s, then $C(s') \subset C(s)$.*

Proof. Define a homotopy H: $|K| \times I \to Y$ in stages on the skeleta of K. Let H_n: $|K| \times \{0, 1\} \cup |K^n| \times I \to Y$ be defined inductively as follows:

$$H_{-1}(p, 0) = f(p) , \quad H_{-1}(p, 1) = g(p) ,$$

(Note: $K^{-1} = \emptyset$.) Suppose H_n has been defined and satisfies

(3) $$H_n(|s'| \times I) \subset C(s') \quad \text{for every } s' \in K^n .$$

If s is any $(n+1)$-simplex of K, then

$$H_n(|s| \times \{0, 1\} \cup |\partial s| \times I) \subset C(s)$$

because of (3) and (2). Then there is a map $|s| \times I \to C(s)$ extending $H_n|_{|\partial(s \times I)|}$. These extensions combine to give the desired H_{n+1}.

The next Corollary follows from Theorem 6.2 and Lemma 6.3.

COROLLARY 6.1. *For any open cover $\mathscr{U}$ of an LC^{n-1} paracompactum X and for any sufficiently fine open cover $\mathscr{V}$ refining $\mathscr{U}$ there is a map t: $|K^n(\mathscr{V})| \to X$ such that $ut \simeq \pi_{\mathscr{U}\mathscr{V}}||K^n(\mathscr{V})|$, when u: $X \to |K(\mathscr{U})|$ is a barycentric map.*

7. Locally well-behaved paracompacta.

THEOREM 7.1. *Any LC^n paracompactum X of (covering) dimension $\leqslant n$ is shape dominated by some polyhedron of dimension $\leqslant n$ (which can be chosen as the polyhedron of the nerve of an open cover of X).*

Proof. Let $\mathscr{W}$ be an open cover with the property that any two maps f, g: $|K| \to X$ of a polyhedron $|K|$ of dimension $\leqslant n$ into X are homotopic, provided that for any simplex s of K there is a $W(s) \in \mathscr{W}$ with $f(|s|) \cup g(|s|) \subset W(s)$. (For a proof of the existence of such a $\mathscr{W}$ see Lemma 2 of [7] and observe that the finiteness condition in its proof is irrelevant.)

Let $\mathscr{R}$ be a star-refinement of $\mathscr{W}$ and let $\mathscr{P}$ be an open cover refining $\mathscr{R}$ such that any partial realization of an n-complex in $\mathscr{P}$ has a full realization in $\mathscr{R}$. Let $\mathscr{U}$ be a locally finite open cover which star-refines $\mathscr{P}$ and whose nerve has dimension $\leqslant n$. We will show that $|K(\mathscr{U})|$ shape dominates X.

The map $t_{\mathscr{U}}$: $|K(\mathscr{U})| \to X$ is the full realization obtained by using Lemma 6.3. Let $\mathscr{C}$ be any locally finite open cover of X. Let $\mathscr{R}(\mathscr{C})$ be a star-refinement of $\mathscr{C}$ and a refinement of $\mathscr{R}$, and let $\mathscr{P}(\mathscr{C})$ be a refinement of $\mathscr{R}(\mathscr{C})$ such that every partial realization of an n-complex in $\mathscr{P}(\mathscr{C})$ has a full realization in $\mathscr{R}(\mathscr{C})$.

Let $\mathscr{V}$ be a star refinement of $\mathscr{P}(\mathscr{C})$ and a refinement of $\mathscr{U}$ whose nerve has dimension $\leqslant n$. By Lemma 6.3 there is a realization $t_{\mathscr{V}}$: $|K(\mathscr{V})| \to X$ in $\mathscr{R}(\mathscr{C})$ such that for any barycentric map c: $X \to |K(\mathscr{C})|$ and any projection $\pi_{\mathscr{C}\mathscr{V}}$: $K(\mathscr{V}) \to K(\mathscr{C})$ we have

(1) $$ct_{\mathscr{V}} \simeq \pi_{\mathscr{C}\mathscr{V}} .$$

If s is any simplex of $K(\mathscr{V})$, then by Lemma 6.3 $t_{\mathscr{V}}(|s|) \subset \bigcup \mathrm{St}(\bigcup s, \mathscr{R}(\mathscr{C}))$, and $t_{\mathscr{U}}(\pi(|s|)) \subset \bigcup \mathrm{St}(\bigcup \pi(s), \mathscr{R})$ where $\pi = \pi_{\mathscr{U}\mathscr{V}}$. Since $\mathscr{R}(\mathscr{C})$ refines $\mathscr{R}$ and $\mathscr{V}$ refines $\mathscr{U}$, $\bigcup s \subset \bigcup \pi(s)$ and accordingly $\bigcup \mathrm{St}(\bigcup s, \mathscr{R}(\mathscr{C})) \subset \bigcup \mathrm{St}(\bigcup \pi(s), \mathscr{R})$. Since $\mathscr{U}$ is a star-refinement of $\mathscr{R}$, $\bigcup \mathrm{St}(\bigcup \pi(s), \mathscr{R}) \subset R^*$ for some $R \in \mathscr{R}$ and, therefore, $\bigcup \mathrm{St}(\bigcup \pi(s), \mathscr{R})$ is contained in some member $W(s)$ of $\mathscr{W}$. Hence, for any simplex s of $K(\mathscr{V})$ the images $t_{\mathscr{V}}(|s|)$ and $t_{\mathscr{U}}\pi(|s|)$ are contained in $W(s)$, and consequently

(2) $$t_{\mathscr{V}} \simeq t_{\mathscr{U}}\pi_{\mathscr{U}\mathscr{V}} .$$

The desired shape domination is induced by any barycentric map u: $X \to |K(\mathscr{U})|$ and the map $t_{\mathscr{U}}$: $|K(\mathscr{U})| \to X$. To prove this we will show that $u^{\#}t_{\mathscr{U}}^{\#}$ is the identity natural transformation on the functor Π_X. This means that for any map f: $X \to P$ of X into any polyhedron P we have $u^{\#}t_{\mathscr{U}}^{\#}[f] = [f]$, or equivalently $ft_{\mathscr{U}}u \simeq f$. Since f has a barycentric factorization (c, f_c) it suffices to show that

(3) $$ct_{\mathscr{U}}u \simeq c$$

for any barycentric map c: $X \to |K(\mathscr{C})|$. Now

(4) $$ct_{\mathscr{U}}u \simeq ct_{\mathscr{U}}\pi_{\mathscr{U}\mathscr{V}}v \simeq ct_{\mathscr{V}}v \simeq \pi_{\mathscr{C}\mathscr{V}}v \simeq c .$$

The first and last homotopies of (4) hold because any barycentric map followed by a projection is a barycentric map and barycentric maps are homotopic; the second and third homotopies are consequences of (2) and (1), respectively.

Since any polyhedron with the metric topology is an ANR (see [5, p. 106]) we have

COROLLARY 7.2. *Any LC^n paracompactum X of dimension $\leqslant n$ is shape dominated by an ANR.*

The next result also follows from Theorems 7.1, 5.4 and 5.3.

THEOREM 7.3. *Any* LC^n *paracompactum* X *of dimension* $\leqslant n$ *is an* ANSE *and therefore an* ANSR.

Remark 7.1. Since an ANSR may behave badly locally there is no chance of extending the compact metric result, ANR$\Rightarrow LC^n$, to paracompacta. On the other hand, an example due C. W. Saalfrank [21] shows that the compact metric result, at most n-dimensional and $LC^n\Rightarrow$ANR, cannot be extended to compact Hausdorff spaces. However, Theorem 7.3 does extend it to paracompacta in shape theory, i.e., at most n-dimensional and $LC^n\Rightarrow$ANSR.

THEOREM 7.4. *Every* LC^{n-1} *paracompactum is uniformly* n*-movable.*

Proof. Let $f\colon X\to P$ be a map into a polyhedron, and let $(u\colon X\to|K(\mathcal{U})|$, $f_u\colon |K(\mathcal{U})|\to P)$ be a barycentric factorization of f. Choose $\mathcal{V}$ and $t\colon |K^n(\mathcal{V})|\to X$ as in Corollary 6.1, and take $Q=|K(\mathcal{V})|$, $\varphi=f_u\pi_{\mathcal{U}\mathcal{V}}$, $\Phi=t^{\#}$, and $g\colon X\to Q$ a barycentric map. These satisfy the conditions of Definition 4.3.

COROLLARY 7.3. *Every* LC^{n-1} *paracompactum of dimension* $\leqslant n$ *is uniformly movable.* (*In the compact metric case this was first obtained by Mardešić* [11] *and in* [20].)

Remark 7.2. The proofs indicate that the natural transformations implicit in Theorem 7.1, Corollary 7.2, Theorem 7.4 and Corollary 7.3 are induced by maps.

8. Summary. We now summarize in diagram form the results of this paper and classical results on locally well-behaved compacta. An arrow ($\to$) indicates class inclusion and a broken arrow ($-n\to$) indicates class inclusion under the additional hypothesis that the dimension of the space in question is $\leqslant n$. Here SDP indicates a space dominated by a polyhedron.

Classically we have for metric spaces:

(I) $$LC^n \overset{\longleftarrow}{\underset{n}{\longrightarrow}} ANE\leftrightarrow ANR$$

and for compacta:

(II) $$LC^n\leftarrow ANE\leftrightarrow ANR\,.$$

In shape theory we have for compacta:

(III) $$\begin{array}{ccc} LC^n & \xrightarrow{\;\;n\;\;} & SDP\leftrightarrow ANSE\leftrightarrow ANSR \\ \downarrow & & \downarrow \\ LC^{n-1} & & \\ \downarrow & & \downarrow \\ \text{uniformly } n\text{-movable} & \overset{\longleftarrow}{\underset{n}{\longrightarrow}} & \text{uniformly movable} \end{array}$$

and for paracompacta

(IV) $$\begin{array}{ccc} LC^n & \xrightarrow{\;\;n\;\;} & SDP\to ANSE\to ANSR \\ \downarrow & & \downarrow \\ LC^{n-1} & & \\ \downarrow & & \downarrow \\ \text{uniformly } n\text{-movable} & \overset{\longleftarrow}{\underset{n}{\longrightarrow}} & \text{uniformly movable} \end{array}$$

We do not know if ANSR$\to$ANSE or if ANSE$\to$SDP for paracompacta.

References

[1] K. Borsuk, *Fundamental retracts and extension of fundamental sequences*, Fund. Math. 64 (1969), pp. 55–85.
[2] — *On movable compacta*, Fund. Math. 66 (1969), pp. 137–146.
[3] — *On the n-movability*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 20 (1972), pp. 859–864.
[4] R. H. Fox, *On shape*, Fund. Math. 74 (1972), pp. 47–71.
[5] S. T. Hu, *Theory of Retracts*, Wayne State University Press 1965.
[6] G. Kozlowski, *Polyhedral limitations on shape*. (Unpublished.)
[7] — *Factorization of certain maps up to homotopy*, Proc. Amer. Math. Soc. 21 (1969), pp. 88–92.
[8] — and J. Segal, *n-movable compacta and* ANR-*systems*, Fund. Math. 85 (1974), pp. 235–243.
[9] K. Kuperberg, *An isomorphism theorem of the Hurewicz type in Borsuk's theory of shape*. Fund. Math. 77 (1972), pp. 21–32.
[10] C. N. Lee and F. Raymond, *Čech extensions of contravariant functors*, Trans. Amer. Math. Soc. 133 (1968), pp. 415–434.
[11] S. Mardešić, *n-dimensional* LC^{n-1} *compacta are movable*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 19 (1971), pp. 25–29.
[12] — *Shapes for topological spaces*, Gen. Top. Appl. 3 (1973), pp. 265–282.
[13] — *Equivalence of two notions of shape for metric spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 21 (1973), pp. 1137–1142.
[14] — *Retracts in shape theory*, Glasnik Mat. 6 (1971), pp. 153–163.
[15] — and J. Segal, *Shape of compacta and* ANR-*systems*, Fund. Math. 72 (1971), pp. 41–59.
[16] — — *Movable compacta and* ANR-*systems*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 18 (1970), pp. 649–654.
[17] J. Milnor, *On spaces having the homotopy type of* CW-*complex*, Trans. Amer. Math. Soc. 90 (1959), pp. 272–280.
[18] M. Moszyńska, *Uniformly movable compact spaces and their algebraic properties*, Fund. Math. 77 (1972), pp. 125–144.
[19] R. H. Overton, *Čech homology for movable compacta*, Fund. Math. 77 (1973), pp. 241–251.
[20] — and J. Segal, *A new construction of movable compacta*, Glasnik Mat. 6 (1971), pp. 361–363.
[21] C. W. Saalfrank, *Retraction properties for normal Hausdorff spaces*, Fund. Math. 36 (1949), pp. 93–108.
[22] E. H. Spanier, *Algebraic Topology*, Mc Graw-Hill 1966.
[23] J. H. C. Whitehead, *Combinatorial Homotopy I*, Bull. Amer. Math. Soc. 55 (1949), pp. 213–245.

THE UNIVERSITY OF WASHINGTON
Seattle, Washington

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