assuming that $X \subset N$, it suffices to observe that $N \setminus X$ cannot posses infinitely many components. But this is an easy consequence of the fact that $X \in \alpha$.

COROLLARY 4.5. *Any compactum X quasi-homeomorphic with an* ANR-*set $Y \subset M$, where M is a surface, is itself an* ANR-*set embeddable into a surface.*

Indeed, since Y is X-like, it follows that X is locally connected, and therefore, by Corollary 4.4, X is an ANR-set embeddable into a surface.

The answer to the following question is not known to the author, but it seems to be positive:

PROBLEM. Can we assert in Corollaries 4.4 and 4.5 that the space X is embeddable into the same surface M which contains Y?

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On models of arithmetic having non-modular substructure lattices

by

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Abstract. A model of arithmetic having the pentagon lattice for its lattice of elementary substructures is constructed, and some related results are proved. This answers a question raised by J. B. Paris in his paper [3].

1. Introduction to the problem. Let T be a complete consistent extension of the Peano axioms P, and M the minimal (i.e. pointwise definable) model of T. We suppose that L, the language of T, contains the set S_n of all n-place Skolem functions for $n \in \omega$, and identify M with S_0. Thus the notion of elementary substructure coincides with that of substructure for models of T. Our aim in this paper is to study the possible complexity of models of T. This we do by letting $\mathcal{S}(M^*)$ be the set of all substructures of M^* partially ordered by the "is a substructure of" relation, $\subseteq$. It is clear that $\mathcal{S}(M^*)$ is a lattice; $M_1 \wedge M_2$ (the infimum of M_1 and M_2 in $\mathcal{S}(M^*)$) being $M_1 \cap M_2$, and $M_1 \vee M_2$ (the supremum of M_1 and M_2 in $\mathcal{S}(M^*)$) being that substructure of M^* generated by $M_1 \cup M_2$ under all functions in $\bigcup_{n \in \omega} S_n$. Our problem can now be stated as: "which lattices occur as $\mathcal{S}(M^*)$ for some $M^* \models T$?"

A complete characterization of such lattices seems a long way off — even if we restrict our attention to finite lattices, as we do in this paper. For all known positive results on the problem we refer the reader to [3]; in particular it is proved there that every finite distributive lattice is an $\mathcal{S}(M^*)$. If M is non-standard (i.e. if T is not true arithmetic) it is still possible that every finite lattice is an $\mathcal{S}(M^*)$, whereas if M is standard there is not even an obvious conjecture. For under this latter assumption it is known (see Lemma 3.3 and [4]) that C_5 (the simplest modular non-distributive lattice — see Fig. (1)) is *not* an $\mathcal{S}(M^*)$ and, as we prove here, neither is H (which is non-modular). However, to confuse matters we also answer in the sequel a question raised in [3] by showing that for any T, P_5 (which is non-modular but somewhat less symmetrical than H) *is* of the form $\mathcal{S}(M^*)$ for some $M^* \models T$!

* The results in this paper were obtained while the author was working for his Ph. D. at Bedford College, London, and many thanks are due to W. A. Hodges for the supervision given during that period, and the Science Research Council for financial support.

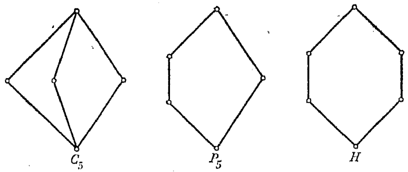

Fig. 1

2. Definable ultrapowers. In order to construct models of T with prescribed substructure lattices we introduce a method similar to Skolem's original construction of a non-standard model of arithmetic.

Let B denote the Boolean algebra of M-definable sets and U any ultrafilter over $\boldsymbol{B}$. (See e.g. [1] for definitions of these concepts.) We define an equivalence relation $\sim_U$ on S_1 by:

$$f \sim_U g \Leftrightarrow \{x \in M\colon\ M \models f(x) = g(x)\} \in U,$$

and set $f^U = \{g \in S_1\colon f \sim_U g\}$ and $M_U = \{f^U\colon f \in S_1\}$.

We turn M_U into an L-structure by defining:

$$F(f_1^U, \ldots, f_n^U) = g^U \Leftrightarrow \{x \in M\colon\ M \models F(f_1(x), \ldots, f_n(x)) = g(x)\} \in U,$$

for $F \in S_n$, $f_1, \ldots, f_n, g \in S_1$.

That F so defined is a function on M_U, and that $\sim_U$ is a congruence relation for this function is easily verified, as is the following theorem, which is a definable analogue of Łoś's theorem on ultrapowers (see [1]).

THEOREM 2.1. *If* $\Phi(x_0, \ldots, x_{n-1}) \in L$ *and* $f_0, \ldots, f_{n-1} \in S_1$, *then*

$$M_U \models \Phi(f_0^U, \ldots, f_{n-1}^U) \quad \textit{iff} \quad \{x \in M\colon\ M \models \Phi(f_0(x), \ldots, f_{n-1}(x))\} \in U.$$

Further, if for each $a \in M$ we denote by $\hat{a}$ that function in S_1 with constant value a, the map $l\colon M \to M_U$ defined by $l(a) = \hat{a}^U$ $(\forall a \in M)$, is an (elementary) embedding of M into M_U.

From now on we shall identify M with its image under l in M_U.

DEFINITION 2.2. If $M^* \models T$, $a \in M^*$, $M^*[a]$ denotes the smallest substructure of M^* containing a, or equivalently that substructure of M^* generated by a under all functions in $\bigcup_{n \in \omega} S_n$.

THEOREM 2.3. *Let* id *denote the identity function on M. Then* $\mathrm{id} \in S_1$ *and we have*:

(i) $M_U[\mathrm{id}^U] = M_U$ *for any ultrafilter U over $\boldsymbol{B}$, and*

(ii) *if $M^* \models T$, $a \in M^*$ and $M^* = M^*[a]$, then there is an ultrafilter U over $\boldsymbol{B}$ s.th. $M^* \simeq M_U$.*

Proof. (i) is obvious. For (ii) let $U = \{A \in B\colon\ M^* \models a \in A\}$. Then U is an ultrafilter and the map taking a to id^U can clearly be extended to an isomorphism of M^* onto M_U. ■

Working towards our aim of constructing models of T with prescribed substructure lattice we introduce the following notions similar to those used by Paris in [3] (p. 253).

For $f, g \in S_1$, $B \in \boldsymbol{B}$, and U an ultrafilter over B define

$$f \underline{\Delta}_B\, g \Leftrightarrow M \models (\forall x, y \in B)\big(g(x) = g(y) \to f(x) = f(y)\big),$$
$$f \equiv_B g \Leftrightarrow f \underline{\Delta}_B\, g \text{ and } g \underline{\Delta}_B f,$$
$$f \underline{\Delta}_U\, g \Leftrightarrow \exists B \in U \text{ s.th. } f \underline{\Delta}_B\, g,$$
$$f \equiv_U g \Leftrightarrow f \underline{\Delta}_U\, g \text{ and } g \underline{\Delta}_U f, \text{ and}$$
$$f \Delta_U\, g \Leftrightarrow f \underline{\Delta}_U\, g \text{ and not } f \equiv_U g.$$

The point of these definitions becomes clear with the following:

LEMMA 2.4. *Let U be an ultrafilter over $\boldsymbol{B}$, and $f, g \in S_1$. Then $M_U[f^U] \subseteq M_U[g^U]$ iff $f \underline{\Delta}_U\, g$.*

Proof. Suppose $M_U[f^U] \subseteq M_U[g^U]$. Then $\exists h \in S_1$ s.th.

$$B = \{x \in M\colon\ M \models h(g(x)) = f(x)\} \in U.$$

Clearly $f \underline{\Delta}_B\, g$, hence $f \underline{\Delta}_U\, g$.

Now suppose $f \underline{\Delta}_U\, g$. Then $\exists B \in U$ s.th. $f \underline{\Delta}_B\, g$, i.e.

(1) $$M \models (\forall x, y \in B)\big(g(x) = g(y) \to f(x) = f(y)\big).$$

Define $h \in S_1$ by

$$h(y) = \begin{cases} f(x), \text{ where } x = \mu t \in B \text{ s.th. } g(t) = y \text{ if } (\exists t \in B)(g(t) = y), \\ \text{(where } \mu t \ldots = \text{the least } t \text{ s.th. } \ldots), \\ 0, \text{ otherwise.} \end{cases}$$

Then I claim

(2) $$B \subseteq A = \{x \in M\colon\ M \models h(g(x) = f(x))\}.$$

For suppose $x \in B$ and let $x_0 = \mu t \in B\colon g(t) = g(x)$. Then $x_0, x \in B$ and $g(x_0) = g(x)$. Therefore, by (1), $f(x) = f(x_0)$. But $h(g(x)) = f(x_0)$, by the definition of h, so $h(g(x)) = f(x)$ from which (2) follows.

Now $B \subseteq A \Rightarrow A \in U$, since $B \in U$. Hence $M_U \models h(g^U) = f^U$ (from (2) and Theorem 2.1) from which it follows, since $h \in S_1$, that $M_U[f^U] \subseteq M_U[g^U]$ as required. ■

Now $\equiv_U$ is an equivalence relation on S_1, as is easily checked, and it is also easy to show that $\underline{\Delta}_U$ induces an upper-semi lattice ordering on the equivalence classes. We denote this upper-semi lattice by L_U and have the following result, analogous to Aczel's theorem in [3] (Lemma 0).

LEMMA 2.5. $\mathcal{S}(M_U) \simeq$ *The ideals of* L_U.

Proof. It follows from Lemma 2.4 that the map θ: $\mathcal{S}(M_U) \to$ The ideals of L_U, given by $\theta(M') = \{f/\equiv_U: f^U \in M'\}$, where $f/\equiv_U$ is the $\equiv_U$ — equivalence class containing $f(\in S_1)$, is the required isomorphism. ■

It is clear that Lemma 2.5 reduces our original problem to one of investigating (definable) partitions of M. Before we do this however, we require a lemma which reduces the complexity of partitions we shall have to consider later and also provides us with the negative results promised earlier.

3. The main lemma. We first require the following definitions and results.

DEFINITION 3.1. (i) If $M^* \models T$, S, $S' \subseteq M^*$, we write $S > S'$ (or $a > S'$ if $S = \{a\}$) if $M^* \models s > s'$ $\forall s \in S$, $\forall s' \in S'$.

(ii) If $M' \subseteq M^* \models T$ and $a > M'$ holds for no $a \in M^*$ we say M' *is cofinal in* M^* or M^* is a *cofinal extension* of M'.

LEMMA 3.2. *Suppose* $M_1, M_2, M^* \models T$, $M_1 \subseteq M^*$, $M_2 \subseteq M^*$ *and* $M_1 \vee M_2$ *is cofinal in* M^*. *Then either* M_1 *or* M_2 *is cofinal in* M^*.

Proof. If the lemma is false $\exists a \in M^*$ s.th. $a > M_1$ and $a > M_2$. Since $M_1 \vee M_2 \not> a$ we may suppose that $\exists a_1 \in M_1$, $a_2 \in M_2$ and $F \in S_2$ s.th. $M^* \models (F(a_1, a_2) = a \wedge a_1 \leqslant a_2)$. Let $G(x) \underset{\mathrm{df}}{=} \max_{y,z \leqslant x} F(y, z)$. Then $G \in S_1$ and $M^* \models G(a_2) \geqslant a$. But $M_2 \subseteq M^*$ so $G(a_2) \in M_2$ which contradicts $a > M_2$. ■

LEMMA 3.3 (Paris, Gaifman). *Suppose* $M^* \models T$ *and* $M^* = M^*[a']$ *for some* $a' \in M^*$. *Suppose further that there is a lattice embedding of* C_5 (see Fig. (1)) *into* $\mathcal{S}(M^*)$ *which takes the least element of* C_5 *onto* M *and the greatest element of* C_5 *onto* M^*. *Then* M^* *is a cofinal extension of* M.

Proof. Suppose the lemma is false. Then we may suppose there are $a_1 < a_2 < a_3 \in M^*[a'] - M$ s. th. $M^*[a_i] \wedge M^*[a_j] = M$ and $M^*[a_i] \vee M^*[a_j] = M^*[a'] \forall i, j$, $1 \leqslant i < j \leqslant 3$, and that $a_2 > M$ (by 3.2).

Now there must be some $F \in S_2$ s. th. $M^* \models F(a_1, a_2) = a_3$.

Define $G \in S_1$ by:

$$G(0) = 0\,,$$
$$G(x+1) = 1 + G(x) + \max\{F(y, z)\colon y \leqslant z \leqslant G(x)\}\,.$$

Working in M^* we see that G is strictly increasing and so we may define $i_0 = \mu x$: $G(x) \geqslant a_2$ and $i_1 = \mu x$: $G(x) \geqslant a_3$. Clearly $i_0 \in M^*[a_2]$ and $i_1 \in M^*[a_3]$. However, by the definition of G we have $i_1 = i_0$ or $i_1 = i_0 + 1$, but in either case $i_0 \in M^*[a_3]$. Hence $i_0 \in M^*[a_2] \wedge M^*[a_3] = M$. So $G(i_0) \in M$ which contradicts $a_2 > M$. ■

DEFINITION 3.4. If $M_1, M_2 \models T$ we write $M_1 \subseteq^m M_2$ if $M_1 \subseteq M_2$, $M_1 \neq M_2$ and $\forall M' \models T$, $M_1 \subseteq M' \subseteq M_2 \Rightarrow M' = M_1$ or $M' = M_2$.

We can now prove the main result of this section.

LEMMA 3.5. *Suppose* $M^* \models T$, $M^* = M^*[a]$ *for some* $a \in M^*$ *and that* M^* *is not a cofinal extension of* M. *Suppose further that* $\exists M_1, M_2, M_3 \subseteq M^*$ *s.th.*

(i) $M \subseteq^m M_1 \subseteq M_2 \subseteq^m M^*$ *and* $M_1 \neq M_2$,

(ii) $M_3 \vee M_1 = M^*$ *and* $M_3 \wedge M_2 = M$,

(iii) $\forall M' \subseteq M_2$, $M' \supseteq M_1$ *or* $M' = M$, *and*

(iv) $\forall M' \supseteq M_1$, $M' \subseteq M_2$ *or* $M' = M^*$.

Then $\forall M' \subseteq M^*$, $M' = M^*$ *or* $M' \subseteq M_2$ *or* $M' = M_3$.

Proof. We first show that $\forall M' \subseteq M^*$ with $M' \neq M^*$, either

(1) $$M' \subseteq M_2 \quad \text{or} \quad M' \wedge M_2 = M \quad \text{and} \quad M' \vee M_1 = M^*.$$

So suppose $M' \subseteq M^*$, $M' \neq M^*$ and $M' \not\subseteq M_2$.

Now $M' \wedge M_2 \subseteq M_2$; therefore by (iii) $M' \wedge M_2 \supseteq M_1$ or $M' \wedge M_2 = M$. But $M' \wedge M_2 \supseteq M_1 \Rightarrow M' \supseteq M_1$ and thus by (iv) $M' \subseteq M_2$ or $M' = M^*$ which is contrary to our supposition above. Hence $M' \wedge M_2 = M$.

Similarly $M' \subseteq M^*$, $M' \neq M^*$ and $M' \not\subseteq M_2 \Rightarrow M' \vee M_1 = M^*$, and (1) is thus proved.

Now let

(2) $$M' \subseteq M^*, \quad M' \neq M^* \quad \text{and} \quad M' \not\subseteq M_2\,.$$

We now claim that

(3) $$M' - M > M_2\,.$$

For suppose (3) false. Then $\exists a \in M' - M$ and $b \in M_2$ s.th. $a < b$. (We work in M^* throughout this proof unless otherwise stated).

Now by (1) and (2) $M' \wedge M_2 = M$. Therefore $M'[a] \wedge M_2 = M$ since $M'[a] \subseteq M'$. But $M'[a] \neq M$, by choice of a, so $M'[a] \not\subseteq M_2$. Hence by (1) we have both

(4) $$M'[a] \wedge M_2 = M\,,$$

and

(5) $$M'[a] \vee M_1 = M^*.$$

Now suppose

(*) $$\exists c \in M_2 - M \text{ s.th. } c < a \ (< b)\,.$$

Then $M_2 \supseteq M_2[c] \supseteq M$ and $M_2[c] \neq M$; so by (iii) $M_2[c] \supseteq M_1$. Using this and (5) we see that there must be some $f \in S_2$ s.th. $f(c, a) = b$. Define $F \in S_1$ by:

$$F(0) = 0\,,$$
$$F(i+1) = i + 1 + \max\{f(j, k)\colon j, k \leqslant F(i)\}\,.$$

Then $f \in S_1$ (by the induction schema in T) and is strictly increasing. Hence we can define i_0, i_1 as follows:

$$i_0 = \mu i\colon F(i) \geqslant b\,.$$
$$i_1 = \mu i\colon F(i) \geqslant a\,.$$

Clearly $i_0 \in M_2[b] \subseteq M_2$, and $i_1 \in M'[a]$. But since $c<a<b$ we have, by the definition of F, that either $i_0 = i_1$ or $i_0 = i_1+1$. In either case $i_0 \in M'[i_1] \subseteq M'[a]$. Therefore $i_0 \in M'[a] \wedge M_2 = M$ (by (4)). Thus we have:

$$(6) \qquad F(i_0) \in M \quad \text{and} \quad F(i_0) \geqslant b > a > c\,.$$

Now from (5) and Lemma 3.2 it follows that either $M'[a]$ or M_1 is cofinal in M^*. Let us first suppose that $M'[a]$ is. Choose $d \in M'[a]$ s.th. $d>M$. (This is possible since M^* and therefore $M'[a]$ is not a cofinal extension of M by the lemma hypotheses). Let $g \in S_1$ be s.th. $g(a) = d$. Define $g^* \in S_1$ by:

$$g^*(x) = \max\{g(y)\colon y \leqslant F(x)\}\,.$$

Then by (6): $g^*(i_0) \geqslant g(a) = d > M_1$ and since $i_0 \in M$, $g^*(i_0) \in M$ — a contradiction.

Now suppose that M_1 is cofinal in M^*. Choose $d \in M_1$ s.th. $d>M$. Now $M_2[c] \subseteq M_2$, therefore by (iii) $M_2[c] \supseteq M_1$ or $M_2[c] = M$. In the former case, choose $g \in S_1$ s.th. $g(c) = d$ and proceed to a contradiction (using (6)) as above. The latter case is impossible by the choice of c (see (∗)).

We have now shown (∗) is impossible, which means that

$$(7) \qquad a < M_2 - M\,.$$

Now choose $a_1 \in M_1 - M$ and $a_2 \in M_2 - M_1$. This is possible by (i), from which it also follows that $M_1 = M_1[a_1]$.

Hence, by (5), $\exists h \in S_2$ s.th. $h(a, a_1) = a_2$. More precisely: $M^* \models h(a, a_1) = a_2$, so by (7):

$$\forall d \in M_2 - M,\ M^* \models (\exists x<d)\big(h(x, a_1) = a_2\big)\,.$$

Therefore,

$$(8) \qquad \forall d \in M_2 - M,\ M_2 \models (\exists x<d)\big(h(x, a_1) = a_2\big)\,.$$

Let $x_0 = \mu x\colon h(x, a_1) = a_2$ (working in M^*). Then $x_0 \in M^*[a_1, a_2] \subseteq M_2$. But from (8) we see that in fact $x_0 \in M = S_0$. Define g by: $g(x) = h(x_0, x)$. Then since $x_0 \in S_0$, $g \in S_1$ and further, $M^* \models g(a_1) = a_2$, — so $a_2 \in M^*[a_1] \subseteq M_1$ — contradicting the choice of a_2.

Thus the supposition that (3) is false is absurd, so $M' - M > M_2$. We must now show that under the assumption (2), $M' = M_3$.

We certainly cannot have $M' \wedge M_3 = M$ and $M' \vee M_3 = M^*$, for this would contradict Lemma 3.3, since $M^* = M^*[a']$, M^* is not a cofinal extension of M and the sublattice $\langle\{M, M', M_1, M_3, M^*\}, \subseteq\rangle$ of $\$(M^*)$ would be isomorphic to C_5. So say $M' \wedge M_3 = M_4 \neq M$ and $M' \neq M_3$. If $M_4 = M_3$, then $M' \supseteq M_3$. Let $a \in M' - M_3$. Then $\exists f \in S_2$, $a_1 \in M_1$ and $b \in M_3$ s.th. $M^* \models f(a_1, b) = a$ (using (ii)). Hence from (3) and (i) it follows that:

$$\forall d \in M' - M,\ M^* \models (\exists x<d) f(x, b) = a\,.$$

Therefore:

$$\forall d \in M' - M,\ M' \models (\exists x<d) f(x, b) = a\,.$$

Arguing as before, this implies that $a \in M'[b] \subseteq M_3$, contradicting the choice of a.

If $M_4 \neq M_3$, then $M \subseteq M_4 \subseteq M_3$, $M \neq M_4 \neq M_3$, and we get a contradiction using (3) with $M' = M_3$.

Using a similar method we can show that both $M' \vee M_3 = M_4 \neq M^*$ and $M' \neq M_3$ is impossible.

Hence we must have $M' = M_3$ whenever M' satisfies (2) and the proof of Lemma 3.5 is complete. ■

Corollary 3.6. *If M is standard and K, K' are finite lattices each having at least two elements, there is no $M^* \models T$ s.th. $\$(M^*) \simeq R$, where R is the lattice represented by the diagram:*

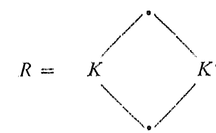

Fig. 2

In particular, there is no $M^* \models T$ s.th. $H \simeq \$(M^*)$ (see Fig. 1).

Proof. If $M^* \models T$ and $\$(M^*) \simeq R$, then clearly $M^* = M^*[a']$ for some $a' \in M^*$ (since $\$(M^*)$ is finite) and M^* is not a cofinal extension of M (since M is standard). A contradiction now follows easily from Lemma 3.5. ■

4. The pentagon lattice. We now show $\exists M^* \models T$ s.th. $\$(M^*) \simeq P_5$, where T is once again an arbitrary complete, consistent extension of P with Skolem functions.

By Lemma 2.5 it is sufficient to find an ultrafilter U over $\boldsymbol{B}$ s.th. $P_5 \simeq L_U$. This, however, we do not do directly as Lemma 3.5 allows us to construct U with apparently weaker properties (and also gives us some information about how we should go about it). To use Lemma 3.5 we must first guarantee that our resulting M_U is not a cofinal extension of M, for which we need the following result.

Lemma 4.1. *Let U be any ultrafilter over $\boldsymbol{B}$. Then M_U is a cofinal extension of M iff U contains an M-finite set.*

Proof. Suppose B is M-finite and $B \in U$. Let f^U $(f \in S_1)$ be any element of M_U. Let $a = \max\{f(x)\colon x \in B\}$ (working in M). Now $M_U \models f^U \leqslant \hat{a}^U$ by Theorem 2.1, so M_U is a cofinal extension of M.

Conversely, $\mathrm{id}^U \in M_U$ and if $M_U \models \mathrm{id}^U \leqslant \hat{a}^U$ for some $a \in M$ we have

$$B \in U \quad \text{s.th.} \quad B = \{x \in M\colon M \models \mathrm{id}(x) \leqslant \hat{a}(x)\} = \{x \in M\colon M \models x \leqslant a\}\,,$$

which is M-finite. ■

We now begin the construction of the required U. For the purposes of clarity however, we shall for the rest of this paper make two omissions (which have in fact been made to some extent already). Firstly, it will be necessary to check that all arithmetic results we use can be proved in P (so that they are true in M). This will usually be clear (though tedious to perform) and is left to the reader. Secondly, we shall be constructing many sets and functions and it will be vital to ensure that they are in $\boldsymbol{B}$ and $\bigcup_{n\in\omega} S_n$, respectively. In cases where this is not immediately clear we refer the reader to [2] where general theorems are proved justifying (in P) the definitions we shall use. We also work in M, and assume all sets mentioned are subsets of M, unless stated otherwise, from now on.

Now let $\lambda x, y: \langle x, y\rangle \in S_2$ be a fixed pairing function and π_1, π_2 be the corresponding projection functions, i.e.

$$\pi_1(\langle x, y\rangle) = x \quad \text{and} \quad \pi_2(\langle x, y\rangle) = y .$$

For $B \in \boldsymbol{B}$ and $\langle x, y\rangle, \langle x', y'\rangle \in B$ define:

$$\langle x, y\rangle \leqslant_B \langle x', y'\rangle \Leftrightarrow y \leqslant y' \wedge x \equiv x' (\operatorname{mod} 2^y),$$

and

$$\langle x, y\rangle \sim_B \langle x', y'\rangle \Leftrightarrow \langle x, y\rangle \leqslant_B \langle x', y'\rangle \wedge \langle x', y'\rangle \leqslant_B \langle x, y\rangle .$$

Then $\sim_B$ is a definable equivalence relation on B. Let

$$\langle x, y\rangle^B = \{\langle x', y'\rangle: \langle x', y'\rangle \sim_B \langle x, y\rangle\},$$

and

$$\mathscr{I}_B = \{\langle x, y\rangle^B: \langle x, y\rangle \in B\} .$$

$\leqslant_B$ induces a partial ordering on $\mathscr{I}_B$ (in fact an M-binary-tree ordering) which we shall also denote by $\leqslant_B$. Also if $B, C \in \boldsymbol{B}$ and $B \subset C$ we have $\leqslant_B = \leqslant_C \upharpoonright B$ (in both senses of $\leqslant_B$ and $\leqslant_C$).

We shall usually regard all sets in $\boldsymbol{B}$ as sets of ordered pairs. Thus we shall speak of the horizontal and vertical lines of $B \in \boldsymbol{B}$, meaning sets of the form $\pi_2^{-1}[s] \cap B$ and $\pi_1^{-1}[s] \cap B$, for some $s \in M$, respectively.

For $A \in \boldsymbol{B}$, let $\operatorname{lev} A$ = the unique y s.th. $\pi_2[A] = \{y\}$, if such a unique y exists, and let $\operatorname{lev} A$ be undefined otherwise. Note that if $\emptyset \neq A \in \mathscr{I}_B$ (for some $B \in \boldsymbol{B}$) then $\operatorname{lev} A$ is defined.

On setting $K = \{\langle x, y\rangle: y \leqslant x\}$ $(\in \boldsymbol{B})$ we can make the following crucial.

DEFINITION 4.2. A set $B \in \boldsymbol{B}$ is called *correct* iff

(i) $B \subseteq K$.

(ii) Every set in $\mathscr{I}_B$ is infinite.

(iii) $\mathscr{I}_B$ has a $\leqslant_B$—least element.

(iv) Every element of $\mathscr{I}_B$ has precisely two immediate $\leqslant_B$—successors in $\mathscr{I}_B$.

(v) If l, h are horizontal lines of B s.th. $\operatorname{lev} l \leqslant \operatorname{lev} h$, then $\pi_1[h] \subseteq \pi_1[l]$.

(vi) If $C, D \in \mathscr{I}_B$ and $\operatorname{lev} C = \operatorname{lev} D$, and if C', D' are immediate $\leqslant_B$—successors of C, D respectively, then $\operatorname{lev} C' = \operatorname{lev} D'$.

We first note that if $B \in \boldsymbol{B}$, there is a sentence of L (depending on B) which is true in M iff B is correct, and also that K is a correct set.

Now let σ be any function in S_1 which is constant on each set in $\mathscr{I}_K$ but takes different values on different numbers of $\mathscr{I}_K$, e.g.

$$\sigma(\langle x, y\rangle) = \begin{cases} \langle \operatorname{rm}(x, 2^y), y\rangle & \text{for} \quad \langle x, y\rangle \in K, \\ 0 & \text{otherwise}, \end{cases}$$

where $\operatorname{rm}(s, t)$ = the remainder when s is divided by t, will suffice.

LEMMA 4.3. *Let $f \in S_1$, and B any correct set. Then there is a correct set $C \subseteq B$, s.th. either*

(i) *f is one-one on every horizontal line of C, or*

(ii) *$f \equiv_C \sigma$, or*

(iii) *$f \equiv_C \pi_2$, or*

(iv) *$f \equiv_C \hat{0}$ (i.e. f is constant on C).*

Before we prove Lemma 4.3 let us show how it implies our main theorem, as immediate justification for these rather obscure definitions.

THEOREM 4.4. *There is an ultrafilter U over $\boldsymbol{B}$ s.th. $\$(M_U) \simeq P_5$.*

Proof. For $A \in \boldsymbol{B}$, define $f_A \in S_1$ by;

$$f_A(x) = \begin{cases} 0 & \text{if} \quad x \in A, \\ 1 & \text{if} \quad x \notin A. \end{cases}$$

Let B be any correct set and apply Lemma 4.3 with $f = f_A$ to obtain a correct $C \subseteq B$ satisfying (i) or (ii) or (iii) or (iv) of that lemma. Now f_A takes only two values, so by the correctness of C it clearly follows that C satisfies (iv). Thus we have shown that if B is any correct set and $A \in \boldsymbol{B}$, then there is a correct $C \subseteq B$ s.th. $C \subseteq A$ or $C \subseteq M - A$.

Now enumerate $S_1 \times \boldsymbol{B}$ as follows:

$$\langle f_1, B_1\rangle, \langle f_2, B_2\rangle, \ldots, \langle f_n, B_n\rangle, \ldots \quad n \in \omega, n \geqslant 1 .$$

We can now construct a sequence $A_0, A_1, \ldots, A_n, \ldots$, $n \in \omega$, of sets in $\boldsymbol{B}$ s.th.

(i) $A_0 = K$,

(ii) $(\forall i \in \omega) A_i \supseteq A_{i+1}$,

(iii) $(\forall i \in \omega) A_i$ is correct,

(iv) $(\forall i \in \omega, i \geqslant 1)$ $A_i \subseteq B_i$ or $A_i \subseteq M - B_i$,

(v) $(\forall i \in \omega, i \geqslant 1)$ either (a) f_i is one-one on every horizontal line of A_i, or (b) $f_i \equiv_{A_i} \sigma$, or (c) $f_i \equiv_{A_i} \pi_2$, or (d) $f_i \equiv_{A_i} \hat{O}$.

It is clear how the A_i are constructed using Lemma 4.3 and the above remarks. (ii) and (iii) imply that $\{A_i: i \in \omega\}$ can be extended to an ultrafilter U over $\boldsymbol{B}$ (which is uinque in view of (iv)) containing no M-finite sets. (Every correct set is M-infinite by Definition 4.2 (ii)). We claim $\mathcal{S}(M_U)\simeq P_5$. In fact we show the elementary substructures of M_U are arranged as follows:

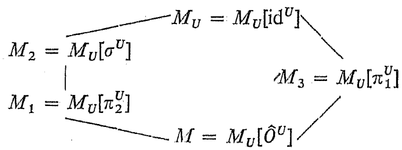

Fig. 3

Firstly, we clearly have $\hat{O}\ \underline{\Delta}_K\ \pi_2\ \underline{\Delta}_K\ \sigma\ \underline{\Delta}_K\ \mathrm{id}$; hence, since $K = A_0 \in U$, $M \subseteq M_1 \subseteq M_2 \subseteq M_U$, by Lemma 2.4. Similarly $M \subseteq M_3 \subseteq M_U$.

Now, by part (iv) of the construction, every set in U contains a correct set, so it follows from Definition 4.2 that

(1) $$M \neq M_1 \neq M_2 \neq M_U,$$

and

(2) $$M \neq M_3 \neq M_U.$$

In order to show the hypotheses of Lemma 3.5 are satisfied let us first suppose that $M' \subseteq M_U$, $M' \supseteq M_1$ and $M' \supseteq M_3$. Then $\pi_1^U \in M'$ and $\pi_2^U \in M'$. But the pairing function $\lambda x, y\colon \langle x, y\rangle \in S_2$, hence $\langle \pi_1^U, \pi_2^U\rangle \in M'$, i.e. $\mathrm{id}^U \in M' = M_U$. Thus

(3) $$M_1 \vee M_3 = M_U.$$

We now show

(4) $$M_2 \wedge M_3 = M.$$

Suppose $\tau \in S_1$ and $\tau^U \in M_2 \wedge M_3$. Then $\tau\ \underline{\Delta}_U\ \sigma$ and $\tau\ \underline{\Delta}_U\ \pi_1$ by Lemma 2.4. Hence

(*) $$\exists B \in U \quad \text{s.th.} \quad \tau\ \underline{\Delta}_B \sigma \quad \text{and} \quad \tau\ \underline{\Delta}_B\ \pi_1,$$

and we may suppose B correct. Let y_0 be the level of the $\leqslant_B$ — least element, D, of $\mathscr{I}_B$, (see Definition 4.2 (iii)). We show that $\langle x, y\rangle, \langle x', y'\rangle \in B \Rightarrow \tau(\langle x, y\rangle) = \tau(\langle x', y'\rangle)$, so that $\tau \equiv_B \hat{O}$ and thus $M_2 \wedge M_3 = M$.

So suppose $\langle x, y\rangle, \langle x', y'\rangle \in B$. Then

(**) $$\pi_1(\langle x, y\rangle) = \pi_1(\langle x, y_0\rangle) \quad \text{and} \quad \pi_1(\langle x', y'\rangle) = \pi_1(\langle x', y_0\rangle).$$

Also $\langle x, y_0\rangle, \langle x', y_0\rangle \in B$ by Definition 4.2 (v). Therefore, by the definition of D, $\langle x, y_0\rangle, \langle x', y_0\rangle \in D$, so $\langle x, y_0\rangle \sim_B \langle x', y_0\rangle$, which implies $\sigma(\langle x, y_0\rangle) = \sigma(\langle x', y_0\rangle)$, by the definition of σ. Therefore, by (*), $\tau(\langle x, y_0\rangle) = \tau(\langle x', y_0\rangle)$. But by (*) and (**), $\tau(\langle x, y\rangle) = \tau(\langle x, y_0\rangle)$ and $\tau(\langle x', y'\rangle) = \tau(\langle x', y_0\rangle)$. Hence $\tau(\langle x, y\rangle) = \tau(\langle x', y'\rangle)$, as required.

Now by the definition of U and Lemma 2.4:

(5) $$M' \subseteq M_2,\ M' \neq M_2 \Rightarrow M' = M_1 \text{ or } M' = M.$$

In particular,

(6) $$M \subseteq^m M_1.$$

Now suppose $M' \supseteq M_1$, $M' \neq M_1$. Choose $f^U \in M' - M_1$. We may suppose $\pi_2\ \underline{\Delta}_U f$; say $\pi_2\ \underline{\Delta}_B f = f_l$ with $B \in U$. Then from the definition of U; either (i) f_l is one-one on every horizontal line of A_l, or (ii) $f_l \equiv_{A_l} \sigma$. But if (i) holds we have, using $\pi_2\ \underline{\Delta}_B f_l$, that f_l is one-one on $B \cap A_l \in U$. Hence $f = f_l \equiv_U \mathrm{id}$, and $M' = M_U$. If (i) holds for no $f^U \in M' - M_1$, then $f \equiv_U \sigma$ for all such f and hence $M' = M_2$. Thus

(7) $$M' \supseteq M_1,\ M' \neq M_1 \Rightarrow M' = M_2 \text{ or } M' = M_U.$$

In particular

(8) $$M_1 \subseteq^m M_2 \subseteq^m M_U.$$

Now U contains no finite sets so, by Lemma 4.1, M_U cannot be a cofinal extension of M. This, Theorem 2.3 and (1)-(8) now imply the hypotheses of Lemma 3.5 with M_U replacing M^*. Hence, $\forall M' \subseteq M_U$, either $M' = M_U$, $M' \subseteq M_2$ or $M' = M_3$, which together with (3), (4), (6) and (8) gives $\mathcal{S}(M_U) \simeq P_5$ as required. ∎

Before we return to the proof of Lemma 4.3 I should like to mention why it was necessary to invoke Lemma 3.5 in the above theorem. It is simply this. A direct construction of the required U would require proving a stronger version of Lemma 4.3, namely with (i) replaced by the condition:

either (ia) $f \equiv_C \mathrm{id}$, or (ib) $f \equiv_C \pi_1$,

and this I could not do. However, Lemma 3.5 tells us that in constructing the U of Theorem 4.4 we only have to guarantee (i) (or (ii) or (iii) or (iv)) to ensure that (ia) or (ib) (or (ii) or (iii) or (iv)) must eventually occur.

Now the proof of Lemma 4.3.

Stage 1. We first construct a correct set $C' \subseteq B$ s.th. $\forall A \in \mathscr{I}_C$ either

(*) (a) f is constant on A, or (b) f is one-one on A.

We define, by induction, sets $l_0, l_1, \ldots, l_i, \ldots$ ($i \in M$) which will be the horizontal lines of C' in ascending order of level. Thus we will put $C' = \bigcup \{l_i\colon i \in M\}$. We simultaneously define sets $A_0^i, \ldots, A_{2^i-1}^i$ ($i \in M$), which are elements of $\mathscr{I}_B$ and are s.th. $l_i \cap A_j^i$ for $j < 2^i$ will be all the elements of $\mathscr{I}_{C'}$ having the same level as l_i. We require the following induction conditions:

A$_i$. $l_i \subseteq$ some horizontal line of B, and $\mathrm{lev}\, l_{i-1} < \mathrm{lev}\, l_i$.

B$_i$. $A_j^i \in \mathscr{I}_B \forall j < 2^i$, and $l_i \subseteq \bigcup \{A_j^i\colon j < 2^i\}$, and $l_i \cap A_j^i$ is infinite $\forall j < 2^i$, and $j \neq k \Rightarrow A_j^i \cap A_k^i = \varnothing$.

C_i. Either $i = 0$ or $\forall j<2^{i-1}$ there are precisely two numbers $j_0, j_1 < 2^i$ s.th. $\pi_1[(A^i_{j_0} \cup A^i_{j_1}) \cap l_i] \subseteq \pi_1[A^{i-1}_j \cap l_{i-1}]$.

D_i. $(\forall j<2^i)$ f is either constant on $l_i \cap A^i_j$ or one-one on $l_i \cap A^i_j$.

E_i. $\forall j<2^i$, $\exists D^i_j \in \mathscr{I}_B$ s.th. $\pi_1[A^i_j \cap l_i] \cap \pi_1[D']$ is infinite $\forall D' \in \mathscr{I}_B$ s.th. $D^i_j \leqslant_B D'$. (This condition is purely to make the inductive step possible.)

First let $B^*(y, s)$ be a formula s.th. as y runs over M, $B^*_y = \{s \in M: M \models B^*(y, s)\}$ runs over all sets in $\mathscr{I}_B$ without repetitions.

DEFINITION OF l_0. Let $l = \leqslant_B$-least element of $\mathscr{I}_B$, and $t_0 = \operatorname{lev} l$. We define the function g by:

$$g(0) = \langle x_0, t_0\rangle \text{ where } x_0 = \mu x\colon \langle x, t_0\rangle \in 1.$$

$$g(y+1) = \begin{cases} \langle x', t_0\rangle, & \text{where } x' = \mu x\colon \big(\langle x, t_0\rangle \in l \wedge x \in \pi_1[B^*_{y+1}] \wedge \\ & \wedge (\forall z \leqslant y)\big(x \neq \pi_1(g(z)) \wedge f(\langle x, t_0\rangle) \neq f(g(z))\big)\big), \\ & \text{if there is such an } x. \\ g(y), & \text{otherwise.} \end{cases}$$

If the range of g is M-infinite, let $l_0 = \operatorname{range} g$, and $A^0_0 = l$, whence $D^0_0 = l$ will satisfy E_0. Conditions A_0-D_0 are easily checked — f being one-one on $l_0 \cap \cap A^0_0 = l_0$. If the range of g is M-finite, there must be some $D \in \mathscr{I}_B$ s.th.

$$f[\{\langle x, t_0\rangle\colon x \in \pi_1[D]\}]$$

is M-finite. It is easy to define, in this case, a set $\bar{D} \in \mathscr{I}_B$ s.th. $D \leqslant_B \bar{D}$ and a set $D'' \subseteq \bar{D}$ s.th. f is constant on $D^* = \{\langle x, t_0\rangle\colon x \in \pi_1[D'']\}$, and s.th. $\forall G \in \mathscr{I}_B$, $\bar{D} \leqslant_B G \Rightarrow \pi_1[D^*] \cap \pi_1[G]$ is infinite. We now put $l_0 = D^*$, $A^0_0 = l$. Condition D_0 is satisfied since f is constant on $D^* = l_0 = l \cap A^0_0$, and E_0 is satisfied with $D^0_0 = \bar{D}$. The other conditions are trivial to check.

Induction step. Now suppose for some $i, l_0, \ldots, l_i, A^i_j$ have been defined $(\forall j<2^i)$ satisfying A_i-E_i. Let D^i_j $(\forall j<2^i)$ be the sets given by E_i. We can suppose all the D^i_j have the same level and E_i still holds. Consider the elements of $\mathscr{I}_B$ which are immediate $\leqslant_B$ — successors of the D^i_j. Each D^i_j has two such $\leqslant_B$ — successors, say G^j_0, G^j_1 and all G^j_k have the same level (i is fixed) say t_0. (This follows from the correctness of B.) For $k \leqslant 1$, $j<2^i$ let $G^{j*}_k = \{\langle x, t_0\rangle \in G^j_k\colon x \in \pi_1[A^i_j \cap 1_i]\}$.

Now each G^{j*}_k generates a correct subset, T^j_k of B in a natural way, namely: $T^j_k = \{\langle x, y\rangle \in B\colon y \geqslant t_0 \wedge x \in \pi_1[G^{j*}_k]\}$. Further, G^{j*}_k is the $\leqslant_B$ $(= \leqslant_{T^j_k})$ — least element of $\mathscr{I}_{T^j_k}$. Hence we can perform the same construction on the T^j_k as we did for B in the first part of the proof, to obtain subsets $*G^j_k$ of G^{j*}_k on which f is either one-one or constant and s.th. A_{i+1}-E_{i+1} hold when we put $A^{i+1}_0, \ldots, A^{i+1}_{2^{i+1}-1}$ equal to $G^0_0, G^0_1, G^1_0, G^1_1, \ldots, G^{2^i-1}_0, G^{2^i-1}_1$ respectively, and $l_{i+1} = \bigcup\{*G^j_k\colon k \leqslant 1, j<2^i\}$, where in C_{i+1}, $A^{i+1}_{j_0} = G^j_0$ and $A^{i+1}_{j_1} = G^j_1$ (i.e. $j_0 = 2j$, $j_1 = 2j+1$).

The induction is now complete and we leave the reader to check that our construction ensures that $C' = \bigcup\{l_i\colon i \in M\}$ is correct and satisfies $(*)$.

Stage 2. We now construct a correct set $C'' \subseteq C'$ s.th. either

$(**)$ (a) f is constant on every set in $\mathscr{I}_{C''}$, or (b) f is one-one on every set in $\mathscr{I}_{C''}$.

First note that one of the following must occur: either

(α) $(\forall A \in \mathscr{I}_{C'})(\exists x \in M)\big(x \geqslant \operatorname{lev} A$ and $(\forall y \geqslant x)\big((\exists A' \in \mathscr{I}_{C'})(\operatorname{lev} A' = y) \Rightarrow (\exists A', A'')(A' \neq A''$ and $\operatorname{lev} A' = \operatorname{lev} A'' = y$ and $A \leqslant_{C'} A', A''$ and f is constant on both A' and $A'')\big)\big)$, or

(β) $(\exists A \in \mathscr{I}_{C'})$ (There are M-infinitely many horizontal lines, l, of C' s.th. $\operatorname{lev} l \geqslant \operatorname{lev} A$ and f is one-one on all but possibly one of the elements of $\mathscr{I}_{C'}$ which are subsets of l and greater (in the $\leqslant_{C'}$ ordering) than A).

It is easy to check that in case (α) one can construct $C'' \subset C'$ to satisfy (a) of $(**)$, or to satisfy (b) in case (β).

Stage 3.

$(***)$ If C'' satisfies $(**)$ (a) I claim we can find a correct set $C \subset C''$ s.th. (ii) or (iii) or (iv) of Lemma 4.3 holds.

Upon observing that $\langle \mathscr{I}_{C''}, \leqslant_{C'}\rangle$ is an M-(full binary tree of height ω) (since C' is correct) it is clear that $(***)$ is equivalent to the following.

LEMMA 4.5. *Suppose $\mathscr{I}$ is an M-(full binary tree of height ω) and the nodes of $\mathscr{I}$ are coloured (i.e. partitioned) in any way (possibly using infinitely many colours). Then there is a subtree of $\mathscr{I}'$ of $\mathscr{I}$ s.th.*

(α) *$\mathscr{I}'$ is an M-(full binary tree of height ω), and*

(β) *any two nodes of $\mathscr{I}'$ having the same $\mathscr{I}'$-level, also have the same $\mathscr{I}$-level and, either*

(γ) *every node of $\mathscr{I}'$ has a different colour, or*

(δ) *two nodes of $\mathscr{I}'$ have the same colour iff they have the same level, or*

(ε) *every node of $\mathscr{I}'$ has the same colour.*

Proof. Denoting the order on $\mathscr{I}$ by $\leqslant$, we first suppose the following holds: $(+)$ $\forall z \in M$, $\forall x \in \mathscr{I}$, $\exists$ level, l of $\mathscr{I}$ above x, s.th. $\forall$ levels, l', above l, $l' \cap \cap \{y \in \mathscr{I}\colon y \gg x\}$ is at least z-coloured (i.e. there are z colours appearing in this set).

We define $\mathscr{I}'$ to satisfy (α), (β) and (γ) by constructing its levels $l_0, l_1, \ldots$ by induction as follows.

$$l_0 = \{\text{least element of } \mathscr{I}\}.$$

Suppose $l_0, \ldots, l_i$ have been constructed s.th.

$(1)_i$ every element of $\bigcup\{l_j\colon j \leqslant i\}$ has a different colour,

$(2)_i$ $(j \leqslant i)$ $l_j \subseteq$ some level of $\mathscr{I}$,

$(3)_i$ $\langle \bigcup\{l_j\colon j \leqslant i\}, \leqslant\rangle$ is an M-(full binary tree of height i), where we use $\leqslant$ to denote its restriction to subsets of $\mathscr{I}$.

To construct l_{i+1} take $z = 2^{i+2}$ in $(+)$ and find a level l of $\mathscr{I}$ s.th. $l \cap \{y \in \mathscr{I}\colon x \ll y\}$ is at least 2^{i+2} — coloured $\forall x \in l_i$. This is possible from $(+)$ since l_i is finite and $\mathscr{I}$ has infinitely many levels. Suppose $l_i = \{x_0, \ldots, x_{2^i-1}\}$, and let $A_j = \{y \in \mathscr{I}: y \geqslant x_j\} \cap l$ $(\forall j < 2^i)$. Then since $\bigcup \{l_j: j \leqslant i\}$ has $2^{i+1}-1$ elements, we may pick two elements, y_j^0 and y_j^1, from each A_j s.th. every element of $\bigcup \{l_j: j \leqslant i\} \cup \{y_j^k: k<2, j<2^i\}$ has a different colour. Putting $l_{i+1} = \{y_j^k: k<2, j<2^i\}$ completes the induction, and it is easy to check that $\mathscr{I}' = \{l_i: i \in M\}$ satisfies (α), (β) and (γ).

If (+) is false, then using the method of Stage 2 we can construct a subtree $\mathscr{I}''$ of $\mathscr{I}$ s.th. $\mathscr{I}''$ satisfies (α) and (β) and s.th. nodes of the same level in $\mathscr{I}''$ have the same colour. It is now a trivility to construct a subtree $\mathscr{I}'$ of $\mathscr{I}''$ satisfying (α) and (β) and either (δ) or (ε).

This completes the proof of Lemma 4.5 and hence of (***).

Stage 4. We complete the proof of Lemma 4.3 by showing that

(****) if C'' satisfies (**) (b), then there is a correct set $C \subseteq C''$ s.th. Lemma 4.3 (i) holds.

Let $l_0, l_1, \ldots, l_i, \ldots$ $i \in M$, be the horizontal lines of C'' in increasing order of level. We define $l'_0, l'_1, \ldots, l'_i, \ldots$, $i \in M$ s.th. $\forall i$:

A_i. $l'_i \subseteq l_i$ and $\pi_1[l'_i] \subseteq \pi_1[l'_{i-1}]$ (or $i = 0$).

B_i. $D \in \mathscr{I}_{C''}$, $D \subseteq l_i \Rightarrow D \cap l'_i$ is infinite.

C_i. f is one-one on l'_i.

D_i. $D \in \mathscr{I}_{C''}$, $D \subseteq l_i \Rightarrow \pi_1[D \cap l'_i] \cap \pi[D']$ is infinite $\forall D' \in \mathscr{I}_{C''}$ s.th. $D \leqslant_{C''} D'$.

Let $l'_0 = l_0$.

Suppose $l'_0, \ldots, l'_i$ have been constructed for some $i \geqslant 0$, satisfying A_j-D_j $\forall j \leqslant i$.

Let $\operatorname{lev} l_{i+1} = t_0$.

Define $G(y) \Leftrightarrow \operatorname{lev} C_y''^{*} \geqslant t_0$ (where the * operator is defined in Stage 1).

Define g as follows:

$$g(0) = \mu x\colon x \in \pi_1[l_{i+1}] \cap \pi_1[l'_i]\,,$$

$$g(y+1) = \mu x\colon \big(x \in \pi_1[l'_i] \cap \pi_1[C_z''^{*}] \text{ where } z = (y+1)\text{st element, } t, \text{ satisfying}$$
$$G(t)\big) \wedge \big((\forall p \leqslant y)\,(f(\langle x, t_0\rangle) \neq f(\langle g(p), t_0\rangle))\big)\,.$$

By the induction hypotheses A_i-D_i, $g(y)$ is always defined and range $g \subseteq \pi_1[l_{i+1}]$ since $G(z) \wedge x \in \pi_1[C_z''^{*}] \Rightarrow x \in \pi_1[l_{i+1}]$, by the correctness of C''. We now put $l'_{i+1} = \{\langle x, t_0\rangle\colon x \in \operatorname{range} g\}$ whence A_{i+1}-D_{i+1} are easily verified.

Put $C = \bigcup \{l'_i\colon i \in M\}$. That C is correct and that f is one-one on every horizontal line of C (i.e. on $l'_i \forall i$) follows from the construction. Thus (****), Lemma 4.3, and hence Theorem 4.4 are finally established.

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