A class α and locally connected continua which can be ε-mapped onto a surface

by

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Abstract. Given two compact, metric spaces X and Y, X is said to be *Y-like* if for every $\varepsilon > 0$ there is an ε-mapping f of X onto Y, where f is an ε-mapping means that $\operatorname{diam} f^{-1}(y) < \varepsilon$ for every $y \in f(X)$. Using a class α defined in an earlier paper of the author, we prove the following theorem: *Each locally connected, 2-dimensional compactum which is M-like, where M is a surface* (*i.e. a closed 2-manifild*), *is homeomorphic with M*. As a corollary we obtain: *Each compactum quasi-homeomorphic with a surface M is homeomorphic with M, where X is quasi-homeomorphic with Y means that X is Y-like and Y is X-like.*

1. Introduction. We shall consider metrizable spaces only. The AR and ANR-spaces will be assumed to be compact. A map f of a compactum X into a space Y is said to be an ε-*mapping* if $\operatorname{diam} f^{-1}(y) < \varepsilon$ for every $y \in f(X)$. Given two compact spaces X and Y, X is said to be Y-like (cf. [12]) if for every $\varepsilon > 0$ there is an ε-mapping of X onto Y. The spaces X and Y are said to be *quasi-homeomorphic* if X is Y-like and Y is X-like. A compactum X is said to be *quasi-embeddable into a space* Y if for every $\varepsilon > 0$ there is an ε-mapping of X into Y.

A compact, connected 2-manifold without boundary will be called a *surface*. Compacta which are M-like, where M is a surface, have already been investigated by Ganea in [8] and by Mardešić and Segal in [12]. It has been proved by Ganea that any 2-dimensional ANR which is M-like is homeomorphic with M. Using other methods, Mardešić and Segal proved that any locally cyclic continuum which is M-like, where M is an orientable surface, is homeomorphic with M. The main purpose of this paper is to prove the following

THEOREM. *Each locally connected* 2-*dimensional compactum which is M-like, where M is a surface, is homeomorphic with M.*

As an easy consequence we shall obtain the following

COROLLARY. *Each compactum which is quasi-homeomorphic with a surface M is homeomorphic with M.*

The following class α, which has been introduced in [16], will be very useful in the present paper.

DEFINITION 1.1. *A locally connected continuum X belongs to the class α* if and only if there is an $\varepsilon>0$ such that no simple closed curve $S\subset X$ with $\operatorname{diam} S<\varepsilon$ is a retract of X.

The structure of this paper is as follows. In Section 2 we shall consider any space Y which is semi-lc_1 (in the sense of homology) and we shall prove that any locally connected continuum X which is Y-like belongs to the class α. In the Section 3 we shall use that result to prove the theorem formulated above. Moreover, we shall prove in that section that any locally connected compactum which is Y-like, where Y is a plane ANR, is itself a plane ANR. In Section 4 we shall prove — generalizing some Borsuk's results [2] — that each locally plane space of that class α is embeddable into a surface. As a corollary of this fact and by the results of Section 3 we shall prove that each locally connected compactum which is Y-like, where Y is an ANR embeddable into a surface, is itself an ANR embeddable into a surface.

2. Locally connected compacts which are Y-like, where Y is a (homologically) semi-lc_1 space. The following lemma, in which the last statement is obtained by an easy modification of the original proof, has been shown by Fort (see [7]).

LEMMA 2.1. *Let S be a simple closed curve which is the union of four arcs L_1, L_2, L_3, L_4 having at most end-points in common and satisfying $L_1\cap L_3=\emptyset=L_2\cap L_4$. If K is a metric space such that $K=\bigcup_{i=1}^{4} A_i$, where each A_i is closed in K, $A_i\supset L_i$ for each i, and $A_1\cap A_3=\emptyset=A_2\cap A_4$, then there is a retraction r of K onto S. Moreover, the retraction r can be chosen so that $r(A_1)=L_1$.*

The following consequence of this lemma will be useful for us:

COROLLARY 2.2. *Let Y be a compactum and let X be a locally connected continuum. If $S\subset X$ is a simple closed curve which is a retract of X, then there is an $\varepsilon>0$ such that, given an arbitrary ε-mapping f of X onto Y, there exist a simple closed curve $S'\subset f(S)$ and a retraction r' of Y onto S'. Moreover, given a (non-degenerate) arc $I_1\subset S$ and a retraction r of X onto S, the simple closed curve S' and the retraction r' can be chosen so that $r'\big(f(r^{-1}(I_1))\big)$ is a proper subset of S'.*

Indeed, find arcs I_2, I_3, I_4 such that I_1, I_2, I_3, I_4 satisfy the hypothesis of Lemma 2.1. Let $A_i=r^{-1}(I_i)$, $i=1,2,3,4$ and let $\varepsilon=\min\big(\varrho(A_1,A_3),\varrho(A_2,A_4)\big)$, where $\varrho(A,B)=\min_{x\in A,\,y\in B}\varrho(x,y)$. Now, let f be any ε-mapping of X onto Y and let $A_i'=f(A_i)$, $I_i'=f(I_i)$. Then $A_1'\cap A_3'=\emptyset=A_2'\cap A_4'$. Since I_i' is a locally connected continuum, we can find an arc $J_i'\subset I_i'$ joining the points $f(a_i)$ and $f(b_i)$, where $(a_i)\cup(b_i)=\dot{I}_i$. One can improve the arcs J_i' so as to obtain the arcs L_i', $i=1,2,3,4$, such that $\bigcup_{i=1}^{4}L_i'$ is a simple closed curve S', where $L_i'\subset J_i'\subset A_i'$, $S'\subset\bigcup_{i=1}^{4}J_i'\subset f(S)$ and L_1', L_2', L_3', L_4' have at most end-points in common. Using Fort's lemma, we conclude the proof of the corollary.

In the sequel we shall use the following definition of the semi-lc_1 spaces:

DEFINITION 2.3. A compactum Y is said to be a *semi-lc_1 space* if there is a $\delta>0$ such that, given a compact set $A\subset Y$ with $\operatorname{diam} A<\delta$, we have $i_*(H_1(A))=0$, where $H_1(A)$ is the first Čech homology group of A with integer coefficients and $i\colon A\to Y$ is the inclusion map.

We shall give an elementary proof of the following natural lemma with reference to [5] for the necessary algebraic topology notions.

LEMMA 2.4. *If Y is a locally connected compactum which is a semi-lc_1 space, then $H_1(Y)$ is a finitely generated abelian group.*

Proof. Since Y contains only finitely many components, it suffices to consider the case where Y is a continuum. Since Y is a semi-lc_1 space, there is a $\delta>0$ such that

(1) *given a compact set $A\subset Y$ with $\operatorname{diam} A<\delta$, we have $i_*(H_1(A))=0$, where $i\colon A\to Y$ is the inclusion map.*

Let ϑ be a finite covering of Y such that each element of ϑ is a region (i.e., an open and connected subset of Y) and $\operatorname{diam}(U)<\frac{1}{2}\delta$ for every $U\in\vartheta$. Let P denote the nerve of ϑ. To establish the lemma it suffices to prove that:

(2) *The group $H_1(Y)$ is a direct factor of the group $H_1(P)$.*

For this purpose, choose for each element $U\in\vartheta$ a fixed point $x_U\in U$. Then find a sequence $\vartheta^1, \vartheta^2, \ldots$ of finite coverings of Y such that:

1^0 $\vartheta^1=\vartheta$.

2^0 ϑ^{n+1} is a refinement of ϑ^n for $n=1,2,\ldots$

3^0 Each element of ϑ^n is a region in Y.

4^0 For any $n>1$ and for each $U\in\vartheta$ there is a fixed $\hat{U}^n\in\vartheta^n$ such that $x_U\in\hat{U}^n\subset U$.

5^0 If $U,V\in\vartheta$ and $U\cap V\neq\emptyset$, then — for any n — there is a sequence $U_1^n,\ldots,U_k^n$ of elements of ϑ^n such that $U_1^n=\hat{U}^n$, $U_k^n=\hat{V}^n$ (as determined by 4^0), each U_i^n is contained either in U or in V and $U_i^n\cap U_{i+1}^n\neq\emptyset$ for $i=1,2,\ldots,k-1$.

6^0 If $V\in\vartheta$, V^n is any element of ϑ^n contained in V and $\hat{V}^n$ is the element of ϑ^n determined by 4^0, then there is a sequence $V_1^n,\ldots,V_l^n$ of elements of ϑ^n such that $V_1^n=V^n$, $V_l^n=\hat{V}^n$, each V_i^n is contained in V and $V_i^n\cap V_{i+1}^n\neq\emptyset$ for $i=1,2,\ldots,k-1$.

The existence of such a sequence of coverings of Y easily follows from the arcwise connectedness of any region in Y. Moreover, we can carry out the construction in such a way that if $U,V\in\vartheta$ and $n<m$ then $\hat{U}^n\supset\hat{U}^m$ and that the sequence constructed in 5^0 for ϑ^m is a refinement of the respective sequence constructed for ϑ^n. If $V^n\supset V^m$, the same can be done for the sequences constructed in 6^0.

Now, let P_n denote the nerve of ϑ^n and let π_m^n be a projection of P_n into P_m, where $n\geqslant m$. For any n, consider the chain complex

$$C(P_n)=\{C_0(P_n)\xleftarrow{\partial_1}C_1(P_n)\xleftarrow{\partial_2}C_2(P_n)\}.$$

The projection π_m^n induces the chain map of $C(P_n)$ into $C(P_m)$, which we shall also denote by π_m^n.

For any n, we shall construct a chain map φ_n of the chain complex $\{C_0(P_1) \overset{\partial_1}{\leftarrow} C_1(P_1)\}$ into the chain complex $\{C_0(P_n) \overset{\partial_1}{\leftarrow} C_1(P_n)\}$. If v is a vertex of P_1 and $\mathrm{Car}_\vartheta(v) = V$ (i.e., V is the element of ϑ corresponding to v), then $\varphi_n(v) = \hat{v}^n$, where $\mathrm{Car}_{\vartheta^n}(\hat{v}^n)$ is the element $\hat{V}^n$ of ϑ^n determined by 4^0.

Now, let σ be an oriented 1-simplex of P_1 such that $\partial_1(\sigma) = v-u$, where $\mathrm{Car}_\vartheta(u) = U$, $\mathrm{Car}_\vartheta(v) = V$. Let $U_1^n, \ldots, U_k^n$ be a sequence of elements of ϑ^n with the properties described in 5^0. Define $\varphi_n(\sigma) = \sigma_1+\ldots+\sigma_{k-1}$, where σ_i is the oriented 1-simplex of P_n such that $\mathrm{Car}_{\vartheta^n}\sigma_i = U_i^n \cap U_{i+1}^n$ and the orientation of σ_i agrees with the succession of the elements of the sequence $U_1^n, \ldots, U_k^n$. Then $\varphi_n(\partial_1\sigma) = \partial_1\varphi_n(\sigma)$ and thus we obtain the desired chain map φ_n.

Next, we shall construct a chain homotopy $D\colon C(P_n)\to C(P_n)$ between the chain maps id and $\varphi_n\pi_1^n$, where $n>1$. Let v^n be a vertex of P_n. Then $\mathrm{id}\,v^n = v^n$ and $\varphi_n\pi_1^n(v^n) = \hat{v}^n$, where $\mathrm{Car}_{\vartheta^n}v^n = V^n$, $\mathrm{Car}_{\vartheta^n}\hat{v}^n = \hat{V}^n$ and V^n, $\hat{V}^n$ are two elements of ϑ^n as in 6^0. Let $V_1^n, \ldots, V_l^n$ be a sequence of the elements of ϑ^n with the properties described in 6^0. Then we define $D(v^n) = \tau_1+\ldots+\tau_{l-1}$, where τ_i is an oriented 1-simplex of P_n such that $\mathrm{Car}_{\vartheta^n}\tau_i = V_i^n \cap V_{i+1}^n$ and the orientation of τ_i agrees with the succession of the elements of the sequence $V_1^n, \ldots, V_l^n$.

Now, let v be an oriented 1-simplex of P_n and let $\partial_1 v = v''-v'$. Let $\mathrm{Car}_{\vartheta^n}v' = V'^n$, $\mathrm{Car}_{\vartheta^n}v'' = V''^n$ and let V', V'' denote the elements of ϑ corresponding to V'^n, V''^n, respectively, under the projection $\pi_1^n\colon P_n\to P_1$. Denote by σ the oriented 1-simplex of P_1 such that $\mathrm{Car}_\vartheta\sigma = V' \cap V''$ and the orientation of σ agrees with the succession of V', V''. Let $\varphi_n(\sigma) = \sigma_1+\ldots+\sigma_{k-1}$, $D(v') = \tau_1'+\ldots+\tau_{l'}'$, $D(v'') = \tau_1''+\ldots+\tau_{l''}''$. If $\varphi_n\pi_1^n(v') = \hat{v}'$, $\varphi_n\pi_1^n(v'') = \hat{v}''$, then it is easy to see from the definitions that $\partial_1 D(v') = \hat{v}'-v'$, $\partial_1\varphi_n(\sigma) = \hat{v}''-\hat{v}'$, $\partial_1 D(v'') = \hat{v}''-v''$. It follows that

$$\zeta = \tau_1'+\ldots+\tau_{l'}'+\sigma_1+\ldots+\sigma_{k-1}-\tau_1''-\ldots-\tau_{l''}''-v$$

is an element of $C_1(P_n)$ such that $\partial_1\zeta = 0$.

The index n being fixed, we shall observe that the homology class $[\zeta]$ is equal to zero. Indeed, we can construct a compact set $A\subset V' \cup V''$ and an element $a\in H_1(A)$ such that $i_*(a)$ is an element of $H_1(Y) = \varprojlim\{H_1(P_q), (\pi_p^q)_*\}$ whose nth coordinate is equal to $[\zeta]$, where $i_*\colon H_1(A)\to H_1(Y)$ denotes the homomorphism induced by the inclusion.

To find A, one can choose a point from the carrier of each 1-simplex which is a summand of ζ and then, for any two successive summands of ζ, join the chosen points by an arc lying in the element of ϑ^n corresponding to their common vertex. The desired element $a\in H_1(A)$ is easily constructed.

Since $V' \cap V'' \neq \emptyset$, we have $\mathrm{diam}(V' \cup V'')\leqslant\mathrm{diam}\,V'+\mathrm{diam}\,V''<\delta$, and therefore, by (1), $i_*(a) = 0$, whence $\zeta\in B_1(P_n) = \mathrm{Im}\,\partial_2$. Thus, there is an element $c\in C_2(P_n)$ such that $\partial_2 c = \zeta$. We define $D(v) = c$.

Then we have

$$\partial_2 D(v)+D(\partial v_1) = \zeta+D(v'')-D(v') = \sigma_1+\ldots+\sigma_{k-1}-v = \varphi_n\pi_1^n(v)-\mathrm{id}\,v$$

and therefore D is the desired chain homotopy.

It follows that $\varphi_n\pi_1^n$ induces the identity homomorphism of $H_1(P_n)$ for any n. It is easy to see from the remarks below the definition of the sequence $\vartheta^1, \vartheta^2, \ldots$ that the chain maps $\varphi_n\colon C_1(P_1)\to C_1(P_n)$ induce the homomorphism of $H_1(P_1) = H_1(P)$ into the inverse system $\{H_1(P_n), (\pi_m^n)_*\}$. Thus, they determine the homomorphism φ_* of $H_1(P)$ into $H_1(Y) = \varprojlim\{H_1(P_n), (\pi_m^n)_*\}$. The homomorphisms π_1^n, $n = 1, 2, \ldots$ determine a homomorphism π_* of $H_1(Y)$ into $H_1(P)$ and we see that $\varphi_*\pi_* = \mathrm{id}_{H_1(Y)}$. This proves (2), and therefore the proof of the lemma is complete.

Remark 2.5. One easily sees that the assumption of Lemma 2.4 that Y is locally connected is essential. Indeed, it suffices to take $Y = \bigcup_{n=0}^{\infty} S(p_n, r_n)$, where $S(p_n, r_n)$ is the circle on the plane E^2 with centre p_n and radius r_n, where $p_n = (1/4+1/4^n, 0)$, $r_n = 1/4+1/4^n$ for $n = 1, 2, \ldots, p_0 = (1/4, 0)$, $r_0 = 1/4$.

THEOREM 2.6. *Let Y be a semi-lc$_1$ space. If X is a locally connected continuum which is Y-like, then $X\in\alpha$. More generally, if X is a locally connected compactum which is Y-like, then each component of X belongs to α.*

Proof. First, consider the case where X is connected. Suppose, proceeding to the contrary, that $X\notin\alpha$. Then there exist a sequence of simple closed curves $S_n\subset X$ with $\lim_{n\to\infty}\mathrm{diam}\,S_n = 0$ and a sequence of retractions $r_n\colon X\to S_n$, where $n = 1, 2, \ldots$ We can assume that there exists a point $x_0\in X$ such that $\operatorname{Lim}_{n\to\infty} S_n = (x_0)$.

Let $I_{n_0}\subset S_{n_0}$ be an arc containing the point $r_{n_0}(x_0)$ as an interior point. Then, for almost all n with $n>n_0$, we have $S_n\subset r_{n_0}^{-1}(I_{n_0})$. Thus, choosing a subsequence of the sequence $S_1, S_2, \ldots$ if necessary, we can assume that for each n_0 the inclusion $S_n\subset r_{n_0}^{-1}(I_{n_0})$ holds for all $n>n_0$.

Since Y is a continuous image of X (as X is Y-like), we infer that Y is locally connected, and therefore it follows from Lemma 2.4 that $H_1(Y)$ is a finitely generated abelian group. Let k_0 be equal to the rank of $H_1(Y)$ plus one.

Consider the finite sequence $S_1, \ldots, S_{k_0}$. By Corollary 2.2, there exists an $\varepsilon>0$ such that, given an ε-mapping f of X onto Y, there are simple closed curves $S_i'\subset f(S_i)$ and retractions $r_i'\colon Y\to S_i'$ such that $r_i'\big(f(r_i^{-1}(I_i))\big)$ is a proper subset of S_i' for $i = 1, 2, \ldots, k_0$. Since X is Y-like, the ε-mapping f of X onto Y exists. If $j>i$, $j\leqslant k_0$, then $S_j\subset r_i^{-1}(I_i)$, whence $S_j'\subset f(S_j)\subset f(r_i^{-1}(I_i))$, and therefore $r_i'(S_j')$ is a proper subset of S_i'.

Now, let λ_i be a generator of the group $H_1(S_i')$ and let $\mu_i = (j_i)_*(\lambda_i)$, where $(j_i)_*\colon H_1(S_i')\to H_1(Y)$ denotes the homomorphism induced by the inclusion $j_i\colon S_i'\to Y$. Let us notice that $\mu_1, \ldots, \mu_{k_0}$ are linearly independent elements of the group $H_1(Y)$. Conversely, if $n_1\mu_{i_1}+\ldots+n_l\mu_{i_l} = 0$, where $n_1, \ldots, n_l$ are non-zero integers and $i_1<i_2<\ldots<i_l$, then $(r_{i_1}')_*(n_1\mu_{i_1}+\ldots+n_l\mu_{i_l}) = n_1 \neq 0$ (because $r_{i_1}'(S_i')$ is a proper subset of S_{i_1}' for $i>i_1$), which is impossible. Thus, we obtain a contradiction, because the rank of $H_1(Y)$ is equal to k_0-1, which completes the proof in the case where X is connected.

If X is not connected, then X has a finite number of components, say $C_1, \ldots, C_p$, and — since X is Y-like — it is easy to see that Y has the same number of components, say $C'_1, \ldots, C'_p$. If f is any ε-mapping of X onto Y, then there is a one-to-one correspondence between C_i's and C'_j's such that $f(C_i) = C'_j$. Of course, there is a sequence f_n, $n = 1, 2, \ldots$, where f_n is an ε_n-mapping of X onto Y with $\lim_{n\to\infty} \varepsilon_n = 0$, for which this correspondence is the same. Thus, we can assume that for each $\varepsilon > 0$ there is an ε-mapping f of X onto Y such that $f(C_i) = C'_i$ for $i = 1, 2, \ldots, p$. Evidently, each C'_i is a semi-lc$_1$ space, and we infer from the first part of the proof that each C_i belongs to α. This completes the proof.

Remark 2.7. One could change Definition 1.1, assuming only that X is a compactum instead of assuming that it is a continuum. It is evident that X satisfies the changed definition if and only if X is a locally connected compactum each component of which belongs to α.

Remark 2.8. It is easy to construct two spaces X, Y, where $X \in \alpha$, $Y \notin \alpha$ and X is Y-like. For instance, it suffices to take X equal to the interval $\langle p, q\rangle$ on the plane E^2, where $p = (-1, 0)$, $q = (0, 0)$, and $Y = X \cup \bigcup_{i=1}^{\infty} S(p_i, r_i)$, where $S(p_i, r_i)$ is the circle on the plane E^2 with centre $p_i = (1/i, 0)$ and radius $r_i = 1/i$.

However, the answer to the following question is not known to the author:

Problem. Is there a locally connected continuum $X \notin \alpha$ which is Y-like, where $Y \in \alpha$? Is the property α a quasi-homeomorphism invariant?

3. Locally connected continua which are M-like, where M is either a plane ANR **or a surface.** The following two graphs, K_1 and K_2, are called the *graphs of Kuratowski*: K_1 is the 1-skelton of a 3-simplex in which the mid-points of a pair of non-adjacent edges are joined by a segment, K_2 is the 1-skelton of a 4-simplex. By an n-umbrella we mean the one-point union of a (topological) n-ball Q and of an arc I relative to a point $p \in \mathring{Q}$ and a point $q \in \dot{I}$.

The following two theorems will be used or generalized later:

Theorem A (see [16], p. 293). *A connected space X is homeomorphic with an* ANR-*set $Y \subset S^2$ if and only if X satisfies the following two conditions*:

1^0 $X \in \alpha$.

2^0 *X does not contain either a 2-umbrella or any homeomorphic images of the graphs K_1 and K_2.*

Theorem B (see [14], p. 313). *A compactum X is quasi-homeomorphic with S^2 if and only if it is homeomorphic with S^2.*

Now let us prove:

Theorem 3.1. *Each locally connected compactum X which is Y-like, where Y is a plane* ANR, *is itself an* ANR *embeddable into E^2.*

Proof. Evidently, Y is a semi-lc$_1$ space, and therefore we infer from Theorem 2.6 that each component of X belongs to α. Let C be any component of X. It suffices to prove that C is an ANR embeddable into E^2.

It is well known (cf. [11], p. 634) that neither any of the sets K_1, K_2 nor the 2-umbrella is quasi-embeddable into E^2, i.e., they cannot be ε-mapped into E^2 with arbitrarily small $\varepsilon > 0$. Since $Y \subset E^2$ and X is Y-like, we infer that that C satisfies also condition 2^0 of Theorem A. Thus, by Theorem A, C is an ANR embeddable into S^2. Since S^2 is not quasi-embeddable into E^2 by Borsuk's well-known antipodal point theorem, we infer that C is not homeomorphic with S^2. Thus C is an ANR embeddable into E^2.

Corollary 3.2. *Any compactum X quasi-homeomorphic with a plane* ANR *Y is itself an* ANR *embeddable into E^2.*

Indeed, since Y is X-like, X is locally connected, and therefore, by Theorem 3.1, X is an ANR embeddable into E^2.

Remark 3.3. We cannot assert in Corollary 3.2 that X and Y (assumed to be quasi-homeomorphic) are homeomorphic. This is not true even if X and Y are dendrons (i.e., 1-dimensional AR's), as shown by Segal in [17]. However, this is true for graphs (i.e., 1-dimensional, compact polyhedra), as shown also in [17]. On the other hand, a student of mine, Mr Lê Xuân Bình, has proved (cf. [10]) that all plane 2-dimensional AR's are quasi-homeomorphic and that this class contains any compactum quasi-homeomorphic with them.

Remark 3.4. It has been shown by Eilenberg (cf. [6]) that if X and Y are quasi-homeomorphic ANR's, then X homotopically dominates Y and Y homotopically dominates X. Thus, the following assertion seems to be true: If X and Y are 2-dimensional plane ANR's, then X and Y are quasi-homeomorphic if and only if X and Y have the same homotopy type.

Remark 3.5. The assumption of Corollary 3.2 that Y is an ANR is essential. Indeed, it has been proved by another student of mine, Mr Tran Trong Canh (cf. [18]), that there are two locally connected continua X and Y which are quasi-homeomorphic and such that X is embeddable into E^2 but Y is not. Namely, $X \subset E^2$ is the Sierpiński universal plane curve and Y is the one-point union of X and of the interval I with respect to a point $p \in X$ which does not belong to the closure of any component of $E^2 \setminus X$ and a point $q \in \dot{I}$.

It has been proved by Bennet (cf. [1]) that the 2-umbrella is not quasi-embeddable either in E^2 or in S^2. Using the theory of covering spaces, we shall prove the following

Theorem 3.6. *The 2-umbrella ∇ is not quasi-embeddable in any 2-dimensional manifold.*

Proof. Of course, it suffices to prove that ∇ is not quasi-embeddable in any connected 2-manifold M without boundary (compact or not). Thus, suppose the contrary and let $\nabla = D \cup L$, where $D = \{(x_1, x_2, x_3) \in E^3: x_1^2+x_2^2 \leqslant 1,\ x_3 = 0\}$, $L = \{(x_1, x_2, x_3) \in E^3: x_1 = 0,\ x_2 = 0,\ 0 \leqslant x_3 \leqslant 1\}$.

Let M' denote the universal covering space for M and let $p\colon M' \to M$ denote the covering projection. Since $\pi_1(M')$ is trivial, it follows that M' is either E^2 or S^2 (cf. [13], p. 135). Let f be any 1/4-mapping of ∇ into M. Since $\pi_1(\nabla)$ is trivial, the map f can be lifted to M', i.e., there is a map $f'\colon \nabla \to M'$ such that $pf' = f$.

Then f' is a 1/4-mapping of ∇ into M'. Indeed, if $y \in f'(\nabla)$ and $p(y) = x$, then $f^{-1}(x) = f'^{-1}(p^{-1}(x))$, whence $f'^{-1}(y) \subset f^{-1}(x)$, and therefore $\operatorname{diam} f'^{-1}(y) \leqslant \operatorname{diam} f^{-1}(x) < 1/4$. However, by Bennet's theorem [1] mentioned above, there is no 1/4-mapping of ∇ into E^2 or S^2.

Remark 3.7. It has been proved by Mardešić and Segal in [11] that the n-umbrella is not quasi-embeddable either in E^n or in S^n. Thus, it is easy to see that we can prove in the same way as above that the n-umbrella is not quasi-embeddable in any n-manifold M such that the universal covering space for M is either E^n or S^n. The following conjecture seems to be true:

CONJECTURE. The n-umbrella is not quasi-embeddable in any n-dimensional manifold.

The rest of this section is devoted to the proof of the theorem formulated in Introduction. First, we shall prove some lemmas.

A space X will be called *cyclic* if it is not separated by any point. The theory of cyclic elements given in [9], § 47, will be useful for us. We shall refer in general to [15], where the definition and some properties of cyclic elements have been listed. A subset, both open and connected, of a space X will be called a *region*. We shall say that a point $x \in X$ *locally separates* X if it separates any region in X.

LEMMA 3.8. *Let X be a space which is locally compact and locally arcwise connected and does not contains any point that locally separates X. Then for any set $A \subset X$ homeomorphic either with K_1 or with K_2 and for any point $a \in A$ there is a set $B \subset X \setminus (a)$ homeomorphic with A.*

Proof. First, find a region $U \subset X$ such that $\overline{U}$ is compact, $a \in U$, $\overline{U} \cap A$ is connected and does not contain any ramification point of A different from a and that the set $(\overline{U} \setminus U) \cap A$ contains at most four points. We shall consider only the case where it consists of four points p_1, p_2, p_3, p_4. Consequently, the set $\overline{U} \cap A$ is the union of four arcs joining these points with a and disjoint everywhere except at a. Hence, to prove the lemma, it suffices to construct four arcs L_i, $i = 1, 2, 3, 4$, disjoint everywhere except at one common point b and such that $L_i \setminus (p_i) \subset U \setminus (a)$.

Since there is no point which locally separates U, the set $U \setminus (a)$ is an arcwise connected region and the points p_i are accessible from it. Thus, one can see that there exist a point $b \in U \setminus (a)$ and three arcs I_1, I_2, I_3 disjoint everywhere except at the common point b and such that $\dot{I}_i = (p_i) \cup (b)$, $I_i \setminus (p_i) \subset U \setminus (a)$ for $i = 1, 2, 3$. Besides, there is an arc I_4 joining the point p_4 with a point $c \in \bigcup_{i=1}^{3} I_i$ and such that $I_4 \setminus (p_4) \subset U \setminus (a)$, $I_4 \cap (I_1 \cup I_2 \cup I_3) = (c)$. If $c = b$, then the proof is already finished, and so we can assume that $c \in I_3 \setminus (b)$. We set $L_1 = I_1$, $L_2 = I_2$ and we shall change the arcs I_3 and I_4 so as to obtain the arcs L_3 and L_4 as desired.

Since $U \setminus (a)$ is separable (as $\overline{U}$ is compact), locally compact, connected and locally connected, for any point $x \in U \setminus (a)$ there is a locally connected continuum which is a neighborhood of x in $U \setminus (a)$ with an arbitrarily small diameter. This follows from an analogous property of locally connected continua and from the existence of the one-point compactification of $U \setminus (a)$, which is a locally connected continuum, because of [9], p. 176, No. 1. Let I_3' denote the subarc of I_3 such that $\dot{I}_3' = (b) \cup (c)$. Thus, we can construct a sequence $C_1, C_2, \ldots$ of locally connected continua contained in $U \setminus (a) \setminus (L_1 \cup L_2)$ and such that $C_i \cap I_3' \neq \emptyset$, $\bigcup_{i=1}^{\infty} C_i$ is a neighborhood of $I_3' \setminus (b)$, and any point of $I_3' \setminus (b)$ belongs only to a finite number of the sets C_i and $\lim_{i\to\infty} \operatorname{diam}(C_i) = 0$. Consequently, $C = I_3' \cup \bigcup_{i=1}^{\infty} C_i$ is a locally connected continuum. Since the set $I_3' \setminus (b)$ is contained in $\operatorname{Int}(C)$, there is a component S of $\operatorname{Int}(C)$ containing it. We infer from the assumptions of the lemma that S is a region such that no point of S separates S. Consequently (cf. [15], p. 292, (3.9)), there is a cyclic element Z of C containing S. Since $Z = \overline{Z}$ (cf. ibidem, (3.6)), it follows that $Z \supset I_3'$.

Since $c \in \operatorname{Int}(Z)$ and $p_3 \notin Z$ (as $p_3 \notin U \supset C \supset Z$), there is a point $q_3 \in I_3 \cap Z$ such that the subarc J_3 of I_3 joining q_3 with p_3 does not intersect Z at any point different from q_3. Analogously, there is a point $q_4 \in I_4 \cap Z$ such that the subarc J_4 of I_4 joining q_4 with p_4 does not intersect Z at any point different from q_4.

Now, we shall find an arc $L \subset Z$ joining q_3 with q_4 and containing b as an interior point. It follows from [9] (p. 244, No. 16) that there is a simple closed curve $S_0 \subset Z$ containing the points b and q_3. If $q_4 \in S_0$, then S_0 contains the required arc L. If this is not the case, observe that the connectedness of $Z \setminus (q_3)$ (cf. [15], p. 292, (3.6)) implies the existence of an arc $I \subset Z \setminus (q_3)$ joining q_4 with $S_0 \setminus (q_3)$ and containing no proper subarc with this property. Then $S_0 \cup I$ contains the required arc L. It is easy to see that $J_3 \cup L \cup J_4$ is an arc joining p_3 with p_4 and containing b as an interior point. Denoting by L_3 (resp. by L_4) the subarc of this arc joining p_3 with b (resp. p_4 with b), we obtain the arcs L_3 and L_4 as required. This completes the proof of the lemma.

In the next lemma, besides the theory of cyclic elements, we shall use the theory of strongly cyclic elements, as developed for the spaces of the class α in [16]. We shall not recall the definition of strongly cyclic elements and their properties here: we only recall some notions and notations which will be useful later. A connected space X containing more than one point will be called *strongly cyclic* if X is not separated by any finite set $F \subset X$. The strongly cyclic elements of X which contain more than one point will be called *true strongly cyclic elements* and abbreviated to t.s.c.e.'s. The set of the points which locally separate a connected space X will be denoted by L_X. We shall also use the following

DEFINITION 3.9. *A locally connected continuum X belongs to the class α_0 if and only if no simple closed curve $S \subset X$ is a retract of X. A space X belongs to the class α'* (to the class α'_0) if and only if $X \in \alpha$ ($X \in \alpha_0$) and X is a cyclic space.

LEMMA 3.10. *Suppose that $X \in \alpha$ and that for every $\lambda > 0$ there is a subset A of X homeomorphic either with K_1 or with K_2 and such that* $\operatorname{diam}(A) < \lambda$. *Then, for each positive integer k_0, there is a sequence $B_1, \ldots, B_{k_0}$ of disjoint subsets of X each of which is homeomorphic either with K_1 or with K_2.*

Proof. It follows from the assumptions that there are a sequence $A_1, A_2, \ldots$ of subsets of X and a point $a \in X$ such that $\operatorname{diam}(A_n) < 1/n$, $\operatorname{Lim}_{n\to\infty} A_n = (a)$ and each A_n is homeomorphic either with K_1 or with K_2.

First, we shall consider the case where the space X is strongly cyclic, i.e., the only t.s.c.e. of X is equal to X. To use Lemma 3.8, we shall find a locally compact and locally arcwise connected set $B \subset X$ which is not locally separated by any point and contains infinitely many of the sets A_n. By [16] (p. 281, (4.3)), the set L_X is finite, and therefore, if $a \notin L_X$, there is a region U in X such that $a \in U$ and $L_X \cap U = \emptyset$. Thus, setting $B = U$, we obtain the required set B. Now, consider the case where $a \in L_X$. It follows from [16] (p. 276, (3.1)) that there is a region $U \subset X$ containing a whose diameter is so small that $U \cap L_X = (a)$, a separates U and the union of (a) and any component of $U \setminus (a)$ is not locally separated by a, and therefore by any other point, either. Of course, the number of the components of $U \setminus (a)$ is finite, because a does not separate X. Since almost all sets A_n are contained in U, it is easy to see that there is a component V of $U \setminus (a)$ such that the set $B = V \cup (a)$ contains infinitely many of the sets A_n. Thus, B is the required set, because it satisfies other requirements, too.

Now, to finish the proof of the lemma in the case under consideration, we shall prove inductively that for each positive integer k there is a sequence $B_1, \ldots, B_k$ of disjoint subsets of $B \setminus (a)$ such that each B_i ($i \leqslant k$) is homeomorphic with some A_n's. Applying Lemma 3.8, we see at once that there is a set $B_1 \subset B \setminus (a)$ homeomorphic to A_{n_1}, where n_1 is the first index n such that $A_n \subset B$. Assume inductively that the sets $B_1, \ldots, B_{k-1}$ have been constructed. Then, there is a region W in B containing a and such that $W \cap \bigcup_{i=1}^{k-1} B_i = \emptyset$. We infer that W satisfies the assumptions of Lemma 3.8 and contains A_{n_l} for a sufficiently great index n_l. Thus, applying Lemma 3.8, we obtain a set $B_k \subset W \setminus (a)$ homeomorphic with A_{n_l}. This completes the induction.

Now, we shall consider the more general case where the space X is cyclic (but not strongly cyclic). Since each t.s.c.e. E of X satisfies the same assumptions as X does and since E is a strongly cyclic space (cf. [16], (4.4) and (4.11)), we can assume that no t.s.c.e. of X contains infinitely many sets A_n. Thus, because $L_X = \overline{L}_X$ and because the t.s.c.e.'s of X coincide with the closures of the components of $X \setminus L_X$ (cf. ibidem (3.4) and (4.2)), we can assume that $a \in L_X$. It follows from [16] ((4.2), (4.6) and (4.9)) that there is a (closed) neighborhood U of a in X which is the union of a finite number of arcs $I_1, \ldots, I_p$, disjoint everywhere except at the common point a, and of all the t.s.c.e.'s of X which intersect these arcs. Moreover, we infer from [16], (4.9) that the neighborhood U can be constructed so that if E is a t.s.c.e. of X intersecting U, then $E \cap \bigcup_{i=1}^{p} I_i$ is a (non-degenerate) subarc J of one of the arcs I_l such that $\mathring{J} \subset \operatorname{Int} E$. Assume that there is at least one t.s.c.e. E of X containing a. (If there is no such t.s.c.e., then the proof will be easier.) Thus, one can find a $q \leqslant p$ and order the arcs $I_1, \ldots, I_p$ in such a way that the arcs I_j with $j \leqslant q$ are all those for which there is a t.s.c.e. E_j of X intersecting I_j on a (non-degenerate) subarc containing a. By [16] ((4.3) and (4.10)), if E is any t.s.c.e. of X, then $\operatorname{Bd} E = E \cap \overline{X \setminus E}$ is a finite subset of E non-separating E, which consists of exactly two points when the diameter of E is sufficiently small. Consequently, one can reduce U to obtain a connected neighborhood V of a equal to $(a) \cup \bigcup_{i=1}^{p} C_i$, where $C_i = \operatorname{Int} E_i$ for $i \leqslant q$ and C_i is the union of a subarc I'_i of I_i containing a and of all the t.s.c.e.'s $E_{i1}, E_{i2}, \ldots$ of X such that $E_{im} \cap I'_i \neq \emptyset$. Moreover, we can assume that the boundary of E_{im} consists of exactly two points of I'_i. Since the interiors of different t.s.c.e.'s of X are disjoint (cf. ibidem, (4.2)), we infer that the sets $C_i \setminus (a)$ with $i \leqslant p$ are the components of $V \setminus (a)$.

Since $\operatorname{Lim}_{n\to\infty} A_n = (a)$ and $\operatorname{diam} A_n < 1/n$, it follows that almost all sets A_n are contained in V. Since no A_n is separated by a point, we infer that each of them is contained in the closure of one component of $V \setminus (a)$. Since we have assumed that no t.s.c.e. of X contains infinitely many sets A_n, we conclude that there is an i with $q < i \leqslant p$ such that C_i contains infinitely many A_n's. As no A_n is separated by a point and the sets $E_{i1}, E_{i2}, \ldots$ are the non-degenerate cyclic elements of C_i, we infer that for each n such that $A_n \subset C_i$ there is an index $j(n)$ such that $A_n \subset E_{ij(n)}$. We can assume that $n \neq n'$ implies $j(n) \neq j(n')$. Moreover, one can see from the construction of C_i that — choosing a subsequence of the sequence of those A_n's which are contained in C_i if necessary — we can assume that $n \neq n'$ and $A_n, A_n \subset C_i$ imply $E_{ij(n)} \cap E_{ij(n')} = \emptyset$. This completes the proof of the lemma in the case where X is a cyclic space.

Finally, we shall consider the general case. As before, one sees that each A_n is contained in a non-degenerate cyclic element Z_n of X. Since each Z_n, being a retract of X (cf. [15], p. 292, (3.6) and (3.4)), satisfies the same assumptions as X and since Z_n is a cyclic space, we can assume that $n \neq n'$ implies $Z_n \neq Z_{n'}$. Moreover, we can assume that no subsequence of the sequence $Z_1, Z_2, \ldots$ consists of disjoint sets, because the proof of the lemma in that case is immediate. Then, it follows from [15] ((3.6), (3.2)) and [9], (p. 238, Remarque) that the proof reduces to the case where there is a point $p \in X$ such that $Z_i \cap Z_j = (p)$ for $i \neq j$. Since $X \in \alpha$, there is an $\varepsilon > 0$ such that no simple closed curve $S \subset X$ with $\operatorname{diam}(S) < \varepsilon$ is a retract of X. By [15], (3.8), there is only a finite number of Z_n's with $\operatorname{diam} Z_n \geqslant \varepsilon$, and since each Z_n is a retract of X, we infer that almost all of them belong to the class α'_0

(cf. Definition 3.9). It follows from [16] (p. 283, (4.8)) that there is an index N such that for $n \geqslant N$ no point of Z_n locally separates Z_n. Thus, Lemma 3.8 can be applied to Z_n (for $n \geqslant N$) and we can find a set $B_n \subset Z_n \setminus (p)$ homeomorphic to A_n. This completes the proof of the lemma.

The next lemma concerns the subsets of E^2.

LEMMA 3.11. *Let $F \subset E^2$ be a cyclic, locally connected continuum and let $x_0 \in F \cap \overline{E^2 \setminus F} = \mathrm{Bd}\,F$. Then, for any neighborhood U of x_0 (in E^2), there is a disk $Q \subset U$ such that $x_0 \in \mathring{Q}$ and that $F \cap \dot{Q}$ is the union of a finite number of disjoint arcs, some of which can degenerate to a point, $F \cap Q$ being a locally connected continuum.*

Proof. (cf. [14], p. 308, Lemma 1). Let $\varepsilon > 0$ be so small that the ε-neighborhood of x_0 in E^2 is contained in U. By [9] (p. 363), there is only a finite number of components of $E^2 \setminus F$, say $C_1, \ldots, C_l$, such that $\operatorname{diam} C_i \geqslant \varepsilon/3$ and $x_0 \notin \overline{C}_i$.

First, consider the case where there is no component C of $E^2 \setminus F$ such that $x_0 \in \overline{C}$. Choose a $\delta > 0$ such that $\delta < \varepsilon/3$ and that the δ-neighborhood of x_0 in E^2 does not intersect the set $\bigcup_{i=1}^{l} \overline{C}_i$. Since $x_0 \in \mathrm{Bd}\,F$, there is a component $\hat{C}$ of $E^2 \setminus F$ lying in the δ-neighborhood of x_0 in E^2. Since F is a cyclic, locally connected continuum, $\overline{\hat{C}}$ is a disk (cf. [9], p. 360) and, by the assumption of the case considered now, $x_0 \notin \overline{\hat{C}}$. Thus, it is easy to see that there is a disk $Q_0 \subset E^2$ such that $x_0 \in \mathring{Q}_0$, $\operatorname{diam} Q_0 < \delta$ and that $\dot{Q}_0 \cap \overline{\hat{C}}$ is a non-degenerate arc I with $\mathring{I} \subset \hat{C}$. Let $J = \dot{Q}_0 \setminus \mathring{I}$. Denote by A the union of Q_0 and the closures of all components C of $E^2 \setminus F$ such that $C \cap \mathring{J} \neq \emptyset$. Since, for every such component C, $\overline{C}$ is a disk and $\overline{C} \cap J$ contains more than one point and because the diameters of these components converge to zero provided their number is infinite, we infer that A is a cyclic, locally connected continuum. Moreover, $\operatorname{diam} Q_0 < \delta$ implies that $\operatorname{diam} A < \varepsilon$. Let Q denote the union of A and of all bounded components of $E^2 \setminus A$. Then Q is a cyclic, locally connected continuum which does not separate E^2, and therefore it is a disk (cf. [9], p. 380). Evidently $x_0 \in \mathring{Q}$ and $\operatorname{diam} Q = \operatorname{diam} A < \varepsilon$, whence $Q \subset U$. One can see from the construction that $\dot{Q} \supset I$ and $\dot{Q} \cap F = \dot{Q} \setminus \mathring{I}$ is an arc, $Q \cap F$ being a perforated disk, and therefore a locally connected continuum, which completes the proof of the lemma in the case in question.

Now, consider the case where there is a component C of $E^2 \setminus F$ such that $x_0 \in \overline{C}$. One can find a disk $Q_0 \subset E^2$ such that $x_0 \in \mathring{Q}_0$, $\operatorname{diam} Q_0 < \varepsilon/3$, $Q_0 \cap \bigcup_{i=1}^{l} \overline{C}_i = \emptyset$ and that $\dot{Q}_0$ intersects at least one component C of $E^2 \setminus F$ such that $x_0 \in \overline{C}$. Thus, there is a finite number, say $C_1', \ldots, C_k'$, of components of $E^2 \setminus F$ such that $\dot{Q}_0 \cap C_i' \neq \emptyset$ and $x_0 \in \overline{C_i'}$. Since each set $\overline{C_i'}$ is a disk, we can assume that Q_0 is so small that any two points belonging to $\dot{Q}_0 \cap \mathrm{Bd}\,C_i'$ can be connected by an arc whose interior is contained in C_i' and which lies in the $\varepsilon/3$-neighborhood of x_0. Thus, by improving the sets $\dot{Q}_0 \cap \overline{C_i'}$ for $1 \leqslant i \leqslant k$, one can construct a disk $Q_1 \subset E^2$ satisfying analogous conditions as Q_0 does and such that for each i, $1 \leqslant i \leqslant k$, the set $I_i' = \dot{Q}_1 \cap \overline{C_i'}$ is an arc whose interior is contained in C_i'. Using the same method as in the preceding case, one obtains the required disk Q by improving each component of $\dot{Q}_1 \setminus \bigcup_{i=1}^{k} \mathring{I}_i'$ which is a non-degenerate arc to get an arc lying entirely in F. This completes the proof.

LEMMA 3.12. *Assume that $X \in \alpha$, X does not contain any 2-umbrella and that there is a $\lambda > 0$ such that X does not contain any homeomorphic images of the graphs K_1 and K_2 with diameter less than λ. Then, for each point $x_0 \in X$ there is a neighborhood of x_0 in X which is a (compact)* AR *embeddable into E^2.*

Proof. Let $x_0 \in X$. It follows from the assumptions that there is a locally connected continuum $F \subset X$ such that $x_0 \in \mathrm{Int}\,F$, F does not contain any homeomorphic images of the graphs K_1 and K_2 and that no simple closed curve $S \subset F$ is a retract of X. It has been proved by Claytor (cf. [4], p. 632) that each cyclic, locally connected continuum which does not contain homeomorphic images of the graphs K_1 and K_2 is embeddable into S^2. Consequently:

(1) *Each cyclic element of F is embeddable into S^2.*

Evidently, we cannot assert that $F \in \alpha_0$. However, in the next part of the proof we shall find a smaller neighborhood H' of x_0 which is a retract of X and therefore belongs to α_0. We shall assume that the sequence of the components of $F \setminus (x_0)$ is infinite, because the proof in the opposite case is similar but easier. Denote these components by $H_1, H_2, \ldots$ We shall construct a subset H_i' of $\overline{H}_i$. If $H_i \subset \mathrm{Int}\,F$ (which holds for almost all i, because F is locally connected), then $H_i' = \overline{H}_i$. If $H_i \cap \overline{X \setminus F} \neq \emptyset$, then we shall distinguish two cases: where $\mathrm{ord}_{x_0} \overline{H}_i = 1$ and where $\mathrm{ord}_{x_0} \overline{H}_i > 1$. In the first case there is a neighborhood U_i, both open and connected, of x_0 in $\overline{H}_i$ such that $\overline{U}_i \subset \mathrm{Int}(F)$ and that $\overline{U}_i \setminus U_i$ consists of exactly one point a_i. We define $H_i' = \overline{U}_i$. Evidently, in this case $H_i' \setminus ((x_0) \cup (a_i))$ is an open subset of X.

Now, consider the case where $\mathrm{ord}_{x_0} \overline{H}_i > 1$. Then there are two arcs $I_1, I_2 \subset \overline{H}_i$ such that $I_1 \cap I_2 = (x_0) = \dot{I}_1 \cap \dot{I}_2$. Since H_i is a component of $F \setminus (x_0)$, we can join the sets $I_1 \setminus (x_0)$ and $I_2 \setminus (x_0)$ by an arc lying in H_i, which implies the existence of a simple closed curve $S \subset \overline{H}_i$ such that $x_0 \in S$. Consequently (cf. [15], p. 292, (3.9)), there is a cyclic element Z of $\overline{H}_i$ such that $Z \supset S$, and therefore $x_0 \in Z$. Evidently, Z is also a cyclic element of F, and therefore, by (1), Z is embeddable into S^2. If there is a disk $Q \subset Z$ such that $x_0 \in \mathring{Q}$, then Q must be a neighborhood of x_0 in X, because X is locally arcwise connected and does not contain a 2-umbrella. Since Q is an AR embeddable into E^2, in this case the lemma is proved. So we can assume that there is no disk $Q \subset Z$ such that $x_0 \in \mathring{Q}$. Thus, Lemma 3.11 can be applied to the set Z and to the point $x_0 \in Z$. Consequently, there are a locally connected continuum $A \subset Z \cap \mathrm{Int}\,F$ and a finite number of (perhaps degenerated) arcs, say $J_1, \ldots, J_k$, contained in A and such that $A \setminus \bigcup_{i=1}^{k} J_i$ is an open neighborhood of x_0 in Z. From the theory of the cyclic elements it is known that any component of $\overline{H}_i \setminus Z$ is bounded by one point (different from x_0, because H_i is a component of $F \setminus (x_0)$) and that the diameters of the components of $\overline{H}_i \setminus Z$ converge to zero, provided their number is infinite (cf. [15], (3.6), (3.2) and (3.3)). We define H_i' to be the union of A and the components C of $\bar{H}_i \setminus Z$ such that $C \subset \operatorname{Int} F$ and that $\bar{C} \setminus C \subset A \setminus \bigcup_{i=1}^{k} J_i$.

Let $H' = \bigcup_{i=1}^{\infty} H_i'$. Then H' is a locally connected continuum, $H' \subset \operatorname{Int} F$ and there is a finite number of (perhaps degenerated) arcs, say $K_1, \ldots, K_l$, contained in H' and such that $H' \setminus \bigcup_{i=1}^{l} K_i$ is an open neighborhood of x_0 in X. One can change the ordering of these arcs in such a way that there is a sequence $j_1, \ldots, j_p$ of indices with $1 = j_1 < j_2 < \ldots < j_p = l+1$ and such that $j_i \leqslant j, j' < j_{i+1}$ implies that K_j and $K_{j'}$ are contained in the same component of $(X \setminus H') \cup \bigcup_{i=1}^{l} K_i$, but $j < j_i$ and $j' \geqslant j_i$ imply that K_j and $K_{j'}$ lie in different components of $(X \setminus H') \cup \bigcup_{i=1}^{l} K_i$.

If $j_i \leqslant j, j' < j_{i+1}$ and C is a component of $X \setminus H'$ such that $\bar{C} \cap K_j \neq \emptyset \neq \bar{C} \cap K_{j'}$, then there is an arc $L \subset \bar{C}$ joining K_j with $K_{j'}'$ and such that $\mathring{L} \subset C$ (cf. [9], p. 194). Thus, for each $i = 1, 2, \ldots, p-1$, there is a sequence of arcs $L_{i1}, \ldots, L_{in(i)}$ such that $\mathring{L}_{im} \subset X \setminus H'$, $\dot{L}_{im} \subset \bigcup \{K_j : j_i \leqslant j < j_{i+1}\}$ and that the set

$$T_i = \bigcup \{L_{im} : 1 \leqslant m \leqslant n(i)\} \cup \bigcup \{K_j : j_i \leqslant j < j_{i+1}\}$$

is connected. Moreover, one can change the arcs L_{im} in such a way that the set T_i is a tree (i.e., a graph which is a dendron).

Now, consider the set $M' = H' \cup \bigcup_{i=1}^{p-1} T_i$. We shall prove that

(2) *M' is a retract of X.*

Indeed, let C be a component of $X \setminus H'$. Then, there is at most one index i with $1 \leqslant i \leqslant p-1$ such that $\bar{C} \cap T_i \neq \emptyset$. Since $T_i \in \mathrm{AR}$, there is a retraction r_i of the union of T_i and all components C of $X \setminus H'$ whose closures intersect T_i onto T_i. If C is an other component of $X \setminus H'$ then, by the construction of H', $C \not\subset \operatorname{Int} F$ and $\bar{C} \setminus C$ consists of one point belonging to H'. Thus, we can retract $\bar{C}$ onto $\bar{C} \setminus C$. It is easy to see from the construction that all these retractions together with the identity on M' determine a retraction r of X onto M', which proves (2).

Next, notice that H' is a retract of M'. Indeed, H' — being a locally connected continuum — is arcwise connected, and therefore it is easy to construct a map of T_i into H' which is the identity on $H' \cap T_i$. These maps determine a retraction of M' onto H'. It follows from (2) that H' is a retract of X. Thus, we have obtained the required neighborhood H' of x_0 such that $H' \subset F$ and H' is a retract of X, as mentioned at the beginning of the proof. We infer from the properties of the set F that $H' \in \alpha_0$, H' does not contain either a 2-umbrella or any homeomorphic images of the graphs K_1 and K_2. Moreover, we see from the construction of H' that H' is not homeomorphic with S^2. We conclude from [16], (p. 293, Corollary) that H' is an AR-set embeddable into E^2, which completes the proof of the lemma.

THEOREM 3.13. *Each locally connected 2-dimensional compactum X which is M-like, where M is a surface, is homeomorphic with M.*

Proof. Since M is connected and since for every $\varepsilon > 0$ there is an ε-mapping of X onto M, it follows that X is connected. Since M is a semi-lc$_1$ space, Theorem 2.6 implies that $X \in \alpha$. By Theorem 3.6, X does not contain a 2-umbrella.

Suppose that for every $\lambda > 0$ there is a set $A \subset X$ homeomorphic either with K_1 or with K_2 and such that $\operatorname{diam} A < \lambda$. It has been proved by Borsuk (cf. [2], p. 75) that the surface M cannot contain the set which is the union of k_0 disjoint subsets, each of them homeomorphic with K_1, where $k_0 = \gamma(M)+1$ and $\gamma(M)$ denotes the genus of M. An easy modification of Borsuk's proof permits us to speak about the sets whose each component is homeomorphic either with K_1 or with K_2, instead of the sets with all components homeomorphic to K_1. It follows from the supposition and from Lemma 3.10 that the space X contains k_0 disjoint subsets, say $B_1, \ldots, B_{k_0}$, each of them homeomorphic either with K_1 or with K_2. Since X is M-like, there is an ε-mapping f of X onto M with $\varepsilon > 0$ which is so small that $f(B_i) \cap f(B_j) = \emptyset$ for $i \neq j$ and $i, j \leqslant k_0$. Moreover, it is not difficult to see that if $\varepsilon > 0$ is sufficiently small then each set $f(B_i)$ contains a homeomorphic image either of K_1 or of K_2. Indeed, the arcwise connectedness of $f(B_i)$ implies that $f(B_i)$ contains a graph which is also an ε-image of B_i. This graph cannot be embeddable into E^2 if ε is sufficiently small, because K_1 and K_2 are not quasi-embeddable into E^2 (cf. for instance [11]). Consequently, by the classical result of Kuratowski, this graph, and therefore also $f(B_i)$, contains a graph homeomorphic either with K_1 or with K_2. Thus we obtain a contradiction of Borsuk's result mentioned above, which proves that there is a $\lambda > 0$ such that X does not contain any homeomorphic images of the graphs K_1 and K_2 with diameter less than λ.

Thus we have proved that X satisfies all the assumptions of Lemma 3.12. Consequently, for each point $x_0 \in X$ there is an AR-set which is a neighborhood of x_0 in X. We infer from Hanner's theorem (see for instance [3], p. 97) that $X \in \mathrm{ANR}$. Finally, we conclude from Ganea's theorem [8] mentioned in Section 1 that X is homeomorphic with M, which completes the proof of the theorem.

COROLLARY 3.14. *Each compactum X quasi-homeomorphic with a surface M is homeomorphic with M.*

Proof. Since M is X-like, it follows that X is a locally connected continuum. Since $\dim M = 2$ and since the dimension is a quasi-homeomorphism invariant (cf. [9], p. 64), we infer that $\dim X = 2$. Since X is M-like, we conclude from Theorem 3.13 that X is homeomorphic with M.

4. Embeddability of the locally plane spaces of class α into surfaces. Let M be any surface. The following properties of M have been proved in [9] for the case where $M = S^2$, but it is almost evident that they also hold for the case where M is an arbitrary surface: Let $X\subset M$ be a locally connected continuum and let C be a component of $M\setminus X$. Then both $\bar{C}$ and $\mathrm{Bd}(C)$ are locally connected continua (cf. [9], p. 360). Each point $p\in \mathrm{Bd}(C)$ is accessible from C by an arc and, moreover, by a disk (cf. [9], p. 365). If the sequence $C_1, C_2, \ldots$ of the components of $M\setminus X$ is infinite, then $\lim_{i\to\infty} \operatorname{diam} C_i = 0$ (cf. [9], p. 363). Moreover, if X is an ANR-set, then $M\setminus X$ has a finite number of components and if X is an AR-set, then $M\setminus X$ is connected (cf. [3], p. 132). A point x_0 belonging to a space X will be called *Euclidean point* of X if there is a disk $Q\subset X$ which is a neighborhood of x_0 in X and is such that $x_0\in \mathring{Q}$.

First, we shall prove the following

LEMMA 4.1. *Let M be a surface and let $X\subset M$ be a locally connected continuum. Suppose that $F=\bigcup_{i=1}^{k}(y_i)$ is a finite subset of X such that each point $y\in F$ belongs to the closure of a component C of $M\setminus X$ and let $<$ be any ordering of F. Then, there are another surface N, an embedding h of X into N, a disk $Q\subset N$ and an orientation of the simple closed curve $\dot{Q}$ such that $Q\cap h(X)=h(F)\subset \dot{Q}$ and that $y_p<y_q<y_r$ implies that $h(y_q)\in \mathring{L}$, where L is the arc from $h(y_p)$ to $h(y_r)$ lying on $\dot{Q}$ and coherent with its orientation.*

Proof. We shall proceed by induction with respect to k. If $k=1$, then the assertion follows easily from the fact that the point $y_1\in F$ is accessible by a disk from the component C of $M\setminus X$ such that $\bar{C}\supset(y_1)$.

Now, given a $k>1$, assume that the assertion is true for $k-1$. Thus, not to complicate the notation, we can assume that there are a disk $Q\subset M$ and an orientation of $\dot{Q}$ such that $Q\cap X=\bigcup_{i=1}^{k-1}(y_i)\subset \dot{Q}$ and that for $p,q,r\leqslant k-1$ the relation $y_p<y_q<y_r$ agrees with the ordering of these points on $\dot{Q}$, as formulated before. Let C denote the component of $M\setminus X$ such that $\bar{C}\supset Q$ and let C_k denote the component of $M\setminus X$ such that $\bar{C}\supset Q$ and let C_k denote the component of $M\setminus X$ such that $y_k\in \bar{C}_k$. We can assume that $C=C_k$. Indeed, if this is not the case, then one improves the situation by removing the interiors of some disks lying in $C\setminus Q$ and C_k and by identifying their boundaries by means of a homeomorphism. Let I denote the arc lying on $\dot{Q}$ whose end-points are y_1 and y_{k-1} (or y_1 and another point of $\dot{Q}$ if $k-1=1$) and such that $\mathring{I}\cap F=\emptyset$. Using the same procedure as before if necessary, we can assume that there is a component P of $C\setminus Q$ such that $\bar{P}\supset(y_k)\cup I$. Now, one can find an arc $J\subset\bar{P}$ joining y_k with $\mathring{I}$ and such that $\mathring{J}\subset P$. By using the arc J one expands the disk Q so as to construct a disk $Q'\subset M$ containing F in the way as required. This completes the proof of the lemma.

Now, we shall prove the main result of this section, as mentioned in Introduction. In the proof, as in Section 3, we shall use both the theories of the cyclic elements and of the strongly cyclic elements. Moreover, we shall make use of the following definition (cf. [9], § 47 and [15], Section 3):

DEFINITION 4.2. A set $A\subset X$ is said to be *entirely arcwise connected* (in X) if $x,y\in A$ and $x\neq y$ imply that each arc (in X) joining x and y is contained in A.

THEOREM 4.3. *Each locally plane space $X\in\alpha$ is embeddable into a surface. Moreover, any space $X\in\alpha$ containing no 2-umbrella and such that each point $x_0\in X$ has a neighborhood containing no homeomorphic images of the graphs K_1 and K_2 is embeddable into a surface.*

Proof. It follows from Lemma 3.12 that for each point $x_0\in X$ there is a neighborhood of x_0 in X which is an AR-set embeddable into E^2. It follows from Hanner's theorem (cf. [3], p. 97) that $X\in$ ANR. We can assume that X does not contain a simple surface, because otherwise the assumption that X does not contain a 2-umbrella and the arcwise connectedness of X would imply that X itself is a simple surface (i.e., that X is homeomorphic with S^2).

First, we shall prove that:

(1) Assume additionally that X is a strongly cyclic space and let F be any finite subset of X such that no point of F is a Euclidean point of X. Then there are an embedding h of X into a surface M and a disk $Q\subset M$ such that $Q\cap h(X)=h(F)\subset\dot{Q}$. Moreover, given an ordering $<$ of the set F, we can choose the manifold M, the disk Q and an orientation of the simple closed curve $\dot{Q}$ so that for all $x,y,z\in F$ the relation $x<y<z$ implies that $h(y)\in\mathring{L}$, where L is the arc from $h(x)$ to $h(z)$ lying on $\dot{Q}$ and coherent with its orientation.

Indeed, let $x_0\in X$ and let A be an AR-set embeddable into E^2 which is a neighborhood of x_0 in X. Let B denote the union of the non-degenerate cyclic elements of A containing x_0. The number of those cyclic elements must be finite, because otherwise almost all of them have arbitrarily small diameters and therefore are contained in $\operatorname{Int} A$, which contradicts the fact that x_0 does not separate X. Each of those cyclic elements, being a cyclic AR-set embeddable into E^2, is a disk (cf. [9], p. 380, No. 11). Since the boundary of each component of $A\setminus B$ consists of one point and since almost all those components have arbitrarily small diameters (cf. [15], (3.2) and (3.3)), it follows that the union of B and the components C of $A\setminus B$ such that $\bar{C}\setminus C\subset\operatorname{Int}A$ is a neighborhood of x_0 in X. For such a component C the point belonging to $\bar{C}\setminus C$ locally separates X, i.e., belongs to L_X. Since X has been assumed in (1) to be a strongly cyclic space, we infer from [16] (p. 281, (4.3)) that the set L_X is finite. Thus, we conclude that B is a neighborhood of x_0 in X, B being the union of a finite number of disks, disjoint everywhere except at x_0. Observe that $x_0\in X\setminus L_X$ implies that B is a disk.

Now, it is not difficult to see that there is a compact, connected 2-manifold (perhaps with boundary), say M_0, such that M_0 contains X (topologically). If $L_X=\emptyset$ then X itself is a 2-manifold, and so assume that $L_X=\bigcup_{i=1}^{k}(p_i)$. Let B_i be a neighborhood of p_i in X which is the union of a finite number $n(i)$ of disks Q_{ij}, $j=1,2,\ldots,n(i)$, with the only common point p_i. We can assume that $B_i\cap B_{i'}=\emptyset$

for $i \neq i'$ and that the boundary of B_i in X is the union of $n(i)$ arcs L_{ij}, $j = 1, \ldots, n(i)$, where $L_{ij} \subset \dot{Q}_{ij} \setminus (p_i)$ and there is a neighborhood of L_{ij} in X which is a disk separated by L_{ij} into two components. Take any disjoint disks $D_1, \ldots, D_k$ and consider the disjoint union $Y = X \cup \bigcup_{i=1}^{k} D_i$. There is a homeomorphism h_i mapping B_i onto a subset $h_i(B_i)$ of D_i such that $h_i(B_i) \cap \dot{D}_i = h_i(\operatorname{Bd} B_i)$. Identifying B_i with $h_i(B_i)$ by means of h_i, one constructs from Y the required 2-manifold M_0.

It is well known that there is a surface $M_1 \supset M_0$. We can assume that $X \subset M_0 \subset M_1$. Consider now the finite set $F \subset X$ mentioned in (1). Since no point of F is a Euclidean point of X and since $M_1 \setminus X$ has a finite number of components as $X \in \mathrm{ANR}$, it follows that for each point $y \in F$ there is a component C of $M_1 \setminus X$ such that $y \in \bar{C}$. Thus, applying Lemma 4.1, we complete the proof of (1).

Next, consider the more general case where the space X is cyclic (but not strongly cyclic). Let $E_1, E_2, \ldots$ denote the sequence of all strongly cyclic elements of X. We shall consider only the case where this sequence is infinite, because the opposite case is similar, but easier. It follows from [16] (p. 283, (4.9)) that there is a connected graph $G \subset X$ containing L_X and such that for each $i = 1, 2, \ldots$ the intersection $E_i \cap G$ is a non-degenerate tree T_i, the set of the end-points of T_i being equal to $\operatorname{Bd} E_i$. Moreover (cf. ibidem, (4.10)), for almost all i the tree T_i is an arc. Since any point $x \in X$ has a neighborhood in X which is an AR-set embeddable into E^2, since $\lim_{i\to\infty} \operatorname{diam} E_i = 0$ (cf. ibidem, (4.6)) and E_i is a retract of X (ibidem, (4.4)), we infer that almost all E_i are AR-sets embeddable into E^2. Since no E_i is separated by a point, we conclude that almost all E_i are disks (cf. [9], p. 380, No. 11). Thus, we can assume that there is an i_0 such that $i \leqslant i_0$ if and only if either E_i is not a disk or T_i is not an arc.

By Borsuk's result (cf. [2], p. 78), there is a homeomorphism h mapping the graph G into a surface M. It follows from [16], (4.2) that $E_i \cap E_j = \operatorname{Bd} E_i \cap \operatorname{Bd} E_j$ for $i \neq j$, and therefore $T_i \cap T_j$ is contained in the set of the end-points both of T_i and of T_j. Consequently, one easily constructs a sequence of disks $Q_1, Q_2, \ldots$ contained in M and such that $Q_i \cap h(G) = h(T_i)$, $Q_i \cap Q_j = h(T_i) \cap h(T_j)$ for $i \neq j$ and $\lim_{i\to\infty} \operatorname{diam} Q_i = 0$. Let $\dot{T}_i$ denote the set of the end-points of T_i. Since $\dot{T}_i = \operatorname{Bd} E_i$ and since X does not contain a 2-umbrella, no point of $\dot{T}_i$ is Euclidean point of E_i. Since $\dot{T}_i$ is equal to the boundary of T_i in G (by [16], (4.2) and (4.9)), we infer from the construction of the disks Q_i that $h(\dot{T}_i) \subset \dot{Q}_i$. Making use of the fact that $\lim_{i\to\infty} \operatorname{diam} E_i = 0$ (cf. ibidem, (4.6)), one easily extends the homeomorphism h to a homeomorphism h' mapping $G \cup \bigcup \{E_i \colon i > i_0\}$ onto $h(G) \cup \bigcup \{Q_i \colon i > i_0\}$.

Since all t.s.c.e.'s of X inherit all the properties of X assumed in the theorem and since they are strongly cyclic (cf. ibidem, (4.4) and (4.11)), we can apply (1) to each set E_i with $i \leqslant i_0$, replacing then the set F by the finite set $\operatorname{Bd} E_i$. Since $h(\operatorname{Bd} E_i) = h(\dot{T}_i) \subset \dot{Q}_i$, there is a natural ordering of $h(\operatorname{Bd} E_i)$ defined by choosing a point of this set and an orientation of the simple closed curve $\dot{Q}_i$. This ordering defines in turn an ordering $<$ of $\operatorname{Bd} E_i$. Now, applying (1), we conclude that for each $i \leqslant i_0$ there are a surface M_i, a disk $D_i \subset M_i$ and a homeomorphism h_i of E_i into M_i such that $h_i(E_i) \cap D_i = h_i(\operatorname{Bd} E_i) \subset \dot{D}_i$. Moreover, there is an orientation of $\dot{D}_i$ such that $x, y, z \in \operatorname{Bd} E_i$ and $x < y < z$ imply that $h(y)$ lies between $h(x)$ and $h(z)$, as formulated in (1).

Now, consider the disjoint union

$$(M \setminus \bigcup \{\mathring{Q}_i \colon i \leqslant i_0\}) \cup \bigcup \{M_i \setminus \mathring{D}_i \colon i \leqslant i_0\} .$$

It follows from the definition of the ordering $<$ of $\operatorname{Bd} E_i$ and from the property of h_i with respect to that ordering described in (1) that there is a homeomorphism f_i of $\dot{Q}_i$ onto $\dot{D}_i$ such that for each $x \in \operatorname{Bd} E_i$ we have $f_i(h(x)) = h_i(x)$, where $i \leqslant i_0$. Identifying $\dot{Q}_i$ with $\dot{D}_i$ by means of f_i for all $i \leqslant i_0$, one constructs a surface N from the disjoint union mentioned above. If φ denotes the natural map of that disjoint union onto N, then defining

$$h^*(x) = \begin{cases} \varphi h'(x) & \text{if} \quad x \in G \cup \bigcup \{E_i \colon i > i_0\} , \\ \varphi h_i(x) & \text{if} \quad x \in E_i, \text{ where } i \leqslant i_0 \end{cases}$$

one obtains an embedding h^* of X into N. Indeed, since the t.s.c.e.'s E_i of X are the closures of the components of $X \setminus L_X$ (cf. [16], (4.2)) and since $G \supset L_X$, it follows that h^* is defined on the whole X. The construction of N and the properties of h', h_i and f_i imply that h^* is a homeomorphism. This proves the theorem for the case where X is a cyclic space.

Finally, consider the general case with no additional assumptions on X. Since each point $x \in X$ has a neighborhood in X which is an AR-set embeddable into E^2, it follows from [15] ((3.8), (3.6) and (3.4)) that almost all non-degenerate cyclic elements of X are cyclic AR-sets embeddable into E^2, and therefore — disks. If all of them are disks, then it follows from [15], (p. 290, Theorem 2) that X is embeddable into S^2. Thus, we shall assume that there are finitely many non-degenerate cyclic elements of X, say $Z_1, \ldots, Z_k$, which are not disks.

Denote by A_l, where $1 \leqslant l \leqslant k$, the least closed and entirely arcwise connected subset of X containing $\bigcup_{i=1}^{l} Z_i$ (cf. Definition 4.2). It follows from [15], (3.5) that if A and B are closed and entirely arcwise connected subsets of X, then the least closed and entirely arcwise connected subset of X containing $A \cup B$ is equal to the union of $A \cup B$ and the least closed and entirely arcwise connected subset of X containing $(a) \cup (b)$, where a is any point of A and b is any point of B. Thus, we infer from [15], (3.13) that the numeration of $Z_1, \ldots, Z_k$ can be changed in such a way that A_l does not contain any Z_i with $l < i \leqslant k$.

Now, we shall prove by induction with respect to l, where $1 \leqslant l \leqslant k$, that:

(2) *the set A_l is embeddable into a surface.*

If $l = 1$, then $A_1 = Z_1$ is a cyclic space satisfying all the assumptions of the theorem that X does, and therefore A_1 is embeddable into a surface by the preceding part of the proof. Now, given an $l>1$, assume that (2) is true for $l-1$. Thus, there is a surface M and an embedding h_{l-1} of A_{l-1} into M. Since $A_{l-1} \not\supset Z_l$, we infer that $A_l = A_{l-1} \cup B \cup Z_l$, where B is the least closed and entirely arcwise connected subset of X containing $(a) \cup (z)$ with $a \in A_{l-1}$, $z \in Z_l$. Considering the structure of the set B (cf. [15], p. 293, (3.13)), one sees that the points a and z can be chosen so that $B \cap A_{l-1} = (a)$ and $B \cap Z_l = (z)$. We can assume that $a \neq z$, because the proof in that case is similar, but easier. Thus $A_{l-1} \cap Z_l = \emptyset$. Notice that the assumption that X does not contain a 2-umbrella implies that a is not a Euclidean point either of B or of A_{l-1} and z is not a Euclidean point either of B or of Z_l. Since A_{l-1}, as a retract of X, is an ANR-set and since h_{l-1} is an embedding of A_{l-1} into the surface M, there is a component C of $M \setminus h_{l-1}(A_{l-1})$ whose closure contains $h_{l-1}(a)$. Consequently, there is a disk $Q_1 \subset M$ such that $Q_1 \cap h_{l-1}(A_{l-1}) = h_{l-1}(a) \subset \dot{Q}_1$.

Now, observe that B, as a closed and entirely arcwise connected subset of X, is also a retract of X, and therefore B is an ANR. Evidently, B does not contains a 2-umbrella. Moreover, one sees from the structure of B (cf. ibidem, (3.13)) that the non-degenerate cyclic elements of B are those non-degenerate cyclic elements of X which are contained in B. Since $B \subset A_l$ and $B \cap A_{l-1} = (a)$, we infer that all those cyclic elements are disks. Thus, we conclude from [15] (p. 290, Theorem 2), that B is embeddable into a disk. Since a is not a Euclidean point of B, it follows that there is an embedding h of B into Q_1 such that $h(B) \cap \dot{Q}_1 = (h_{l-1}(a)) = (h(a))$. Since z is not a Euclidean point of B, we infer that there is a disk $Q_2 \subset Q_1$ such that $Q_2 \cap h(B) = (h(z))$.

Next, consider the cyclic element Z_l. Since Z_l satisfies all the assumptions of the theorem that X does and since Z_l is a cyclic set, we infer by the preceding part of the proof of the theorem concerning the case where X is a cyclic space that there are a surface N and an embedding h_l of Z_l into N. Since Z_l, as a retract of X, is an ANR-set and since z is not a Euclidean point of Z_l, it follows that there is a disk $D \subset N$ such that $D \cap h_l(Z_l) = (h_l(z)) \subset \dot{D}$.

Finally, consider the disjoint union $(M \setminus \mathring{Q}_2) \cup (N \setminus \mathring{D})$ and identify $\dot{Q}_2$ with $\dot{D}$ by means of a homeomorphism mapping $h(z)$ onto $h_l(z)$. It is easy to see that we obtain a surface and that the homeomorphisms h_{l-1}, h and h_l induce an embedding of $A_l = A_{l-1} \cup B \cup Z_l$ into this surface, which completes the inductive proof of (2).

It follows from (2) that there are a surface P and a homeomorphism h_k mapping A_k into P. We can assume that $X \setminus A_k \neq \emptyset$. Since A_k is a closed and entirely arcwise connected subset of X, the set $X \setminus A_k$ has at most countably many components, the boundary of each being a point. Let $c_1, c_2, \ldots$ denote the sequence of all points of A_k bounding some components of $X \setminus A_k$. We can assume that this sequence is infinite, because the opposite case is similar, but easier. Denote by C_i the union of the closures of all components C of $X \setminus A_k$ such that $\bar{C} \setminus C = (c_i)$. Then $\lim_{i\to\infty} \operatorname{diam} C_i = 0$ (cf. [15], (3.3)). Notice that each set C_i is a retract of X, and therefore it is a connected ANR. Moreover, the non-degenerate cyclic elements of C_i are the non-degenerate cyclic elements of X which are contained in C_i. Since they are different from Z_i with $i \leqslant k$, it follows that all of them are disks. Since C_i, as a subset of X, does not contain a 2-umbrella, we conclude from [15] (p. 290, Theorem 2) that each C_i is embeddable into a disk. Since, for each i, $C_i \cap A_k = (c_i)$ and since X does not contain a 2-umbrella, it follows that c_i is not a Euclidean point either of C_i or of A_k.

Since h_k is an embedding of A_k into the surface P and since $A_k \in$ ANR, it follows that for each $i = 1, 2, \ldots$ there is a disk $Q_i \subset P$ such that $Q_i \cap h_k(A_k) = (h_k(c_i))$, $Q_i \cap Q_{i'} = \emptyset$ for $i \neq i'$ and $\lim_{i\to\infty} \operatorname{diam}(Q_i) = 0$. Consequently, for each $i = 1, 2, \ldots$ there is an embedding f_i of C_i into Q_i such that $f_i(c_i) = h_k(c_i)$. Thus, defining

$$h'(x) = \begin{cases} h_k(x) & \text{if} \quad x \in A_k, \\ f_i(x) & \text{if} \quad x \in C_i, \quad i = 1, 2, \ldots, \end{cases}$$

we obtain a homeomorphism h' mapping X into P, which completes the proof of the theorem.

Now, we shall prove a corollary to Theorem 4.3, mentioned in Introduction (Section 1), which is a generalization of Theorem 3.1.

COROLLARY 4.4. *Let X be a locally connected compactum which is Y-like, where Y is an* ANR*-set, $Y \subset M$ and M is a surface. Then X is an* ANR*-set embeddable into a surface.*

Proof. Since Y has a finite number of components and since X is Y-like, we easily infer that X and Y have the same number of components. Moreover, one easily sees (cf. the proof of Theorem 2.6 for the case where X is not connected) that for each component C of X there is a component C' of Y such that C is C'-like. Consequently, we can assume in the sequel that both X and Y are connected.

The next part of the proof is similar to the proof of Theorem 3.13. Indeed, since Y is a semi-lc_1 space, it follows from Theorem 2.6 that $X \in \alpha$. By Theorem 3.6, X does not contain a 2-umbrella. Suppose that for every $\lambda > 0$ there is a set $A \subset X$ homeomorphic either with K_1 or with K_2 and such that $\operatorname{diam} A < \lambda$. Let $k_0 = 1 + \gamma(M)$, where $\gamma(M)$ denotes the genus of M. It follows from Lemma 3.10 that X contains k_0 disjoint subsets, say $B_1, \ldots, B_{k_0}$, each of which is homeomorphic either with K_1 or with K_2. We infer in the same way as in the proof of Theorem 3.13 that if f is an ε-mapping of X onto Y and if $\varepsilon > 0$ is sufficiently small, then $f(B_i)$, where $1 \leqslant i \leqslant k_0$, are disjoint subsets of Y each of which contains a graph homeomorphic either with K_1 or with K_2. This contradicts Borsuk's result (cf. [2], p. 75), and therefore there is a $\lambda > 0$ such that X does not contain any homeomorphic images of the graphs K_1 and K_2 with diameters less than λ. Now, we conclude from Therem 4.3 that X is embeddable into a surface N. Since $X \in \alpha$, it is not difficult to see that X must be an ANR-set. Indeed, since X is a locally connected continuum, assuming that $X \subset N$, it suffices to observe that $N \setminus X$ cannot posses infinitely many components. But this is an easy consequence of the fact that $X \in \alpha$.

COROLLARY 4.5. *Any compactum X quasi-homeomorphic with an* ANR-*set* $Y \subset M$, *where M is a surface, is itself an* ANR-*set embeddable into a surface.*

Indeed, since Y is X-like, it follows that X is locally connected, and therefore, by Corollary 4.4, X is an ANR-set embeddable into a surface.

The answer to the following question is not known to the author, but it seems to be positive:

PROBLEM. Can we assert in Corollaries 4.4 and 4.5 that the space X is embeddable into the same surface M which contains Y?

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On models of arithmetic having non-modular substructure lattices

by

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Abstract. A model of arithmetic having the pentagon lattice for its lattice of elementary substructures is constructed, and some related results are proved. This answers a question raised by J. B. Paris in his paper [3].

1. Introduction to the problem. Let T be a complete consistent extension of the Peano axioms P, and M the minimal (i.e. pointwise definable) model of T. We suppose that L, the language of T, contains the set S_n of all n-place Skolem functions for $n \in \omega$, and identify M with S_0. Thus the notion of elementary substructure coincides with that of substructure for models of T. Our aim in this paper is to study the possible complexity of models of T. This we do by letting $\mathcal{S}(M^*)$ be the set of all substructures of M^* partially ordered by the "is a substructure of" relation, $\subseteq$. It is clear that $\mathcal{S}(M^*)$ is a lattice; $M_1 \wedge M_2$ (the infimum of M_1 and M_2 in $\mathcal{S}(M^*)$) being $M_1 \cap M_2$, and $M_1 \vee M_2$ (the supremum of M_1 and M_2 in $\mathcal{S}(M^*)$) being that substructure of M^* generated by $M_1 \cup M_2$ under all functions in $\bigcup_{n \in \omega} S_n$. Our problem can now be stated as: "which lattices occur as $\mathcal{S}(M^*)$ for some $M^* \models T$?"

A complete characterization of such lattices seems a long way off — even if we restrict our attention to finite lattices, as we do in this paper. For all known positive results on the problem we refer the reader to [3]; in particular it is proved there that every finite distributive lattice is an $\mathcal{S}(M^*)$. If M is non-standard (i.e. if T is not true arithmetic) it is still possible that every finite lattice is an $\mathcal{S}(M^*)$, whereas if M is standard there is not even an obvious conjecture. For under this latter assumption it is known (see Lemma 3.3 and [4]) that C_5 (the simplest modular non-distributive lattice — see Fig. (1)) is *not* an $\mathcal{S}(M^*)$ and, as we prove here, neither is H (which is non-modular). However, to confuse matters we also answer in the sequel a question raised in [3] by showing that for any T, P_5 (which is non-modular but somewhat less symmetrical than H) *is* of the form $\mathcal{S}(M^*)$ for some $M^* \models T$!

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