ally adjacent with u and v, since G is outerplanar. Because u and v may be adjacent, G contains at least $2a_2-2$ vertices. However, there exists an outerplanar graph G of order $2a_2-2$ with $\mathcal{D}_G = \{2, a_2\}$ (see Fig. 2); therefore, $\mu_0(2, a_2) = 2a_2-2$.

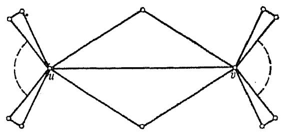

Fig. 2

We note in closing that $\mu_0(S)$ has been completely determined for $|S| = 3$, and the result will be presented elsewhere.

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Accepté par la Rédaction le 21. 4. 1975

Models of arithmetic and the 1-3-1 lattice

by

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Abstract. In this paper we show that if T is any complete theory in the language of number theory extending Peano's Axioms then there is a model M of T such that the 1–3–1 lattice can be embedded in the lattice of elementary substructures of M.

Introduction. Let T be a complete theory in the language of number theory extending Peano's Axioms. For M a model of T, let $\mathcal{S}(M)$ be the lattice of elementary substructures of M. In this paper we show that there is a model M of T such that the 1-3-1 lattice can be embedded in $\mathcal{S}(M)$.

This result continues investigations started in [1]. Related work also appears in [2] and we adopt the notation of that paper. Thus for M a model of T, $a_1, \ldots, a_n \in M$, $M[a_1, \ldots, a_n]$ is the smallest elementary substructure of M containing $a_1, \ldots, a_n$. Since M is a model of Peano's Axioms, $M[a_1, \ldots, a_n]$ consists exactly of those elements of M definable in M from $a_1, \ldots, a_n$.

THEOREM. *There is a model M of T such that the* 1-3-1 *lattice can be embedded in $\mathcal{S}(M)$.*

Proof. Fix M to be an ω_1-saturated model of T and identify N, the natural numbers, with an initial segment of M. We shall show that M satisfies the properties of the theorem.

Before proceeding further it will be useful to have the following crude estimate.

LEMMA 1. *Let $r, q \in M$, $s \in N$ and $s \geqslant 2$. Let x_i, y_i, $1 \leqslant i \leqslant q$ be sequences of elements of M definable in M and let*

$$\sum_{i=1}^{q} x_i = \sum_{i=1}^{q} y_i = r \quad (\textit{sums taken in } M).$$

Then

$$\sum_{i=1}^{q} x_i y_i - (\textit{the sum of the } s \textit{ largest } x_i y_i) \leqslant \frac{r^2}{4(s-1)}.$$

* This paper was written when the author was working at Manchester University and the University of California, Berkeley.

Proof. We work in M. We shall simplify the proof by working with rationals in the sense of M, hereafter just called rationals. It is easy to check that the rational arithmetic required in the proof can be carried out in M.

Given $x_i y_i$ as above we may assume,

$$x_1 y_1 \geqslant x_2 y_2 \geqslant \ldots \geqslant x_q y_q ,$$

and $x_{s+1} y_{s+1} > 0$, since otherwise the result is trivial. By removing rational fractions of the x_i for $i \leqslant s$ we can obtain positive rational z_i, t_i for $i \leqslant s$ such that

$$\sum_{i=1}^{s} z_i + \sum_{i=s+1}^{q} x_i \leqslant r ,$$

$$\sum_{i=1}^{s} t_i + \sum_{i=s+1}^{q} y_i \leqslant r$$

and $z_1 t_1 = z_2 t_2 = \ldots = z_s t_s = x_{s+1} y_{s+1}$.

By further redistributing rational fractions of x_{n+i} to x_i and y_{n+i} to y_i for $s+1 < i < q$ we can obtain non-negative rationals z_i, t_i for $s+1 \leqslant i \leqslant m$, where $m \leqslant q$, such that

$$\sum_{i=1}^{m} z_i \leqslant r , \quad \sum_{i=1}^{m} t_i \leqslant r ,$$

$$x_{s+1} y_{s+1} = z_{s+1} t_{s+1} = z_{s+2} t_{s+2} = \ldots = z_{m-1} t_{m-1} \geqslant z_m t_m$$

and

$$\sum_{i=s+1}^{m} z_i t_i \geqslant \sum_{i=s+1}^{q} x_i y_i .$$

Put

$$a = \frac{\sum_{i=1}^{m-1} z_i}{m-1} , \quad b = \frac{\sum_{i=1}^{m-1} t_i}{m-1} .$$

By Cauchy's inequality for $1 \leqslant i \leqslant m-1$,

$$ab \geqslant z_i t_i \geqslant z_m t_m ,$$

so

$$\sum_{i=s+1}^{q} x_i y_i \leqslant \sum_{i=s}^{m} z_i t_i \leqslant (m-s) ab .$$

Since

$$a(m-1) + z_m = \sum_{i=1}^{m} z_i \leqslant r ,$$

$$b(m-1) + t_m = \sum_{i=1}^{m} t_i \leqslant r$$

$$ab \leqslant \frac{r^2}{(m-1)^2} ,$$

so

$$\sum_{i=s+1}^{q} x_i y_i \leqslant \frac{(m-s) r^2}{(m-1)^2} \leqslant \frac{r^2}{4(s-1)} ,$$

as required. ■

Now pick $p \in M - N$ and set

$$A = \{\langle a_1, a_2, a_3 \rangle |\ a_1 + a_2 + a_3 = 0 \bmod p,\ 0 \leqslant a_1, a_2, a_3 < p\} .$$

A is definable in M from p, hereafter shortened to p-Def. For $C \subseteq A$ and C p-Def let $|C|$ be that $a \in M$ such that

$$M \models C \text{ has exactly } a \text{ elements} .$$

Thus $|A| = p^2$. Let F_n, $n \in N$ enumerate the p-Def functions from M into M and let π_n, $n \in N$, enumerate all 6-tuples $\langle i, j, k, s, t, u \rangle$ with i, j, k, distinct, $1 \leqslant i, j, k \leqslant 3$ and $s, t, u \in N$.

Lemma 2. *Let $A_n \subseteq A$ be p-Def and $|A_n| \geqslant p^2/m$, some $m \in N$. Let $\pi_n = \langle i, j, k, s, t, u \rangle$. Then $\exists$ p-Def $A_{n+1} \subseteq A_n$ such that $|A_{n+1}| \geqslant p^2/q$, some $q \in N$ and if $\langle a_1, a_2, a_3 \rangle \in A_{n+1}$ then*

i) *either $F_s(a_i) \neq F_t(a_j)$ or $F_s(a_i) = F_t(a_j) \in M[p]$,*

ii) *$F_u(p) \neq a_k$.*

Proof. First notice that if $m \in N$ then $m < p$ so p^2/m is large. Set

$$B = \{\langle a_1, a_2, a_3 \rangle \in A_n |\ F_u(p) = a_k\} .$$

Since there are at most p elements $\langle a_1, a_2, a_3 \rangle \in A_n$ such that $F_u(p) = a_k$, $|B| \leqslant p$.

Now for $v \in M$ set

$$I_v = \{a |\ 0 \leqslant a < p \ \&\ F_s(a) = v\} ,$$

$$J_v = \{a |\ 0 \leqslant a < p \ \&\ F_t(a) = v\} .$$

Thus in M,

$$\sum_v |I_v| = \sum_v |J_v| = p .$$

Now assume that $|I_{v_q}| \cdot |J_{v_q}|$, $1 \leqslant q \leqslant m+1$ are the $m+1$ largest elements of $\{|I_v| \cdot |J_v| |\ v \in M\}$. Since $m \in N$, we may assume that $v_1, \ldots, v_{m+1}$ are definable in M from p so $v_1, \ldots, v_{m+1} \in M[p]$.

Set

$$K = \{\langle a_1, a_2, a_3 \rangle \in A |\ F_s(a_i) \neq F_t(a_i) \text{ or } F_s(a_i) = F_t(a_j) = v_q, \text{ some } 1 \leqslant q \leqslant m+1\} .$$

Then

$$|K| = \sum_{e_1 \neq e_2} |\{\langle a_1, a_2, a_3\rangle \in A|\ F_s(a_i) = e_1 \ \&\ F_t(a_j) = e_2\}| + $$
$$+ \sum_{q=1}^{m+1} |\{\langle a_1, a_2, a_3\rangle \in A|\ F_s(a_i) = F_t(a_j) = v_q\}|$$
$$= \sum_{e_1 \neq e_2} |I_{e_1}|\cdot|J_{e_2}| + \sum_{q=1}^{m+1} |I_{v_q}|\cdot|J_{v_q}|$$
$$= \sum_{e_1, e_2} |I_{e_1}|\cdot|J_{e_2}| - [\sum_{v} |I_v|\cdot|J_v| - \sum_{q=1}^{m+1} |I_{v_q}|\cdot|J_{v_q}|]$$
$$\geqslant p^2 - p^2/4m \text{ by Lemma 1.}$$

Set

$$A_{n+1} = A_n \cap K - B = A - (A - A_n) \cup (A - K) \cup B .$$

Then

$$|A_{n+1}| \geqslant p^2 - (p^2 - p^2/m) - (p^2 - p^2 + p^2/4m) - p \geqslant p^2/4m .$$

Furthermore for $\langle a_1, a_2, a_3\rangle \in A_{n+1}$,

α) Since $A_{n+1} \subseteq K$ either $F_s(a_i) \neq F_t(a_j)$ or $F_s(a_i) = F_t(a_j) = v_q \in M[p]$ some $1 \leqslant q \leqslant m+1$,

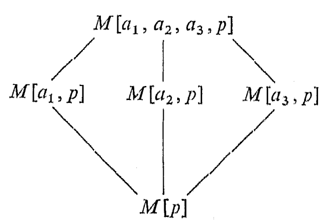
β) Since $A_{n+1} \cap B = \emptyset$, $F_u(p) \neq a_k$.

We are now ready to construct the required sublattice of M.

Set $A_0 = A$ and having found A_n such that $|A_n| \geqslant p^2/m$, some $m \in N$, find, by Lemma 2, $A_{n+1} \subseteq A_n$ such that $|A_{n+1}| \geqslant p^2/q$ some $q \in N$. Since all the A_n are non-empty and p-Def, and since M is ω_1-saturated, we can find

$$\langle a_1, a_2, a_3\rangle \in \bigcap_{n \in N} A_n .$$

We now claim that we have the following sublattice of M:



To see this, let $1 \leqslant i, j, k \leqslant 3$ and i, j, k distinct. Then,

$$e \in M[a_i, p] \wedge M[a_j, p] \leftrightarrow \exists s, t, F_s(a_i) = F_t(a_j) = e \leftrightarrow e \in M[p] \text{ by } \alpha),$$

so

$$M[a_i, p] \wedge M[a_j, p] = M[p] .$$

By β) $F_u(p) \neq a_k$ for all $u \in N$ so $M[p] \neq M[p, a_k]$. Finally $a_i =$ the least z such that $0 \leqslant z \leqslant p$ and $z + a_j + a_k = 0 \bmod p$, so

$$a_i \in M[a_j, p] \vee M[a_k, p] .$$

Thus

$$M[a_i, a_j, a_k, p] = M[a_j, p] \vee M[a_k, p] .$$

Concluding remarks. It may be hoped that this result could be improved to: There is a model M of T such that $\mathcal{S}(M)$ is isomorphic to the 1-3-1 lattice. However, if T is the theory of N then this is impossible, by an unpublished result of Gaifman and the author. (This result is implicit in work of Wilkie, [2].) We do not know if the improvement is possible in the case when T is not the theory of N.

It is known that the pentagon lattice can be embedded in the model M of the main theorem (see [2]). Thus M is both non-distributive and non-modular. We do not know if there is a model M' of T such that $\mathcal{S}(M)$ is modular but non-distributive, that is a model M' such that the 1-3-1 lattice can be embedded in $\mathcal{S}(M')$ but the pentagon lattice cannot.

We finally remark that a very similar proof to the above will show the embeddability of the 1-n-1 lattice in M for all $n \in N$, $n \geqslant 3$.

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[2] A. J. Wilkie, *On models of arithmetic with non-modular substructure lattices*, Fund. Math. 95 (1977), pp. 223–238.

Accepté par la Rédaction le 21. 4. 1975