Models of arithmetic and the 1-3-1 lattice

by

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Abstract. In this paper we show that if T is any complete theory in the language of number theory extending Peano’s Axioms then there is a model M of T such that the 1-3-1 lattice can be embedded in the lattice of elementary substructures of M.

Introduction. Let T be a complete theory in the language of number theory extending Peano’s Axioms. For M a model of T, let \( \mathcal{S}(M) \) be the lattice of elementary substructures of M. In this paper we show that there is a model M of T such that the 1-3-1 lattice can be embedded in \( \mathcal{S}(M) \).

This result continues investigations started in [1]. Related work also appears in [2] and we adopt the notation of that paper. Thus for M a model of T, \( a_1, \ldots, a_n \in M \) and \( M[a_1, \ldots, a_n] \) is the smallest elementary substructure of M containing \( a_1, \ldots, a_n \). Since M is a model of Peano’s Axioms, \( M[a_1, \ldots, a_n] \) consists exactly of those elements of M definable in M from \( a_1, \ldots, a_n \).

Theorem. There is a model M of T such that the 1-3-1 lattice can be embedded in \( \mathcal{S}(M) \).

Proof. Fix M to be an \( \alpha^2 \)-saturated model of T and identify \( N \), the natural numbers, with an initial segment of M. We shall show that M satisfies the properties of the theorem.

Before proceeding further it will be useful to have the following crude estimate.

Lemma 1. Let \( r, q \in M \), \( s \in \mathbb{N} \) and \( s \geq 2 \). Let \( x_1, y_1, 1 \leq i \leq q \) be sequences of elements of M definable in M and let

\[
\sum_{i=1}^{s} x_i = \sum_{i=1}^{s} y_i = r \quad \text{(sums taken in M)}.
\]

Then

\[
\sum_{i=1}^{s} x_i y_i = \text{(the sum of the s largest } x_i y_i) \leq \frac{r^2}{4(s-1)}.
\]

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Proof. We work in $M$. We shall simplify the proof by working with rationals in the sense of $M$, hereafter called rationals. It is easy to check that the rational arithmetic required in the proof can be carried out in $M$.

Given $x_iy_i$ as above we may assume,

$$x_1y_1 \geq x_2y_2 \geq \ldots \geq x_{r+1}y_{r+1},$$

and $x_{r+1}y_{r+1} > 0$, since otherwise the result is trivial. By removing rational fractions of the $x_i$ for $i \leq r$ we can obtain positive rational $z_i, t_i$ for $i \leq r$ such that

$$\sum_{i=1}^r z_i + \sum_{i=1}^r x_i \leq r,$$

$$\sum_{i=1}^r t_i + \sum_{i=1}^r y_i \leq r$$

and $z_1t_1 = z_2t_2 = \ldots = z_rt_r = x_{r+1}y_{r+1}.

By further redistributing rational fractions of $x_{i+1}$ to $x_i$ and $y_{i+1}$ to $y_i$ for $s+1 \leq i \leq q$ we obtain non-negative rationals $z_s, t_s$ for $s+1 \leq i \leq m$, where $m \leq q$, such that

$$\sum_{i=s}^m z_i \leq r,$$

$$\sum_{i=s}^m t_i \leq r$$

and

$$x_{s+1}y_{s+1} = z_{s+1}t_{s+1} = z_{s+2}t_{s+2} = \ldots = z_m t_m \geq x_{s+1}y_{s+1}$$

and

$$\sum_{i=s+1}^m z_i y_i.$$

Put

$$a = \sum_{i=s}^m z_i, \quad b = \sum_{i=s}^m t_i$$

By Cauchy's inequality for $1 \leq i \leq m-1$,

$$ab \geq z_i t_i \geq z_m t_m,$$

so

$$\sum_{i=s+1}^m z_i y_i \leq m - z_m t_m \cdot ab.$$
Then

$$|K| = \sum_{\alpha \in \mathcal{A}} |\langle a_1, a_2, a_3 \rangle \in \mathcal{A} | F_{\alpha}(a_1) = e_1 \& F_{\alpha}(a_2) = e_2| +$$

$$\sum_{e \in \mathcal{O}} |\langle a_1, a_2, a_3 \rangle \in \mathcal{A} | F_{\alpha}(a_3) = F_{\alpha}(a_1) = e|$$

$$= \sum_{\alpha \in \mathcal{A}} |L_{\alpha}||J_{\alpha}| + \sum_{e \in \mathcal{O}} |L_{\alpha}||J_{\alpha}|$$

$$= \sum_{\alpha \in \mathcal{A}} |L_{\alpha}||J_{\alpha}| - \sum_{e \in \mathcal{O}} |L_{\alpha}||J_{\alpha}| - \sum_{\alpha \in \mathcal{A}} |L_{\alpha}||J_{\alpha}|$$

$$\geq p^3 - p^2/4m$$

by Lemma 1.

Set

$$A_{n+1} = A_n \cap K - B = A - (A - A_n) \cup (A - K) \cup B.$$  

Then

$$|A_{n+1}| \geq p^3 - (p^2 - p^2/m) - (p^2 - p^2 + p^2/4m) = p^3 - p^2/4m .$$

Furthermore for $\langle a_1, a_2, a_3 \rangle \in A_{n+1},$

a) Since $A_{n+1} \subseteq K$ either $F_{\alpha}(a_1) \neq F_{\alpha}(a_2)$ or $F_{\alpha}(a_1) = F_{\alpha}(a_2) = e \in M[p]$ some $1 \leq p \leq m + 1$.  

b) Since $A_{n+1} \cap B = \emptyset$, $F_{\alpha}(p) \neq a_n.$

We are now ready to construct the required sublattice of $M.$

Set $A_0 = A$ and having found $A_0$ such that $|A_0| \geq p^2/m$, some $m \in \mathcal{N}$, find, by Lemma 2, $A_1 \subseteq A_0$ such that $|A_{1+1}| \geq p^3/p$ some $q \in \mathcal{N}$. Since all the $A_k$ are non-empty and $p$-Def, and since $M$ is $a_0$-saturated, we can find

$$\langle a_1, a_2, a_3 \rangle \in \mathcal{A}_{n+1}.$$  

We now claim that we have the following sublattice of $M$:

$$M[a_1, a_2, a_3, p]$$

$$M[a_1, p] \quad M[a_2, p] \quad M[a_3, p]$$

$$M[p]$$

To see this, let $1 \leq i, j, k \leq 3$ and $i, j, k$ distinct. Then,

$$e \in M[a_1, p] \land M[a_j, p] \land \exists z, t, F_{\alpha}(a_i) = F_{\alpha}(a_j) = e \land e \in M[p] \text{ by } a_k,$$

so

$$M[a_1, p] \land M[a_j, p] = M[p].$$

By $\beta)$ $F_{\alpha}(p) \neq a_k$ for all $u \in N$ so $M[p] \neq M[p, a_k].$ Finally $a_i = \text{ the least } z \text{ such that } 0 \leq z \leq p$ and $z + a_j + a_k = 0 \text{ mod } p$, so

$$a_i \in M[a_j, p] \lor M[a_k, p].$$

Thus

$$M[a_1, a_j, a_k, p] = M[a_j, p] \lor M[a_k, p].$$

Concluding remarks. It may be hoped that this result could be improved to:

There is a model $M$ of $T$ such that $\mathcal{S}(M)$ is isomorphic to the 1-3-1 lattice.

However, if $T$ is the theory of $N$ then this is impossible, by an unpublished result of Gaifman and the author. (This result is implicit in work of Wilkie, [2].)

We do not know if the improvement is possible in the case when $T$ is not the theory of $N$.

It is known that the pentagon lattice can be embedded in the model $M$ of the main theorem (see [2]). Thus $M$ is both non-distributive and non-modular.

We do not know if there is a model $M'$ of $T$ such that $\mathcal{S}(M')$ is modular but non-distributive, that is a model $M'$ such that the 1-3-1 lattice can be embedded in $\mathcal{S}(M')$ but the pentagon lattice cannot.

We finally remark that a very similar proof to the above will show the embeddability of the 1-1-1 lattice in $M$ for all $n \in N$, $n \geq 3$.

References


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