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Degree sets for graphs

by

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Abstract. For a graph G, the degree set $\mathfrak{D}_G$ of G is the set of degrees of the vertices of G. For a finite, nonempty set S of positive integers, it is shown that there exists a graph G such that $\mathfrak{D}_G = S$. Furthermore, the minimum order of such a graph G is determined. Degree sets are also investigated for trees, planar graphs, and outerplanar graphs.

1. Introduction. For a vertex v of a graph G, the *degree* of v in G, denoted $\deg v$, is the number of edges of G incident with v. We denote the *degree set* of G (i.e., the set of degrees of the vertices of G) by $\mathscr{D}_G$. For example, the graph H of Figure 1 has degree set $\mathscr{D}_H = \{2, 4, 5\}$.

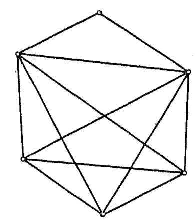

Fig. 1

Given a finite, nonempty set S of positive integers, we show that there exists a graph G such that $\mathscr{D}_G = S$ and determine the minimum order (number of vertices) of such a graph G. In addition to investigating degree sets for graphs, we discuss degree sets for planar graphs (including the subclasses of trees and outerplanar graphs).

2. Degree sets for graphs. Before proceeding to our first result, we present some definitions and establish some notation.

We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$, respectively. The *complement* $\overline{G}$ of a graph G is that graph for which $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. The *union* $G_1 \cup G_2$ of disjoint graphs G_1

and G_2 is that graph whose vertex set is $V(G_1) \cup V(G_2)$ and whose edge set is $E(G_1) \cup E(G_2)$. For disjoint graphs G_1 and G_2, the *join* G_1+G_2 has $V(G_1) \cup V(G_2)$ as its vertex set, while $E(G_1+G_2) = E(G_1) \cup E(G_2) \cup X$, where $X = \{v_1 v_2 |\ v_1 \in V(G_1) \text{ and } v_2 \in V(G_2)\}$. We denote the complete graph of order p by K_p and the complete n-partite graphs by $K(p_1, p_2, \ldots, p_n)$.

For a set S of positive integers, we shall write $\mu(S)$ to represent the minimum order of a graph G such that $\mathscr{D}_G = S$. (If no such graph G exists, then we write $\mu(S) = +\infty$.) If $S = \{a_1, a_2, \ldots, a_n\}$, $n \geqslant 1$, where $a_1 < a_2 < \ldots < a_n$, we shall often find it convenient to write $\mu(S) = \mu(a_1, a_2, \ldots, a_n)$. Since every graph which contains a vertex of degree a_n has order at least a_n+1, it follows immediately that $\mu(a_1, a_2, \ldots, a_n) \geqslant a_n+1$ for every set $S = \{a_1, a_2, \ldots, a_n\}$ of positive integers, with $a_1 < a_2 < \ldots < a_n$. We show that $\mu(S) = a_n+1$ in all cases.

THEOREM 1. *For every set $S = \{a_1, a_2, \ldots, a_n\}$, $n \geqslant 1$, of positive integers, with $a_1 < a_2 < \ldots < a_n$, there exists a graph G such that $\mathscr{D}_G = S$, and furthermore,*

$$\mu(a_1, a_2, \ldots, a_n) = a_n+1 .$$

Proof. We proceed by induction on n. For $n = 1$, we observe that every vertex of the complete graph K_{a_1+1} has degree a_1 so that $\mu(a_1) = a_1+1$. For $n = 2$, the vertices of the graph $F = K_{a_1}+(\overline{K}_{a_2-a_1+1})$ have degrees a_1 and a_2, and since F has order a_2+1, we conclude that $\mu(a_1, a_2) = a_2+1$.

Let $n \geqslant 2$. Assume for every set S containing m positive integers, where $1 \leqslant m \leqslant n$, that $\mu(S) = a_m+1$, where a_m is the largest element of S. Let $S_1 = \{b_1, b_2, \ldots, b_{n+1}\}$ be a set of $n+1$ positive integers such that $b_1 < b_2 < \ldots < b_{n+1}$. By the induction hypothesis, $\mu(b_2-b_1, b_3-b_1, \ldots, b_n-b_1) = (b_n-b_1)+1$. Hence, there exists a graph H of order $(b_n-b_1)+1$ such that

$$\mathscr{D}_H = \{b_2-b_1, b_3-b_1, \ldots, b_n-b_1\} .$$

The graph

$$G = K_{b_1}+(\overline{K}_{b_{n+1}-b_n} \cup H)$$

has order $b_{n+1}+1$, and $\mathscr{D}_G = \{b_1, b_2, \ldots, b_{n+1}\}$; hence, $\mu(b_1, b_2, \ldots, b_{n+1}) = b_{n+1}+1$, which completes the proof.

The proof of the preceding theorem also provides the following result.

COROLLARY 1a. *For every set $S = \{a_1, a_2, \ldots, a_n\}$, $n \geqslant 1$, of positive integers, with $a_1 < a_2 < \ldots < a_n$, there exists a connected graph G of order a_n+1 such that $\mathscr{D}_G = S$.*

3. Degree sets for trees. We now turn our attention to an important subclass of graphs, namely *trees* (connected graphs containing no cycles).

THEOREM 2. *Let $S = \{a_1, a_2, \ldots, a_n\}$, $n \geqslant 1$, be a set of positive integers. There exists a nontrivial tree T with $\mathscr{D}_T = S$ if and only if $1 \in S$. Moreover, if $1 \in S$, then the minimum order of a nontrivial tree T with $\mathscr{D}_T = S$ is $\sum_{i=1}^{n} (a_i-1)+2$.*

Proof. It is well-known that every nontrivial tree contains at least two vertices of degree 1. Let $S = \{a_1, a_2, \ldots, a_n\}$, $n \geqslant 1$, where $1 = a_1 < a_2 < \ldots < a_n$. For $n = 1$, the nontrivial tree K_2 has the degree set $\{1\}$. For $n = 2$, the star T_{a_2} (consisting of a central vertex adjacent with a_2 mutually nonadjacent vertices) has degree set $\{1, a_2\}$. For $n \geqslant 3$, we consider stars G_i $(2 \leqslant i \leqslant n)$, where $G_i = T_{a_i-1}$ for $i = 2$ and $i = n$ and $G_i = T_{a_i-2}$ for $2 < i < n$. A tree G is constructed from these stars by joining the central vertex of G_i to the central vertex of G_{i+1} for $i = 2, 3, \ldots, n-1$. The order of this tree G is $N = \sum_{i=1}^{n} (a_i-1)+2$ and the degree set of G is $\{a_1, a_2, \ldots, a_n\}$.

Now suppose that T is any tree with p vertices and q edges such that $\mathscr{D}_T = S = \{a_1, a_2, \ldots, a_n\}$. Necessarily, T contains at least one vertex of degree a_i for $2 \leqslant i \leqslant n$ and contains at least $p-n+1$ vertices of degree at least $a_1 = 1$. Furthermore, since the sum of the degrees of the vertices of T is $2q$ and since $q = p-1$, it follows that

$$2(p-1) = 2q \geqslant \sum_{i=1}^{n} a_i+(p-n)\cdot 1 .$$

Hence,

$$p \geqslant \sum_{i=1}^{n} (a_i-1)+2 = N .$$

Therefore, the minimum order of a tree T with $\mathscr{D}_T = S$ is $\sum_{i=1}^{n} (a_i-1)+2$.

4. Degree sets for planar graphs. A *planar graph* is a graph which can be embedded in the plane. First, we verify the following result.

THEOREM 3. *Let $S = \{a_1, a_2, \ldots, a_n\}$, $n \geqslant 1$, be a set of positive integers with $a_1 < a_2 < \ldots < a_n$. Then there exists a planar graph G with $\mathscr{D}_G = S$ if and only if $1 \leqslant a_1 \leqslant 5$.*

Proof. It is well-known (see [1], p. 104, for example) that if G is a planar graph, then G contains a vertex of degree at most five. Hence, if the positive integer a_1 is the minimum degree among the vertices of G, then $1 \leqslant a_1 \leqslant 5$.

Conversely, suppose $S = \{a_1, a_2, \ldots, a_n\}$, $n \geqslant 1$, is a set of positive integers such that $a_1 < a_2 < \ldots < a_n$ and $1 \leqslant a_1 \leqslant 5$. We show there exists a planar graph G such that $\mathscr{D}_G = S$. First, if $a_1 = 1$, then by Theorem 2, there exists a tree T (which, of course, is a planar graph) such that $\mathscr{D}_T = S$. Denote the end-vertices of T by $v_1, v_2, \ldots, v_k$. Let T' be another copy of T, embedded in the plane so that it is the "mirror-image" of T. Let v_i' be the end-vertex of T' which corresponds to v_i. If $a_1 = 2$, then we construct a planar graph G by joining v_i and v_i' for each i, $1 \leqslant i \leqslant k$. If $a_1 = 3, 4$, or 5, then we construct G by beginning with T and T' and graphs of the tetrahedron, octahedron, and icosahedron, respectively, embedded in the plane. In each case let $v_i v_i'$ be an edge on the exterior region of the graph of each polyhedron. Then a planar graph G with $\mathscr{D}_G = S$ is obtained by deleting $v_i v_i'$ and identifying the two vertices v_i and identifying the two vertices v_i'.

In view of Theorem 3, we can make the following definition. Let $S = \{a_1, a_2, \ldots, a_n\}$, $n \geq 1$, be a set of positive integers such that $a_1 < a_2 < \ldots < a_n$, and $1 \leq a_1 \leq 5$. Then $\mu_p(S) = \mu_p(a_1, a_2, \ldots, a_n)$ denotes the minimum order of a planar graph G for which $\mathscr{D}_G = S$. The value of $\mu_p(S)$ is well-known for $n = 1$; in fact, $\mu_p(1) = 2$, $\mu_p(2) = 3$, $\mu_p(3) = 4$, $\mu_p(4) = 6$, and $\mu_p(5) = 12$. (As noted earlier, the planar graphs giving these values for $a_1 = 3$, 4 and 5 are the graphs of the tetrahedron, octahedron and icosahedron.) However, for an arbitrary set S of positive integers, it appears to be very difficult to ascertain the value of $\mu_p(S)$. In order to present a result dealing with the case $n = 2$, it is convenient to have two additional definitions.

The *complete n-partite graph* $K(p_1, p_2, \ldots, p_n)$, $n \geq 2$, for positive integers $p_1, p_2, \ldots, p_n$, is that graph G whose vertex set can be partitioned into subsets $V_1, V_2, \ldots, V_n$ in such a way that $|V_i| = p_i$ for $1 \leq i \leq n$ and uv is an edge of G if and only if $u \in V_j$ and $v \in V_k$, for $j \neq k$. Hence, $K(p_1, p_2, \ldots, p_n) = K_n$ if $p_i = 1$ for each i, $1 \leq i \leq n$.

A planar graph G is called *outerplanar* if it is possible to embed G in the plane in such a way that every vertex lies on the boundary of the exterior region. One important fact concerning such graphs is that every outerplanar graph contains a vertex of degree at most two.

THEOREM 4. *Let a_1 and a_2 be positive integers with $a_1 < a_2$. Then*

$$\text{(i)} \quad \mu_p(a_1, a_2) = \begin{cases} a_2+1 & \text{for} \quad 1 \leq a_1 \leq 3, \\ a_2+2 & \text{for} \quad a_1 = 4, \end{cases}$$

$$\text{(ii)} \quad \mu_p(a_1, a_2) \leq 2a_2+2 \quad \text{for} \quad a_1 = 5.$$

Proof. We first consider (i). Clearly, $\mu_p(a_1, a_2) \geq a_2+1$. Hence, in order to show that $\mu_p(a_1, a_2) = a_2+1$ for $1 \leq a_1 \leq 3$, it suffices to give an example of a planar graph G of order a_2+1 such that $\mathscr{D}_G = \{a_1, a_2\}$. For $a_1 = 1$, the star $K(1, a_2)$ is the appropriate graph. For $a_1 = 2$, the complete tripartite graph $K(1, 1, a_2-1)$ is planar and has degree set $\{2, a_2\}$. For $a_1 = 3$, the "wheel" formed by joining a vertex to each vertex of a cycle of lenght a_2 has the desired properties.

Next, we verify the equality $\mu_p(4, a_2) = a_2+2$, where $a_2 > 4$. If $\mu_p(4, a_2) = a_2+1$, then there exists a planar graph G of order a_2+1 such that $\mathscr{D}_G = \{4, a_2\}$. Let v be a vertex of degree a_2 in G. Since v is adjacent to all other vertices of G, it follows that $G-v$ is outerplanar. However, every vertex of $G-v$ has degree at least 3, contradicting the fact that $G-v$ is outerplanar. Therefore, $\mu_p(4, a_2) \geq a_2+2$. To show that $\mu_p(4, a_2) = a_2+2$, we need only observe that the graph formed by joining two nonadjacent vertices to every vertex of a cycle of length a_2 has order a_2+2, is planar, and has degree set $\{4, a_2\}$.

Now we consider (ii). We construct a planar graph G of order $2a_2+2$ having degree set $\{5, a_2\}$ by beginning with disjoint cycles $C\colon u_1, u_2, \ldots, u_{a_2}, u_1$ and $C'\colon u_1', u_2', \ldots, u_{a_2}', u_1'$ such that for $i = 1, 2, \ldots, a_2$, $u_i u_i'$ and $u_i u_{i+1}'$ are edges of G (where the subscripts are expressed modulo a_2). The construction of G is completed by adding a vertex v adjacent to each vertex of C and adding a vertex v' adjacent to each vertex of C'. Thus, $\mu_p(a_1, a_2) \leq 2a_2+2$, for $a_1 = 5$.

We remark that it is not difficult to verify that equality holds in Theorem 4 (ii) for $a_2 = 6$. There are, of course, numerous sets S of positive integers for which $\mu_p(S)$ is not known. There appears to be no simple formula, however.

5. Degree sets for outerplanar graphs. In this final section, we discuss degree sets as they relate to outerplanar graphs. We begin with the following result.

THEOREM 5. *Let $S = \{a_1, a_2, \ldots, a_n\}$, $n \geq 1$, be a set of positive integers with $a_1 < a_2 < \ldots < a_n$. Then there exists an outerplanar graph G with $\mathscr{D}_G = S$ if and only if $a_1 = 1$ or $a_1 = 2$.*

Proof. As noted earlier, if G is an outerplanar graph, then G contains a vertex of degree at most two. Hence, if the positive integer a_1 is the minimum degree among the vertices of G, then $a_1 = 1$ or $a_1 = 2$.

Conversely, suppose $S = \{a_1, a_2, \ldots, a_n\}$, $n \geq 1$, is a set of positive integers such that $a_1 < a_2 < \ldots < a_n$ and $a_1 = 1$ or $a_1 = 2$. We show there exists an outerplanar graph G such that $\mathscr{D}_G = S$. If $a_1 = 1$, then, by Theorem 2, there exists a tree T (which is outerplanar) such that $\mathscr{D}_T = S$.

Next, suppose that $a_1 = 2$. For $i = 2, 3, \ldots, n$, we construct an outerplanar graph G_{a_i} as follows. If a_i is even, we begin with the graph $K(1, a_i)$. The graph G_{a_i} is then constructed by joining the a_i vertices of degree 1 in $K(1, a_i)$ in pairs, resulting in an addition of $\frac{1}{2}a_i$ edges. If a_i is odd, then G_{a_i} consists of two disjoint copies of G_{a_i-1} (just described) together with an edge joining the vertices of degree a_i-1. If we let

$$G = \bigcup_{\substack{a_i \in S \\ i \neq 1}} G_{a_i},$$

then G is outerplanar and $\mathscr{D}_G = S$.

On the basis of Theorem 5, we may make the following definition. Let $S = \{a_1, a_2, \ldots, a_n\}$, $n \geq 1$, be a set of positive integers such that $a_1 < a_2 < \ldots < a_n$, and $a_1 = 1$ or $a_1 = 2$. Then $\mu_0(S) = \mu_0(a_1, a_2, \ldots, a_n)$ denotes the minimum order of an outerplanar graph G for which $\mathscr{D}_G = S$. For $n = 1$, the situation is particularly easy, since $\mu_0(1) = 2$ and $\mu_0(2) = 3$. For $n = 2$, the results are given below.

THEOREM 6. (i) *For $a_2 > 1$, $\mu_0(1, a_2) = a_2+1$.* (ii) *For $a_2 > 2$,*

$$\mu_0(2, a_2) = \begin{cases} a_2+1 & \text{if } a_2 \text{ is even}, \\ 2a_2-2 & \text{if } a_2 \text{ is odd}. \end{cases}$$

Proof. For (i), we need only observe that the graph $K(1, a_2)$ is outerplanar, has order a_2+1, and has degree set $\{1, a_2\}$.

For (ii), we note that if a_2 is even, the graph G_{a_2} described in the preceding proof shows that $\mu_0(2, a_2) = a_2+1$. Now, if G is a graph with $\mathscr{D}_G = \{2, a_2\}$, where a_2 is odd, then G contains at least two vertices of degree a_2. Let u and v be vertices of G having degree a_2. There are at most two vertices in G which are mutually adjacent with u and v, since G is outerplanar. Because u and v may be adjacent, G contains at least $2a_2-2$ vertices. However, there exists an outerplanar graph G of order $2a_2-2$ with $\mathscr{D}_G = \{2, a_2\}$ (see Fig. 2); therefore, $\mu_0(2, a_2) = 2a_2-2$.

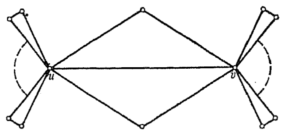

Fig. 2

We note in closing that $\mu_0(S)$ has been completely determined for $|S| = 3$, and the result will be presented elsewhere.

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Models of arithmetic and the 1-3-1 lattice

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Abstract. In this paper we show that if T is any complete theory in the language of number theory extending Peano's Axioms then there is a model M of T such that the 1–3–1 lattice can be embedded in the lattice of elementary substructures of M.

Introduction. Let T be a complete theory in the language of number theory extending Peano's Axioms. For M a model of T, let $\mathcal{S}(M)$ be the lattice of elementary substructures of M. In this paper we show that there is a model M of T such that the 1-3-1 lattice can be embedded in $\mathcal{S}(M)$.

This result continues investigations started in [1]. Related work also appears in [2] and we adopt the notation of that paper. Thus for M a model of T, $a_1, \ldots, a_n \in M$, $M[a_1, \ldots, a_n]$ is the smallest elementary substructure of M containing $a_1, \ldots, a_n$. Since M is a model of Peano's Axioms, $M[a_1, \ldots, a_n]$ consists exactly of those elements of M definable in M from $a_1, \ldots, a_n$.

THEOREM. *There is a model M of T such that the* 1-3-1 *lattice can be embedded in $\mathcal{S}(M)$.*

Proof. Fix M to be an ω_1-saturated model of T and identify N, the natural numbers, with an initial segment of M. We shall show that M satisfies the properties of the theorem.

Before proceeding further it will be useful to have the following crude estimate.

LEMMA 1. *Let $r, q \in M$, $s \in N$ and $s \geqslant 2$. Let x_i, y_i, $1 \leqslant i \leqslant q$ be sequences of elements of M definable in M and let*

$$\sum_{i=1}^{q} x_i = \sum_{i=1}^{q} y_i = r \quad (\textit{sums taken in } M)\,.$$

Then

$$\sum_{i=1}^{q} x_i y_i - (\textit{the sum of the } s \textit{ largest } x_i y_i) \leqslant \frac{r^2}{4(s-1)}\,.$$

* This paper was written when the author was working at Manchester University and the University of California, Berkeley.