Π^1_2 singletons and $O^{\#}$

by

Leo Harrington and **Alexander S. Kechris** (Pasadena, Cal.)

Abstract. A conjecture of Solovay states: Assuming that for every real α, $\alpha^{\#}$ exists, the constructibility degrees of Π^1_2 singletons are wellordered and the successor steps in this wellordering are given by the sharps. In this paper we prove among others things that (assuming $\forall\alpha$ ($\alpha^{\#}$ exists)) for every Π^1_2 singleton α either $O^{\#}$ is constructible from α or $\alpha^{\#}$ is constructible from $O^{\#}$. From a relativized version of this result it follows that the constructibility degrees of $O^{\#}, O^{\#\#}, O^{\#\#\#}, \ldots$ are the first ω constructibility degrees of *sharps* of Π^1_2 singletons.

§ 1. **Preliminaries.** Let $\omega = \{0, 1, 2, \ldots\}$ be the set of natural numbers and ${}^{\omega}\omega$ the set of all functions from ω to ω or for simplicity, *reals*. We use letters $i, j, k, \ldots$ to denote natural numbers and $\alpha, \beta, \gamma, \ldots$ to denote reals.

We shall use without explicit reference standard facts about the theory of indiscernibles for the models $L[\alpha]$, as developed in [4] and [5]. At a key point in our proof in § 2 we shall nevertheless use a recent result of Paris [3] which we now proceed to review. Let $\mathscr{I}^{\alpha}$ denote the class of Silver Indiscernibles for $\mathrm{L}[\alpha]$ and $\{i^{\alpha}_{\xi}\}_{\xi\in\mathrm{ORD}}$ its increasing enumeration. We omit the superscripts if $\alpha \in \mathrm{L}$. Let ν^{α}_{ξ} = order type of $\mathscr{I} \cap (i^{\alpha}_{\xi}, i^{\alpha}_{\xi+1})$, where for any ordinals $\varkappa < \lambda$, $(\varkappa, \lambda) = \{\xi \colon \varkappa < \xi < \lambda\}$. We then have

THEOREM (Paris [3]). *Assume for all α, $\alpha^{\#}$ exists. If for some ξ, η $\nu^{\alpha}_{\xi} \neq \nu^{\alpha}_{\eta}$, then* $O^{\#} \in \mathrm{L}[\alpha]$.

§ 2. **The Main Lemma.** Our results will follow easily from a main lemma which we shall establish in this section and which seems to be interesting in its own right. We need first some terminology and notation.

A *tree* T on a set X is a set of finite sequences from X closed under subsequences. A *path* through T is a sequence $f \in {}^{\omega}X$ such that for every n, $(f(0), f(1), \ldots, f(n)) \in T$. We denote the set of all paths through T by $[T]$. If X is of the form $Y \times Z$, we represent a path $f \in {}^{\omega}(Y \times Z)$ through T by the unique pair $(g, h) \in {}^{\omega}Y \times {}^{\omega}Z$ such that for each n, $f(n) = (g(n), h(n))$. We then let

$$p[T] = \{g \colon \exists h (g, h) \in [T]\} = \textit{first projection of } [T].$$

THE MAIN LEMMA. *Assume* $\forall\alpha(\alpha^\#$ *exists*). *Let* $\varphi(x)$ *be a formula of set theory. Then for some tree* T *on* $\omega\times\lambda$ (*where* λ *is some ordinal*), $T\in L[O^\#]$, *we have*

$$p[T]=\{\alpha^\#:\ L[\alpha]\models\varphi(\alpha)\ \&\ O^\#\notin L[\alpha]\}.$$

Proof. Let ZFL($\dot\alpha$) be the theory in the language of set theory, with a constant $\dot\alpha$ added, which contains the axioms of ZF and also the two axioms: $\dot\alpha\in{}^\omega\omega$ and $V=L[\dot\alpha]$. Let $\tau_0,\tau_1,\ldots$ be a recursive enumeration of all the definable terms in ZFL($\dot\alpha$) and assume τ_i has the n_i free variables $v_1,v_2,\ldots,v_{n_i}$. If φ is a formula in the language of ZFL($\dot\alpha$), we denote by $\ulcorner\varphi\urcorner$ its Gödel number. For each α, $\alpha^\#\in{}^\omega 2$ and $\alpha^\#(\ulcorner\psi\urcorner)=0\Leftrightarrow L[\alpha]\models\psi(\aleph_1\ldots\aleph_n)$, if ψ has the free variables $v_1\ldots v_n$.

Let $\ulcorner\varphi(\dot\alpha)\urcorner=k_0$ and assume n_0 is such that:

$$O^\#\notin L[\alpha]\Leftrightarrow\alpha^\#(n_0)=0.$$

Finally let J be a recursive tree on ω such that $\beta\in[J]\Leftrightarrow\beta$ satisfies all the syntactical properties for being a remarkable (with respect to some real) character; see for example [4]. If $\beta\in[J]$, then we denote by $\Gamma(\beta,\aleph_1)$ the model generated by $\aleph_1$ indiscernibles on the basis of β. Thus

$$\exists\alpha(\beta=\alpha^\#)\Leftrightarrow\beta\in[J]\ \&\ \Gamma(\beta,\aleph_1)\text{ is well-founded}.$$

Define now the following tree T *in* $L[O^\#]$, where $\aleph_1$ is the first uncountable ordinal *in the world* and $[\aleph_1]^n=\{(\xi_1\ldots\xi_n):\ \xi_1<\ldots<\xi_n<\aleph_1\}$:

$$\big((\beta(0),f_0),\ldots,(\beta(n),f_n)\big)\in T\Leftrightarrow$$

(i) $(\beta(0),\ldots,\beta(n))\in J\ \&\ (k_0\leqslant n\Rightarrow\beta(k_0)=0)\ \&\ (n_0\leqslant n\Rightarrow\beta(n_0)=0)$,

(ii) $\forall i\leqslant n\ (f_i:\ [\aleph_1]^{n_i}\to\aleph_1)$,

(iii) $\forall i,j\leqslant n\,\exists\mathscr{C}\ (\mathscr{C}\in L[O^\#]\ \&\ \mathscr{C}\subseteq\aleph_1$ is closed and unbounded and for any $\xi_1<\ldots<\xi_{n_i},\eta_1<\ldots<\eta_{n_j}$ all in $\mathscr{C}$:

$$f_i(\xi_1\ldots\xi_{n_i})\lesseqgtr f_j(\eta_1\ldots\eta_{n_j})\Leftrightarrow\beta(\ulcorner\tau_i(\vec v)\lesseqgtr\tau_j(\vec v\,')\urcorner)=0),$$

where $\vec v,\vec v\,'$ are appropriate sequences of variables interwoven the same way as $\xi_1\ldots\xi_{n_i},\eta_1\ldots\eta_{n_j}$ and $\ulcorner\tau_i(\vec v)\lesseqgtr\tau_j(\vec v\,')\urcorner\leqslant n$.

We shall prove now that

$$p[T]=\{\alpha^\#:\ L[\alpha]\models\varphi(a)\ \&\ O^\#\notin L[\alpha]\},$$

which completes the proof of the lemma, since T can be easily replaced by a tree on $\omega\times\lambda$ for some λ, with the same first projection.

Let $\beta\in p[T]$. To prove that $\beta=\alpha^\#$ where $L[\alpha]\models\varphi(\alpha)\ \&\ O^\#\notin L[\alpha]$ we only have to show that $\Gamma(\beta,\aleph_1)$ is well founded. Let $\{c_\xi\}_{\xi<\aleph_1}$ be generating indiscernibles for $\Gamma=\Gamma(\beta,\aleph_1)$. It is enough to find $\mathscr{C}\subseteq\aleph_1$ unbounded such that the Skolem Hull of $\{c_\xi:\ \xi\in\mathscr{C}\}$ in Γ is well founded. Pick f such that $(\beta,f)\in[T]$. For each n and each $i,j\leqslant n$ pick $\mathscr{C}^n_{i,j}$ a closed unbounded subset of $\aleph_1$ which demonstrates that $\big((\beta(0),f_0),\ldots,(\beta(n),f_n)\big)\in T$. Then if $\mathscr{C}=\bigcap_n\bigcap_{i,j\leqslant n}\mathscr{C}^n_{i,j}$, the mapping

$$f(\tau_i^\Gamma(c_{\xi_1}\ldots c_{\xi_{n_i}}))=f_i(\xi_1\ldots\xi_{n_i})$$

(where $\tau_i^\Gamma(\ldots)$ is the interpretation of τ_i in Γ) proves that the Skolem Hull of $\{c_\xi:\ \xi\in\mathscr{C}\}$ is well founded.

Conversely assume $L[\alpha]\models\varphi(\alpha)\ \&\ O^\#\notin L[\alpha]$ and let $\beta=\alpha^\#$. Since $O^\#\notin L[\alpha]$ Paris' theorem (see § 1) implies that $(i^\alpha_\xi,i^\xi_{\xi+1})\cap\mathscr{I}$ has fixed order type $\nu<\aleph_1$. Consider now τ_i and assume, to simplify the notation, that $n_i=1$. Then since

$$\tau_i^{L[\alpha]}(i^\alpha_{\aleph_1})<i^\alpha_{\aleph_1+1},$$

we can find a finite sequence $\vec\sigma$ of countable indiscernibles for L, a finite sequence $\vec\aleph$ of large enough cardinals and $\delta_1<\delta_2<\ldots<\delta_k<\nu$ such that if $(\varkappa)_\delta=$ the δth indiscernible of L bigger than $\varkappa$, then for some i^*

$$\tau_i^{L[\alpha]}(i^\alpha_{\aleph_1})=\tau^L_{i^*}(\vec\sigma,i^\alpha_{\aleph_1},(i^\alpha_{\aleph_1})\delta_1,\ldots,(i^\alpha_{\aleph_1})\delta_k,\vec\aleph).$$

Then one can check (see for example Paris [3]) that for all large enough countable ξ_1 we have

$$\tau_i^{L[\alpha]}(i^\alpha_{\xi_1})=\tau^L_{i^*}(\vec\sigma,i^\alpha_{\xi_1},(i^\alpha_{\xi_1})\delta_1\ldots(i^\alpha_{\xi_1})\delta_k,\vec\aleph).$$

So if we put

$$f_i(\xi_1)=\tau^L_{i^*}(\vec\sigma,i^\alpha_{\xi_1},(i^\alpha_{\xi_1})\delta_1,\ldots,(i^\alpha_{\xi_1})\delta_k,\vec\aleph),$$

clearly $f_i\in L[O^\#]$ and $\big((\beta(0),f_0),(\beta(1),f_1),\ldots\big)\in[T]$; so $\beta\in p[T]$. ∎

A basic consequence of our lemma is of course the following (assuming $\forall\alpha(\alpha^\#$ exists)): If φ is a formula of set theory and $(\exists\alpha)(L[\alpha]\models\varphi(\alpha)\ \&\ O^\#\notin L[\alpha])$, then $(\exists\alpha)(L[\alpha]\models\varphi(\alpha)\ \&\ \alpha^\#\in L[O^\#])$. For example, if there is a nongeneric real α with $O^\#\notin L[\alpha]$, then there is one such in $L[O^\#]$. Solovay has conjectured that no such real exists.

2. $O^\#$ and Π^1_2 singletons. We now apply our main lemma to get some information about the constructibility degrees of Π^1_2 singletons, which partially confirms Solovay's conjecture. Put for convenience

$$\alpha\leqslant_c\beta\Leftrightarrow\alpha\in L[\beta],$$

$$\alpha=_c\beta\Leftrightarrow\alpha\leqslant_c\beta\ \&\ \beta\leqslant_c\alpha.$$

THEOREM. *Assume* $\forall\alpha(\alpha^\#$ *exists*). *If* α *is a* Π^1_2 *singleton then* $O^\#\leqslant_c\alpha$ *or* $\alpha^\#=_c O^\#$.

THEOREM. *Assume* $\forall\alpha(\alpha^\#$ *exists*). *The constructibility degrees of* $O^\#$, $O^{\#\#}$, $O^{\#\#\#}$, ... *are the first* ω *constructibility degrees of* $\{\alpha^\#:\ \{\alpha\}\in\Pi^1_2\}$.

The proofs of these two results are easy consequences of our main lemma and its obvious relativization. The second of them seems to be a very strong evidence for the truth of the straightforward modification of Solovay's conjecture which asserts that the constructibility degrees of sharps of Π^1_2 singletons are well ordered, with the successor steps given by the sharps. In fact, using an extension of our result, Robert Van Wesep has proved that this modification of Solovay's conjecture is consistent with $\forall\alpha(\alpha^\#$ exists) (provided $\forall\alpha(\alpha^\#$ exists) is consistent).

The next theorem follows from our main lemma, Mansfield's theorem on perfect sets [2], and a result of Friedman [1]. Recall that a set of reals is called *thin* if it contains no perfect subset.

THEOREM. *Assume* $\forall\alpha(\alpha^{\#}$ *exists*). *Let* $A\subseteq{}^{\omega}\omega$ *be* Π^1_2 *and put* $A^* = \{\alpha^{\#}: \alpha\in A \;\&\; O^{\#}\notin \mathrm{L}[\alpha]\}$. *Then*

(i) A^* *is thin* $\Leftrightarrow A^*$ *is countable* $\Leftrightarrow A^*\subseteq \mathrm{L}[O^{\#}]$,

(ii) $\emptyset\neq A^*$ *is countable* $\Rightarrow A^*$ *contains a* Π^1_2 *singleton.*

Proof. (i) By Mansfield's Theorem [2] if M is a standard model of set theory and T (a tree on $\omega\times\lambda$) is in M and $p[T]$ is thin then $p[T]\subseteq M$. Since $A^* = p[T]$ with $T\in \mathrm{L}[O^{\#}]$ (by our Main Lemma) if A^* is thin then $A^*\subseteq \mathrm{L}[O^{\#}]$, thus A^* is countable.

(ii) Find an integer n_0 such that

$$(*)\qquad \gamma\in A^*\Leftrightarrow f(\gamma)\in A \;\&\; \gamma = (f(\gamma))^{\#} \;\&\; \gamma(n_0) = 0\,,$$

where $f(\gamma)$ is a total recursive function such that if $\gamma=\alpha^{\#}$ then $f(\gamma)=\alpha$ (clearly n_0 is the Gödel number of a sentence σ such that $O^{\#}\notin \mathrm{L}[\alpha]\Leftrightarrow \mathrm{L}[\alpha]\models\sigma$). If $\emptyset\neq A^*$ is countable, then by (i) $A^*\subseteq \mathrm{L}[O^{\#}]$ and clearly A^* is also defined in $\mathrm{L}[O^{\#}]$ by the Π^1_2 formula (*) as above. By a result of Friedman [1] every subset of $\mathrm{L}[O^{\#}]$ which is Π^1_2 in $\mathrm{L}[O^{\#}]$ contains a real γ_0 such that γ_0 is (in $\mathrm{L}[O^{\#}]$) a Π^1_2 singleton in $O^{\#}$, i.e., there is a Π^1_2 predicate $P(\gamma,\delta)$ such that $\mathrm{L}[O^{\#}]\models\gamma_0 =$ the unique γ such that $P(\gamma, O^{\#})$. Now there is a total recursive function g such that $g(\alpha^{\#}) = O^{\#}$ for all α; thus $\mathrm{L}[O^{\#}]\models\gamma_0 =$ the unique γ such that $(\gamma\in A^* \;\&\; P(\gamma, g(\gamma)))$. But then $\gamma_0 =$ the unique γ such that $\gamma\in A^*$ and $P(\gamma, g(\gamma))$, since if for some γ' we also have $\gamma'\in A^*$ and $P(\gamma', g(\gamma'))$, then $\gamma'\in \mathrm{L}[O^{\#}]$, so by absoluteness $\mathrm{L}[O^{\#}]\models\gamma'\in A^* \;\&\; P(\gamma', g(\gamma))$, thus $\gamma'=\gamma_0$. So γ_0 is a Π^1_2 singleton and we are done. ∎

Part (ii) of the above theorem seems to be relevant to the following open problem: Does every countable Π^1_2 set contain a Π^1_2 singleton?

§ 4. Some final remarks. It follows easily from Paris' theorem that if $\lambda^{\alpha} = i^{\alpha}_{\aleph_1+\aleph_1}$ then

$$(*)\qquad \alpha\leqslant_c\beta\Rightarrow[\lambda^{\alpha}<\lambda^{\beta}\Leftrightarrow\alpha^{\#}\leqslant_c\beta]\,,$$

i.e., the assignment $\alpha\to\lambda^{\alpha}$ satisfies the "Spector Criterion" for constructibility degrees, where sharps play the role of jumps. One can nevertheless use a much smaller ordinal assignment, namely

$\lambda^{\alpha} =$ next cardinal in $\mathrm{L}[\alpha]$ beyond (the true) $\aleph_1$.

That this works is immediate from the following (unpublished) result of Kunen: If $\varkappa$ is weakly compact and $\varkappa^+>(\varkappa^+)^{\mathrm{L}}$, then $O^{\#}$ exists. We do not know if λ^{α} can be still lowered so that it satisfies the Spector criterion, for example if we can take $\lambda^{\alpha} =$ next α-admissible beyond $\aleph_1$. It seems in any case to us that the use of a suitable assignment $\alpha\to\lambda^{\alpha}$ satisfying (*) may be instrumental in a positive solution of Solovay's conjecture.

Added in proof. In view of recent work of Jensen, Solovay now feels that his conjecture is most likely false. Jensen has also disproved the conjecture stated at the end of § 1 (granting $O^{\#}$ exists).

References

[1] H. Friedman, *A basis theorem for* L, mimeographed notes.

[2] M. Mansfield, *Perfect subsets of definable sets of real numbers*, Pacific J. Math. 35 (1970), pp. 451–457.

[3] J. Paris, *Patterns of indiscernibles*, Bull. London Math. Soc. 6 (1974), pp. 183–188.

[4] J. Silver, *Some applications of model theory to set theory*, Ann. Math. Logic 3 (1971), pp. 45–110.

[5] R. M. Solovay, *A non-constructible* Δ^1_3 *set of integers*, Transactions of the Amer. Math. Soc. 127 (1967), pp. 50–75.

MASS. INSTITUTE OF TECHNOLOGY AND STATE UNIVERSITY OF NEW YORK AT BUFFALO
MASS. INSTITUTE OF TECHNOLOGY AND CALIFORNIA INSTITUTE OF TECHNOLOGY

Accepté par la Rédaction le 1. 4. 1975