II\textsuperscript{1} singletons and $O^*$

by

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Abstract. A conjecture of Solovay states: Assuming that for every real $a$, $a^*$ exists, the constructibility degrees of $\II^1$ singletons are wellordered and the successor steps in this wellordering are given by the sharps. In this paper we prove among other things that (assuming $\exists a (a^* \text{ exists})$) for every $\II^1$ singleton $a$ either $O^*$ is constructible from $a$ or $a^*$ is constructible from $O^*$. From a relativized version of this result it follows that the constructibility degrees of $O^*, O^{**}, O^{***}, ...$ are the first $\omega$ constructibility degrees of sharps of $\II^1$ singletons.

§ 1. Preliminaries. Let $\omega = \{0, 1, 2, ...\}$ be the set of natural numbers and $^{\omega}\omega$ the set of all functions from $\omega$ to $\omega$ or for simplicity, reals. We use letters $i, j, k, ...$ to denote natural numbers and $a, b, c, ...$ to denote reals.

We shall use without explicit reference standard facts about the theory of indiscernibles for the models $L[a]$, as developed in [4] and [5]. At a key point in our proof in § 2 we shall nevertheless use a recent result of Paris [3] which we now proceed to review. Let $\mathfrak{A}$ denote the class of Silver Indiscernibles for $L[a]$ and $\{a_\xi\}_{\xi \in \text{ORD}}$ its increasing enumeration. We omit the superscripts if $a \in L$. Let $v^*_\xi = \text{order type of } \mathfrak{A} \cap (a_\xi, a_\xi+1)$, where for any ordinals $x < \lambda$, $(x, \lambda) = \{\xi: x < \xi < \lambda\}$. We then have

\textbf{Theorem} (Paris [3]). Assume for all $a$, $a^*$ exists. If for some $\xi, \eta v^*_\xi \neq v^*_\eta$, then $O^* \in L[a]$.

§ 2. The Main Lemma. Our results will follow easily from a main lemma which we shall establish in this section and which seems to be interesting in its own right. We need first some terminology and notation.

A tree $T$ on a set $X$ is a set of finite sequences from $X$ closed under subsequences. A path through $T$ is a sequence $f \in {}^nX$ such that for every $n$, $(f(0), f(1), ..., f(n)) \in T$. We denote the set of all paths through $T$ by $[T]$. If $X$ is of the form $Y \times Z$, we represent a path $f \in {}^n(Y \times Z)$ through $T$ by the unique pair $(g, h) \in {}^nY \times {}^nZ$ such that for each $n$, $f(n) = (g(n), h(n))$. We then let

\[ p[T] = \{g: \exists h (g, h) \in [T]\} = \text{first projection of } [T]. \]
The Main Lemma. Assume $\forall s(α^s \exists x)$. Let $φ(s)$ be a formula of set theory. Then for some tree $T$ on $ω \times λ$ (where $λ$ is some ordinal), $T ∈ \mathbb{L}[O^s]$, we have

$$p[T] = \{x^α : L[x] ⊨ φ(α) & O^s \not\subseteq L[x]\}.$$

Proof. Let $ZF[\alpha]$ be the theory in the language of set theory, with a constant $\alpha$ added, which contains the axioms of $ZF$ and also the two axioms: $\forall α \in ω$ and $V = L[\alpha]$. Let $τ_0, τ_1, ...$ be a recursive enumeration of all the definable terms in $ZF[\alpha]$ and assume $τ_1$ has the $n_1$ free variables $v_1, v_2, ..., v_{n_1}$. If $φ$ is a formula in the language of $ZF[\alpha]$, we denote by $⌜φ⌝$ its Gödel number. For each $α, α^s ∈ 2^ω$ and $α^s(⌜φ⌝) = 0 = L[\alpha][α] ⊨ φ(v_1, ..., v_{n_1})$, if $ψ$ has the free variables $v_1, ..., v_{n_1}$.

Let $φ(α^s) = k_0$ and assume $n_0$ is such that

$$O^s \not\subseteq L[x] ⇔ α^s(n_0) = 0.$$

Finally let $J$ be a recursive tree on $ω$ such that $β ∈ J[α] ⇔ β$ satisfies all the syntactical properties for being a remarkable (with respect to some real) character; see for example [8]. If $β ∈ J[α]$, then we denote by $Γ(β, x)$ the model generated by $x_0$ indiscernibles on the basis of $β$. Thus

$$∃x(β = α^s) ⇔ β ∈ J[α] ⊆ Γ(β, x) \text{ is well-founded}.$$

Define now the following tree $T$ in $L[O^s]$, where $x_0$ is the first uncountable ordinal in the world and $[n_0]^α = \{τ_1 < ... < τ_{n_0} < τ\}$.

$$[β(0), f_0], ..., [β(ν), f_0] ∈ T ⇔$$

(i) $[β(0), ... , β(ν)] ∈ J & (ν ≤ n_0 ⇒ β(ν) = 0) & (ν < n_0 ⇒ β(ν) = n_0)$,

(ii) $∀v ∈ [f_0], [x_0]^α \models v ∈ O^s \& \forall v \models [x_0]^α$,

(iii) $∀v ∈ [f_0], [x_0]^α \models v ∈ O^s \& \forall v \models [x_0]^α$.

Then one can check (see for example Paris [3]) that for all large enough countable $τ_1$ we have

$$Γ(β, x) \models [f(τ_1), ..., f(τ_{n_0})] = \tau_0(τ_1, ..., τ_{n_0}) ≡ [f(τ_1), ..., f(τ_{n_0})].$$

So if we put

$$f(τ_1) = f(τ_1), ..., f(τ_{n_0}) = (τ_1), ..., (τ_n),$$

clearly $f ∈ L[O^s]^α$ and $[β(0), f_0], [β(1), f_1], ... \in T$; so $β ∈ p[T].$

A basic consequence of our lemma is of course the following (assuming $\forall α(α^s \exists x)$); if $φ$ is a formula of set theory and $(∃x)(L[x] ⊨ φ(α) & O^s \not\subseteq L[x])$,

then $∃x(L[x] ⊨ φ(α) & α^s \in L[O^s])$. For example, if there is a nongeneric real $α$ with $O^s \not\subseteq L[x]$, then there is one such in $L[O^s]$. Solovay has conjectured that such real exists.

2. $O^s$ and $I_1^s$ singletons. We now apply our main lemma to get some information about the constructibility degrees of $I_1^s$ singletons, which partially confirms Solovay’s conjecture. Put for convenience

$$α ≜ β = α \in L[β],$$

$$α = β = α \leq β = β =_α α.$$
The next theorem follows from our main lemma, Mansfield’s theorem on perfect sets [2], and a result of Friedman [1]. Recall that a set of reals is called thin if it contains no perfect subset.

**Theorem.** Assume $\text{Va}(a^*)$ exists. Let $A \subseteq \omega$ be $\Pi^1_2$ and put $A^* = \{a^* : a \in A \& O^* \neq L[a]\}$. Then

(i) $A^*$ is thin $\iff A^*$ is countable $\iff A^* \subseteq L[O^*]$,

(ii) $\emptyset \notin A^*$ is countable $\iff A^*$ contains a $\Pi^1_2$ singleton.

**Proof.** (i) By Mansfield’s Theorem [2] if $M$ is a standard model of set theory and $T$ (a tree on $\omega \times \omega$) is in $M$ and $p[T]$ is thin then $p[T] \subseteq M$. Since $A^* = p[T]$ with $T \subseteq L[O^*]$ (by our main lemma) if $A^*$ is thin then $A^* \subseteq E[L[O^*]]$, thus $A^*$ is countable.

(ii) Find an integer $n_0$ such that

$$\gamma \in A^* \iff f(\gamma) \in A \& \gamma = (f(\gamma))^2 \& \gamma(n_0) = 0,$$

where $f(\gamma)$ is a total recursive function such that if $\gamma = a^*$ then $f(\gamma) = a$ (clearly $n_0$ is the Gödel number of a sentence $c$ such that $O^* \neq L[a] \equiv L[a] \vdash c$). If $\emptyset \notin A^*$ is countable, then by (i) $A^* \subseteq E[L[O^*]]$ and clearly $A^*$ is also defined in $L[O^*]$ by the $\Pi^1_2$ formula ($\ast$) above. By a result of Friedman [1] every subset of $L[O^*]$ which is $\Pi^1_2$ in $L[O^*]$ contains a real $\gamma_0$ such that $\gamma_0$ is in $L[O^*]$; a $\Pi^1_2$ singleton in $O^*$, i.e., there is a $\Pi^1_2$ predicate $P(\gamma, \delta)$ such that $L[O^*] \vdash \delta = \gamma$ the unique $\gamma$ such that $P(\gamma, O^*)$. Now there is a total recursive function $g$ such that $g(a^*) = O^*$ for all $a^*$; thus $L[O^*] \vdash g_0 = \gamma_0$ is the unique $\gamma$ such that $\gamma \in A^* \& P(\gamma, g(\gamma))$. But then $g_0 = \gamma$ is the unique $\gamma$ such that $\gamma \in A^* \& P(\gamma, g(\gamma))$, since if for some $\gamma'$ we also have $\gamma' \in A^* \& P(\gamma', g(\gamma'))$, then $\gamma' \in L[O^*]$, so by absoluteness $L[O^*] \vdash \gamma' = \gamma_0$. So $\gamma_0$ is a $\Pi^1_2$ singleton and we are done.

Part (ii) of the above theorem seems to be relevant to the open problem: Does every countable $\Pi^1_2$ set contain a $\Pi^1_2$ singleton?

### § 4. Some final remarks.  

It follows easily from Paris’ theorem that if $\lambda^* = \delta_{\text{H}}$, then

$$\varphi \Rightarrow \varphi \iff \varphi \Rightarrow \varphi \iff \varphi \Rightarrow \varphi ,$$

i.e., the assignment $\varphi \mapsto \lambda^*$ satisfies the “Spector Criterion” for constructibility degrees, where sharps play the role of jumps. One can nevertheless use a much smaller ordinal assignment, namely

$$\lambda^* = \text{next cardinal in } L[\varphi] \text{ beyond (the true) } \omega_1 .$$

That this works is immediate from the following (unpublished) result of Kunen:

If $\varphi$ is weakly compact and $\varphi^+ > (\varphi^+)^\varphi$, then $\text{O}^*$ exists. We do not now if $\lambda^*$ can be still lowered so that it satisfies the Spector criterion, for example if we can take $\lambda^* = \text{next } \varphi\text{-admissible beyond } \omega_1$. It seems in any case to us that the use of a suitable assignment $\varphi \mapsto \lambda^*$ satisfying $(\ast)$ may be instrumental in a positive solution of Solovay’s conjecture.

**References**


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