Dimension and decompositions *

by

Craig F. Zemke (Moscow, Idaho)

Abstract. Suppose that G is an upper semi-continuous decomposition of an n-manifold X. In this paper we investigate conditions that are sufficient to ensure that the dimension of X/G is less than or equal to n, or is at least finite. We obtain the following results:

1) If G is a nondegenerately continuous decomposition of E^n into points and polyhedral sets, then $\dim E^n/G \leqslant 2n+1$.

2) If G is a monotone upper semi-continuous decomposition of a 2-manifold X (with or without boundary), then $\dim X/G \leqslant 2$.

3) If G is a cellular decomposition of E^3 into points and polyhedral sets, then $\dim E^3/G \leqslant 3$.

4) If G is a monotone upper semi-continuous decomposition of E^n into convex sets, then $\dim E^n/G \leqslant n$.

5) If G is an upper semi-continuous decomposition of E^n into points and compact $(n-1)$-manifolds, then $\dim E^n/G \leqslant n+1$.

1. Introduction. Hurewicz has shown that if X is any compact metric space, then there exists a monotone upper semi-continuous decomposition of E^3, such that X may be embedded in the resulting decomposition space [5]. Anderson has given an example of a monotone continuous decomposition of a compact 1-dimensional subset of E^3 such that the resulting decomposition space is the Hilbert cube [1]. Consequently, monotone upper semi-continuous, or even monotone continuous decompositions can raise dimension considerably.

A result of Dyer [2] shows that if G is an acyclic decomposition of a compact n-dimensional metric space X and if $\dim X/G<\infty$, then $\dim X/G\leqslant n$. This result can be generalized to locally compact metric spaces and in fact it can be shown that if X/G is a σ_0-space then $\dim X/G\leqslant n$ [3].

The purpose of this paper is to establish upper bounds for the dimension of the decomposition space without requiring that the decomposition space be finite dimensional or even a σ_0-space.

2. Definitions and notation. Suppose that G is an upper semi-continuous decomposition of a topological space X. Then H_G will denote the collection of non-

(*) This paper forms a portion of the author's doctoral dissertation, prepared under the supervision of Professor W. Voxman at the University of Idaho.

degenerate elements of G and H_G^* the union of the members of H_G. Furthermore, X/G will denote the associated decomposition space and $p\colon X \to X/G$ will denote the natural projection. Sets that are unions of members of G are said to be *saturated with respect to* G, or when the context is clear simply *saturated.* The *saturated interior* of a set U is $\bigcup \{g \in G |\ g \subset \operatorname{Int} U\}$.

Suppose that X is a metric space and G is an upper semi-continuous decomposition of X. Then G is said to be *continuous at* g if for each positive number ε, there exists an open set V in X containing g such that if $g' \in G$ and $g' \cap V \neq \emptyset$, then g is contained in the ε-neighborhood of g', and g' is contained in the ε-neighborhood of g. The decomposition G is *continuous* if G is continuous at each $g \in G$, and G is *nondegenerately continuous* if H_G is continuous at each $g \in G$.

E^n will denote Euclidean n-space, B^n the unit n-ball, $\{x \in E^n |\ \|x\| \leqslant 1\}$ in E^n, and S^{n-1} the unit sphere, $\{x \in E^n |\ \|x\| = 1\}$. An *$n$-cell* is any space homeomorphic to B^n and an *open n-cell* is any space homeomorphic to the interior of B^n.

If M is a manifold, by a *triangulation* of M is meant a simplicial complex T such that 1) $M = \bigcup \{t |\ t \in T\}$ and 2) T is locally finite in the sense that each point of M has a neighborhood which intersects only finitely many of the sets of T. If x is a vertex of the triangulation, then the *star of* x $(\operatorname{St}(x, T))$ is the union of all simplexes of T having x as a vertex, with the faces opposite x removed.

If M is an n-manifold and $g \subset M$, then g is said to be *cellular in* M (or *cellular*) if there exists a sequence of n-cells $C_1, C_2, \ldots$ in M such that $C_i \subset \operatorname{Int} C_{i-1}$ and $\bigcap_{i=1}^{\infty} C_i = g$. An upper semi-continuous decomposition is *cellular* if for each $g \in G$, g is cellular in M. An upper semi-continuous decomposition G is *monotone* if for each $g \in G$, g is compact and connected. An upper semi-continuous decomposition is *acyclic* if for each $g \in G$ and for each nonnegative integer k, $H_k(g) = 0$ ($H_k(g)$ is the kth Čech homology group of g).

Small inductive dimension will be denoted dim. A separable metric space is σ_0 if it can be written as a countable union of 0-dimensional spaces ([8], p. 83).

Suppose that O_1 and O_2 are coverings of a space X. O_1 is said to *refine* (or *be a refinement of*) O_2 if for each $U \in O_1$, there exists $V \in O_2$ such that $U \subset V$. Let O be a covering of the space X and $A \subset X$. Then the *star of A with respect to* O, written $\operatorname{St}(A, O)$, is $\bigcup \{U \in O |\ A \cap U \neq \emptyset\}$. If O_1 and O_2 are coverings of X, then O_1 is a *star refinement* of O_2 if the covering $\{\operatorname{St}(U, O_1) |\ U \in O_1\}$ refines O_2.

If (X, d) is a metric space and $A \subset X$, then the *diameter* of A ($\operatorname{diam} A$ (is $\sup \{d(x, y) |\ x, y \in A\}$. A continuous function f of a metric space (X, d) into a space Y is an ε-mapping ($\varepsilon > 0$) if for each $y \in Y$, $\operatorname{diam} f^{-1}(y) < \varepsilon$. $S(x)$ will denote the ε-neighborhood of x, $\{y \in X |\ d(x, y) < \varepsilon\}$.

A set A is *dense in itself* if for every $x \in A$, x is an accumulation point of $A - \{x\}$.

3. Decompositions into convex sets. A set S in a real linear space X is said to be *convex* if for every pair of points $\{x, y\}$ in S and for any real number α with $0 \leqslant \alpha \leqslant 1$, $\alpha x + (1-\alpha)y$ is an element of S. If S is a convex subset of X, then z is an *extreme point* of S if whenever $z = \alpha x + (1-\alpha)y$ with $\{x, y\} \subset S$ and $0 \leqslant \alpha \leqslant 1$, then $\alpha \in \{0, 1\}$. The *convex hull* of a set S is the intersection of all convex sets containing S and the *closed convex hull* of S is the intersection of all closed convex sets containing S.

Recall that if S is a subset of E^n, then the closed convex hull of S is the closure of the convex hull of S. If S is compact, then the closed convex hull of S and the convex hull of S are the same. Furthermore, the closed convex hull of S is the closed convex hull of its extreme points ([1], pp. 17, 40, 138).

Suppose that g is a compact subset of E^n. Then g' will denote the set of extreme points of the convex hull (which is the same as the closed convex hull) of g. By ([11], p. 138), $g' \neq \emptyset$ if $g \neq \emptyset$ and it is easy to see that $g' \subset g$.

THEOREM 3.1. *Suppose that G is a monotone nondegenerately continuous decomposition of E^n such that for all $g \in g'$ is not dense in itself. Then* $\dim E^n/G \leqslant 2n+1$.

Proof. The proof is based on the following two lemmas.

LEMMA 1. *Suppose that G is a monotone nondegenerately continuous decomposition of E^n and that $\{g_i\}_{i=1}^{\infty}$ is a sequence of elements of H_G converging to $g \in H_G$. Let $x \in g'$. Then there exists a sequence $\{x_{n_i}\}_{i=1}^{\infty}$ converging to x where for each i, $x_{n_i} \in g'_{n_i}$ and $\{g_{n_i}\}_{i=1}^{\infty}$ is a subsequence of $\{g_i\}_{i=1}^{\infty}$.*

Proof. Since G is nondegenerately continuous, there exists a sequence $\{x_i\}_{i=1}^{\infty}$ converging to x, where $x_i \in g_i$ for each i. The closed convex hull of g'_i is the same as the convex hull of g_i. Since $g_i \subset$ in its closed, convex hull, $x_i \in$ closed convex hull of g'_i. Then by ([11], p. 15) for each i, there exists a simplex Δ_i containing x'_i, such that the vertices of Δ_i lie in g'_i. Since $\Delta_i \subset E^n$, there exists a subsequence $\{\Delta_{n_i}\}_{i=1}^{\infty}$ of $\{\Delta_i\}_{i=1}^{\infty}$ such that for all i, $\dim \Delta_{n_i} = k$ for some integer k. Denote the vertices of Δ_{n_i} by $\{x_0^i, x_1^i, \ldots, x_k^i\}$ with $x_j^i \in g'_{n_i}$ for all i and j. Since G is upper semi-continuous, there exists a subsequence of $\{x_0^i\}_{i=1}^{\infty}$ that converges to some element x_0 of g. By continuing to choose appropriate subsequences whenever required and reindexing when necessary, we may assume that for each j the sequence $\{x_j^i\}_{i=1}^{\infty}$ converges to a point x_j in g. Clearly then x lies in the convex hull of $\{x_0, x_1, \ldots, x_k\}$, which is contained in the convex hull of g. Since x is an extreme point of the convex hull of g, it follows that $x = x_j$ for some j, and the lemma is established.

LEMMA 2. *Suppose that G is a monotone nondegenerately continuous decomposition of E^n and $p\colon E^n \to E^n/G$ is the natural projection. Let $X = \bigcup \{g' |\ g \in H_G\}$. Then $q = p|_X\colon X \to p(H_G^*)$ is open and onto.*

Proof. Lemma 2 follows easily from Lemma 1.

Proof of Theorem 3.1. Since G is a monotone upper semi-continuous decomposition of E^n, it follows from [10] that E^n/G is a separable metric space. Let $X = \bigcup \{g' |\ g \in H_G\}$. Then by Lemma 2, we have that $q = p|_X\colon X \to p(H_G^*)$ is open and onto. If $x \in p(H_G^*)$, then $q^{-1}(x) = g'$, where $p^{-1}(x) = g$. Therefore since g' is not dense in itself for each g, we have, by [4], that $\dim p(H_G^*) \leqslant \dim X \leqslant n$. Since p is a homeomorphism on $E^n - H_G^*$, it must be the case that $\dim(E^n/G - p(H_G^*)) \leqslant n$. Therefore, $\dim E^n/G \leqslant \dim(E^n/G - p(H_G^*)) + \dim p(H_G^*) + 1 \leqslant 2n+1$.

COROLLARY 1. *Suppose that G is a monotone nondegenerately continuous decomposition of E^n into points and polyhedral sets. Then* $\dim E^n/G<2n+1$.

Proof. For any $g\in H_G$, g' is a subset of the set of vertices of g. Hence, g' is not dense in itself for each $g\in H_G$.

COROLLARY 2. *Suppose that in the hypothesis of Theorem* 3.1 *or Corollary* 1, *"nondegenerately continuous decomposition" is replaced by "continuous decomposition". Then* $\dim E^n/G\leqslant n$.

Proof. If G is continuous, then $E^n/G-p(H_G^*)$ is closed in E^n/G. Therefore, by ([6], p. 32), $\dim E^n/G\leqslant n$.

The above results can be sharpened by replacing the condition that G be continuous with the condition that G be acyclic.

COROLLARY 3. *Suppose that G is a monotone, nondegenerately continuous, acyclic decomposition of E^n such that g' is not dense in itself. Then* $\dim E^n/G\leqslant n$.

Proof. The proof follows directly from Theorem 3.1 and [2].

4. Decompositions of 2-manifolds. Results of Moore [7] and Roberts and Steenrod [9] imply that if G is a monotone decomposition of a compact 2-manifold X, then $\dim X/G\leqslant 2$. The principal results of this section show that if G is a monotone upper semi-continuous decomposition of a 2-manifold (with or without boundary), then $\dim X/G\leqslant 2$.

THEOREM 4.1. *Suppose that X is a compact 2-manifold with boundary and G is a monotone upper semi-continuous decomposition of X. Then* $\dim X/G\leqslant 2$.

Proof. There exists a compact 2-manifold Y such that X can be obtained from Y by removing the interiors of a finite disjoint collection of closed disks. Let G' be the decomposition of Y consisting of members of G and singletons in the interiors of the above mentioned disks. Then clearly G' is a monotone upper semi-continuous decomposition of Y. Hence, $\dim Y/G'\leqslant 2$ [9]. Let $p\colon Y\to Y/G'$ be the natural projection. Clearly, X/G is homeomorphic to $p(X)\subset Y/G$. Therefore, $\dim X/G\leqslant 2$.

THEOREM 4.2. *Suppose that X is a 2-manifold (with or without boundary) and G is a monotone upper semi-continuous decomposition of X. Then* $\dim X/G\leqslant 2$.

Proof. We will need the following lemma.

LEMMA 1. *Suppose that X is a 2-manifold (with or without boundary) and g is a compact connected subset of X. Then there exists a compact 2-manifold K (with or without boundary) such that K is a neighborhood of g in X.*

Proof. Let α be a triangulation of X and β the collection of all members of α that intersect g. Note that β^* (the union of the members of β) is a neighborhood of g in X. Let $\{x_1, x_2, \ldots, x_r\}$ be the collection of vertices of members of β that lie in β^*-g. Then for sufficiently small ε, $\beta^*-\bigcup_{i=1}^{r} S_\varepsilon(x_i)$ is the required compact 2-manifold (with or without boundary).

Proof of Theorem 4.2. Suppose that $g\in G$. By Lemma 1, there exists a compact 2-manifold (with or without boundary) K such that K is a neighborhood of g in X. Let M be the saturated interior of K. There exists a saturated open set U in X such that $g\subset U\subset\overline{U}\subset M\subset K$. Let G_g be the decomposition of K such that $H_{G_g}=\{h\in H_G|\ h\cap\overline{U}\neq\emptyset\}$. It is not difficult to show that G_g is an upper semi-continuous decomposition. Since G_g is clearly monotone, we have that $\dim K/G_g\leqslant 2$. Suppose that $p\colon X\to X/G$ and $p_g\colon K\to K/G_g$ are the natural projections. Let $W=p^{-1}(p(\overline{U}))$. It is clear that $p(W)$ is homeomorphic to $p_g(W)$ and hence, $\dim p(W)\leqslant 2$. Since $p(W)$ is a neighborhood of $p(g)$ in X/G, $p(g)$ has arbitrarily small neighborhoods whose closure are contained in $p(W)$ and whose boundaries have dimension less than or equal to 1. Therefore, $\dim X/G\leqslant 2$.

5. Decompositions into convex sets. Suppose that X is a topological space and $U=\{U_1, U_2, \ldots, U_r\}$ is an open cover of X. Associate with each nonempty U_i an abstract point p_i. We construct the complex $N(U)$, called the *nerve of* U, with vertices $p_1, p_2, \ldots, p_r$ as follows: define $\langle p_{i_1}, p_{i_2}, \ldots, p_{i_k}\rangle$ to be a simplex of $N(U)$ if and only if $\bigcap_{j=1}^{k} U_{ij}\neq\emptyset$. Note that $\dim N(U)$ is the order of U. The *geometric realization of* $N(U)$, denoted by $P(U)$, is a polytope in some Euclidean space such that $P(U)$ and $N(U)$ have the same *vertex scheme*, i.e., the vertices of $N(U)$ and $P(U)$ are in a 1-1 correspondence in such a way that simplexes of $N(U)$ are described by vertices of $P(U)$ which span a cell in $P(U)$.

Suppose that X is a separable metric space and U is a finite open cover of X. Let P be a polytope whose vertices $p_1, p_2, \ldots, p_r$ are in a 1-1 correspondence with the nonempty members of U. Let Z_i be the star of p_i with respect to P. A mapping $g\colon X\to P$ is a barycentric U-mapping if $g^{-1}(Z_i)=U_i$ for each i.

THEOREM 5.1. *Suppose that X is a closed convex n-cell in E^n and G is a monotone upper semi-continuous decomposition of X into points and convex sets such that if $g\in H_G$, then $g\subset \operatorname{Int}X$. Then* $\dim X/G\leqslant n$.

Proof. We will need the following lemma.

LEMMA 1. *Suppose that X is a compact n-dimensional metric space and G is a monotone upper semi-continuous decomposition of X such that for every $\varepsilon>0$ there exists an ε-mapping of X/G into X. Then* $\dim X/G\leqslant n$.

Proof. By ([6], p. 72), it suffices to show that, for every $\varepsilon>0$, there exists an ε-mapping of X/G into a polyhedron of dimension less than or equal to n. Suppose that $\varepsilon>0$. By hypothesis, there exists an ε-mapping f of X/G into X. It is easy to show that there exists a positive real number δ such that if $A\subset f(X/G)$ with $\operatorname{diam}A<\delta$, then $\operatorname{diam}f^{-1}(A)<\varepsilon$. Since $f(X/G)$ is compact and of dimension less than or equal to n, there exists a δ-mapping g of $f(X/G)$ into a polyhedron of dimension less than or equal to n. Then the composition $g\circ f$ is the required ε-mapping.

Proof of Theorem 5.1. Let $\varepsilon>0$. We will show that there exists an ε-mapping f of X/G into X.

Let O_1 be a finite open cover of X/G consisting of sets whose diameter is less than or equal to $\frac{1}{2}\varepsilon$, and let $O_1' = p^{-1}(O_1)$. Then O_1' is a saturated open cover of X. Let O_2' be a finite open refinement of O_1' consisting of convex open sets with the property that for each $g \in G$ there exists an element $U \in O_2'$ such that $g \subset U$. Let O_3' be the collection of saturated interiors of members of O_2'. Then O_3' is a saturated finite open cover of X, and hence, if $O_3 = p(O_3')$, then O_3 is a finite open cover of X/G. Let O_4 be a finite star refinement of O_3 and let f be a barycentric O_4-mapping of X/G into the geometric realization of $N(O_4)$, $P(O_4)$. Denote $P(O_4)$ by K. If $O_4 = \{U_1, U_2, \ldots, U_k\}$, let $p_1, p_2, \ldots, p_k$ be the corresponding vertices of K with respect to f. We define inductively a mapping $g\colon K \to X$ as follows:

For each i, a mapping g_i from the i-skeleton K_i of K into X will be constructed with the property that

1) $g_i = g_{i-1}$ on K_{i-1},

2) if Δ is a simplex in K_i, then $g_i(\Delta)$ is contained in the intersection of all the members of O_2' that contain g_0 (vertices of Δ).

The mappings g_i are defined inductively. For each j, pick $x_j \in p^{-1}(U_j)$ and define $g_0(p_j) = x_j$. Suppose that a continuous function g_i has been defined on K_i satisfying 1) and 2) and define g_{i+1} as follows. If $x \in K_i$, define $g_{i+1}(x) = g_i(x)$. Suppose that Δ is an $(i+1)$-simplex of K_{i+1}, and that $p_{n_1}, p_{n_2}, \ldots, p_{n_r}$ are the vertices of Δ. Since Δ is a simplex in K, we have that $\bigcap_{i=1}^{r} U_{n_i} \neq \emptyset$, and hence, there exists an element U of O_3 such that $\bigcup_{i=1}^{r} U_{n_i} \subset U$. Therefore, $\{x_{n_1}, x_{n_2}, \ldots, x_{n_r}\} \subset p^{-1}(U) \in O_3'$. Consequently, $p^{-1}(U)$ is the saturated interior of some $W \in O_2'$. Thus, $\{x_{n_1}, x_{n_2}, \ldots, x_{n_r}\} \subset W$. Let V be the intersection of all the members of O_2' which contain $\{x_{n_1}, x_{n_2}, \ldots, x_{n_r}\}$. The boundary of Δ is a collection of i-simplexes, each of which is mapped by g_i into V. Therefore, g_{i+1} maps the boundary of Δ into the convex set V, and hence, g_{i+1} has a continuous linear extension over all of Δ. Since there are only a finite number of $(i+1)$-simplexes in K, we obtain a continuous extension of g_i to all of K_{i+1} which clearly satisfies 1) and 2).

Now define $g\colon K \to X$ by $g|_{K_i} = g_i$. Then g is the desired continuous function. Next we show that $g \circ f$ is an ε-mapping of X/G into X. Suppose that $x \in X/G$ and that $U_{n_1}, U_{n_2}, \ldots, U_{n_r}$ are precisely those sets in O_4 which contain x. Then $\bigcap_{i=1}^{r} U_{n_i} \neq \emptyset$. Let Δ be the simplex of K spanned by $p_{n_1}, p_{n_2}, \ldots, p_{n_r}$. There exists an element U of O_3 such that $\bigcup_{i=1}^{r} U_{n_i} \subset U$. Therefore, $\{x_{n_1}, x_{n_2}, \ldots, x_{n_r}\} \subset p^{-1}(U) \in O_3'$ and $p^{-1}(U)$ is the saturated interior of $W \in O_2'$. Since f is a barycentric O_4-mapping, we have that $f(x) \in \Delta$, and by construction $g(\Delta) \subset W$. However, O_2' refines O_1' and, consequently, there exists an element V of O_1' such that $W \subset V$. Therefore, we have

$$g\big(f(x) \cup p^{-1}(\bigcup_{i=1}^{r} U_{n_i}) \subset V\big),$$

and, hence, $p\big(g\big(f(x)\big)\big) \cup x \subset p(V)$. Since the diameter of any element of O_1 is less than $\frac{1}{2}\varepsilon$ and since $p(V) \in O_1$, we have that $d\big(p\big(g\big(f(x)\big)\big), x\big) < \frac{1}{2}\varepsilon$. Therefore, $g \circ f$ is an ε-mapping.

Now by Lemma 1, we can conclude that $\dim X/G \leqslant n$.

Using Theorem 5.1, we obtain the principal result of this section.

THEOREM 5.2. *If G is a monotone upper semi-continuous decomposition of E^n into points and convex sets, then* $\dim E^n/G \leqslant n$.

Proof. For each $g \in G$, there exists a closed convex n-cell C_g that contains g in its interior. Let U be the saturated interior of C_g. There exists an open set W such that $g \subset W \subset \overline{W} \subset U$. Let G_g be the decomposition of C_g such that

$$H_{G_g} = \{h \in H_G \mid h \cap \overline{W} \neq \emptyset\}.$$

It is easy to see that G_g satisfies the hypothesis of Theorem 5.1, and hence $\dim C_g/G_g \leqslant n$. Let $p\colon E^n \to E^n/G$ and $p_g\colon C_g \to G_g$ be the natural projections and let $K = p^{-1}\big(p(\overline{W})\big)$. It is not difficult to show that $p(K)$ is homeomorphic to $p_g(K)$, and therefore $\dim p(K) \leqslant n$. Since $p(K)$ is a neighborhood of $p(g)$ in E^n/G, we have that $p(g)$ has arbitrarily small neighborhoods whose boundaries have $\dim \leqslant n-1$. Therefore $\dim E^n/G \leqslant n$.

6. Further results.

THEOREM 6.1. *Suppose that G is an upper semi-continuous decomposition of E^n into points and compact $(n-1)$-manifolds. Then* $\dim E^n/G \leqslant n+1$.

Proof. Let $p(g) \in E^n/G$ and let W be an open neighborhood of $p(g)$ in E^n/G. There exists a connected open neighborhood V of $p(g)$ such that $p(g) \in V \subset \overline{V} \subset W$ and such that $\overline{V}$ is compact (E^n/G is locally connected and locally compact). Let $U = p^{-1}(V)$. Then U is an open, saturated, connected neighborhood of g in E^n and $\overline{U}$ is compact. For each $x \in (\operatorname{Bd} U) \cap H_G^*$ let g_x be the member of H_G which contains x and let I_x be the bounded complementary domain of g_x.

First we show that there exist at most countably many elements of H_G which intersect $\operatorname{Bd} U$. Since U is saturated and connected, it follows that $I_x \cap U \neq \emptyset$ if and only if $U \subset I_x$. Furthermore, since $I_x \cap I_y \neq \emptyset$ if and only if either $I_x \subset I_y$ or $I_y \subset I_x$, it must be the case that at most one of the I_x's can contain U. This implies that for any $x \in (\operatorname{Bd} U) \cap H_G^*$, I_x can contain at most one additional I_y for $y \in (\operatorname{Bd} U) \cap H_G^*$. Since there can be at most a countable collection of disjoint I_x's, each of which can contain at most one additional I_x, it must be the case that there are at most countably many elements of H_G which intersect $\operatorname{Bd} U$.

Note that $\operatorname{Bd} V = p\big((\operatorname{Bd} U) - H_G^*\big) \cup \{p(g_y) \mid y \in (\operatorname{Bd} U) \cap H_G^*\}$. Since $\operatorname{Bd} U$ is a closed subset of E^n with no interior, it follows that $\dim \operatorname{Bd} U \leqslant n-1$ ([6], p. 44). Since p is a homeomorphism on the complement of H_G^*, it follows that

$$\dim p(\operatorname{Bd} U - H_G^*) \leqslant n-1 .$$

Then since the collection $\{p(g_y) | y \in (\mathrm{Bd}\, U) \cap H_G^*\}$ is a countable collection of points, and hence 0-dimensional, we have that $\dim \mathrm{Bd}\, V \leqslant n$. This implies that $\dim E^n/G \leqslant n+1$.

COROLLARY 1. *Suppose that G is a continuous decomposition of E^n into points and compact $(n-1)$-manifolds. Then $\dim E^n/G \leqslant n$.*

Proof. If G is continuous, then $p(\mathrm{Bd}\, U - H_G^*)$ is closed in E^n/G. Hence, by ([6], p. 32), $\dim \mathrm{Bd}\,\mathrm{Bd}\, V \leqslant n-1$. Therefore, $\dim E^n/G \leqslant n$.

THEOREM 6.2. *Suppose that G is a monotone upper semi-continuous decomposition of a 3-manifold X and suppose that, for every $g \in G$ and for every neighborhood V of g, there exists a 3-cell U such that 1) $g \subset \mathrm{Int}\, U \subset U \subset V$ and 2) for each $h \in G$, the collection of components of $h \cap \mathrm{Bd}\, U$, when considered as points, is not dense in itself. Then $\dim X/G \leqslant 3$.*

Proof. Suppose that $p(g) \in X/G$ and V is any open neighborhood of $p(g)$ in X/G. Then $p^{-1}(V)$ is a saturated open neighborhood of g in X. Therefore, there exists a closed 3-cell U with $g \subset \mathrm{Int}\, U \subset U \subset p^{-1}(V)$ such that for each $h \in G$ the collection of components of $h \cap \mathrm{Bd}\, U$, when considered as points, is not dense in itself. Let $H = \{h \subset \mathrm{Bd}\, U |\ h$ is a component of $k \cap \mathrm{Bd}\, U$ for some $k \in G\}$. Then H is a monotone upper semi-continuous decomposition of the 2-sphere $\mathrm{Bd}\, U$. Thus, $\dim (\mathrm{Bd}\, U)/H \leqslant 2$. Let $p_H\colon \mathrm{Bd}\, U \to (\mathrm{Bd}\, U)/H$ and $p\colon E^3 \to E^3/G$ be the natural projections and set $K = \{p_H(g \cap \mathrm{Bd}\, U) |\ g \in G$ and $g \cap \mathrm{Bd}\, U \neq \emptyset\}$. It is easy to show that K is an upper semi-continuous decomposition of $(\mathrm{Bd}\, U)/H$ and, by hypothesis, members of K are not dense in themselves. It is also not difficult to show that $p(\mathrm{Bd}\, U)$ is homeomorphic to $((\mathrm{Bd}\, U)/H)/K$

By [10], all of the above mentioned spaces are separable metric spaces. Since $\dim (\mathrm{Bd}\, U)/H \leqslant 2$ and since members of K are not dense in themselves, we have, by ([8], p. 85), that $((\mathrm{Bd}\, U)/H)/K$, and consequently $p(\mathrm{Bd}\, U)$, is a compact σ_0 space. Let W be the saturated interior of $\mathrm{Int}\, U$ with respect to G. Then $p(W)$ is an open neighborhood of $p(g)$ and $p(g) \in p(W) \subset \overline{p(W)} \subset V$. Furthermore, $\mathrm{Bd}\, p(W) \subset p(\mathrm{Bd}\, U)$. Hence, $\mathrm{Bd}\, p(W)$ is a compact σ_0 space. G is cellular and, therefore, $p|_{p^{-1}(\mathrm{Bd}\, p(W))}$ is acyclic. This implies by [2] that $\mathrm{Bd}\, p(W)$ can contain no compact subset of finite dimension larger than 3. Therefore, by [3], it must be the case that $\dim \mathrm{Bd}\, p(W) \leqslant 3$. Consequently, by the definition of dim, $\dim X/G \leqslant 4$. Using [2] once again, we have that $\dim X/G \leqslant 3$.

COROLLARY 1. *Suppose that G is a cellular upper semi-continuous decomposition of E^n into points and polyhedral sets. Then $\dim E^3/G \leqslant 3$.*

Proof. Suppose that $g \in G$ and that V is a neighborhood of g in E^3. There exists a polyhedral 3-cell U such that $g \subset \mathrm{Int}\, U \subset U \subset V$. Then U satisfies the hypothesis of Theorem 6.2, and the result follows.

References

[1] R. D. Anderson, *Monotone interior dimension-raising mapping*, Duke Math. J. 19 (1952), pp. 359–366.

[2] E. Dyer, *Certain transformations with lower dimension*, Annals Math. 63 (1956), pp. 15–19.

[3] D. W. Henderson, *Finite dimensional subsets of infinite dimensional spaces*, Topology Seminar Wisconsin, 1965, Princeton, New Jersey, 1966, pp. 141–146.

[4] R. Hodel, *Open functions and dimension*, Duke Math. J. 30 (1963), pp. 461–467.

[5] W. Hurewicz, *Über oberhalf-stetige Zerlegungen von Punktmengen in Kontinua*, Fund. Math. 15 (1930), pp. 57–60.

[6] — and H. Wallman, *Dimension Theory*, Princeton, New Jersey, 1948.

[7] R. L. Moore, *Concerning upper semi-continuous collections*, Monatsh. Math. Phys. 36 (1929), pp. 81–88.

[8] K. Nagami, *Dimension Theory*, New York–London 1970.

[9] J. H. Roberts and N. E. Steenrod, *Monotone transformations of two-dimensional manifolds*, Annals Math. 39 (1938), pp. 851–862.

[10] A. H. Stone, *Metrizability of decomposition spaces*, Proc. Amer. Math. Soc. 7 (1956), pp. 690–700.

[11] F. Valentine, *Convex Sets*, New York 1964.

UNIVERSITY OF IDAHO
Moscow, Idaho

Accepté par la Rédaction le 26. 3. 1975